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Localized beams and dielectric barriers

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Abstract
Recalling the similarities between the Maxwell equations for a transverse electric wave in a stratified medium and the quantum mechanical Schrödinger equation in a piecewise potential, we investigate the analogue of the so-called particle limit in quantum mechanics. It is shown that in this limit the resonance phenomena are lost since individual reflection and transmission terms no longer overlap. The result is a stationary zebra-like response with the intensity in each stripe calculable.

Keywords: propagation, diffusion, tunneling for localized beams

1. Introduction

There exist many unanswered questions in potential theory quantum mechanics. Amongst these are the existence of multiple diffusion phenomena [1, 2], the Hartman effect with its apparent violation of causality [3, 4] and the importance, if any, of wavepackets in oscillation phenomena [5, 6]. Most of these lack direct experimental measurements. It is therefore extremely instructive to study that class of Maxwell equations which are analogous to the Schrödinger equation. These analogies are well known in optics, e.g. we often find references to tunneling and resonance phenomena. However, their relevance to quantum mechanics has not been fully exploited. This paper studies an example of the above analogy.

From the Maxwell equations, we can obtain differential equations which the electric and the magnetic vectors must separately satisfy [7]. For example, for the electric field $\boldsymbol{E}$, in the case of no charges or currents, one has

$$\nabla^2 \boldsymbol{E} - \frac{\epsilon\mu}{c^2}\,\partial_{tt}\boldsymbol{E} + (\nabla \ln\mu) \times (\nabla \times \boldsymbol{E}) + \nabla(\boldsymbol{E}\cdot\nabla\ln\epsilon) = 0. \qquad (1)$$

The corresponding equation for the magnetic field $\boldsymbol{H}$ is obtained by making the changes $\epsilon \leftrightarrow \mu$ and $\boldsymbol{E} \rightarrow \boldsymbol{H}$. We shall confine our attention to the study of a medium characterized by a real (no attenuation) refractive index whose properties are constant throughout each plane perpendicular to the chosen direction $\boldsymbol{z}$, stratified medium [8]:

$$n(z) = \{n_{\mathrm{I}} \text{ for } z < 0,\, n_{\mathrm{II}} \text{ for } 0 < z < L,\; n_{\mathrm{III}} = n_{\mathrm{I}} \text{ for } z > L\}, \qquad (2)$$

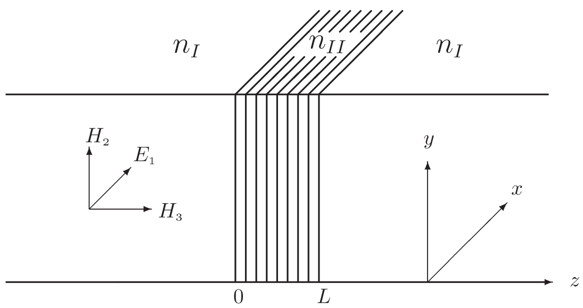
and for which μ assumes the same value in all three regions. By taking the plane of incidence to be the y–z plane, for a monochromatic, $\exp(-\mathrm{i}\omega t)$, transverse electric wave ($E_{2,3} = 0$), equation (1) reduces to

$$\partial_{yy}E_1(y,z) + \partial_{zz}E_1(y,z) + n^2(z)k^2E_1(y,z) = 0, \qquad (3)$$

with $k = \omega/c$ and where we have used the fact that $\nabla\cdot(\epsilon\boldsymbol{E}) = 0$ implies that E_1 is a function of y and z only. The components of the magnetic vector can be determined by using $\nabla\times\boldsymbol{E} = -\partial_t(\mu\boldsymbol{H})/c$:

$$\{H_2(y,z),\, H_3(y,z)\} = \frac{\mathrm{i}}{k\mu}\left\{-\partial_z E_1(y,z),\, \partial_y E_1(y,z)\right\}. \qquad (4)$$

The geometry of our problem is schematically represented in the following picture:



Looking for a separable solution of the form

$$E_1(y,z) = U(z)\exp(\mathrm{i}n_{\mathrm{I}}\sin\theta ky),$$

with θ representing the incidence angle, we obtain the following second-order linear differential equation for $U(z)$:

$$U''(z) + [n^2(z) - n_{\rm I}^2 + n_{\rm I}^2 \cos^2\theta] k^2 U(z) = 0. \quad (5)$$

This equation is formally identical to the one-dimensional Schrödinger equation when the factor which multiplies $U(z)$ is replaced by $2m(E - V)/\hbar^2$. As for Schrödinger the solutions are oscillatory (traveling waves) or evanescent (tunneling) according to whether the term in square brackets is positive or negative, respectively. A particular plane wave solution of equation (5), corresponding to an incoming wave in region I, is

$$z < 0 : \exp(\mathrm{i}n_{\rm I}\cos\theta kz) + R\exp(-\mathrm{i}n_{\rm I}\cos\theta kz),$$
$$0 < z < L : F\exp\left(\mathrm{i}n_{\rm I}\sqrt{\tilde{n}^2 - \sin^2\theta}kz\right) + G\exp\left(-\mathrm{i}n_{\rm I}\sqrt{\tilde{n}^2 - \sin^2\theta}kz\right),$$
$$z > L : T\exp(\mathrm{i}n_{\rm I}\cos\theta kz),$$

with $\tilde{n} = n_{\rm II}/n_{\rm I}$ and R, F, G and T determined by the boundary conditions. The boundary conditions can be set as in quantum mechanics by noting that for piecewise discontinuities in the 'potential' $n(z)$ the function $U(z)$ and its first derivative $U'(z)$ must be continuous at the boundaries or, equivalently, since the magnetic field is proportional to the derivative of the electric field, by the continuity of these fields across the boundaries. Note that the exponentials in region II are oscillatory when $\tilde{n}^2 > \sin^2\theta$ and evanescent for $\tilde{n}^2 < \sin^2\theta$. Thus, if $\tilde{n} > 1$ the solutions will always yield a propagation wave in region II of the y–z plane with direction given by Snell's law. When $\tilde{n} < 1$ both types of solutions exist, with tunneling occurring when $\sin\theta > \tilde{n}$, i.e. for incident angles greater than a critical value $\theta_{\rm c}$ ($\sin\theta_{\rm c} = \tilde{n}$).

The comparison of $E_1(y, z)$ with $\psi(t, z)$ in non-relativistic quantum mechanics also implies that the y dependence replaces the time dependence of the latter. This accounts for the constant term in front of $U(z)$ in equation (5). Another identity is the condition (which we prove in the next section) that

$$|R|^2 + |T|^2 = 1.$$

In quantum mechanics this implies conservation of probability [9] while here it implies conservation of energy [10].

2. Reflection and transmission coefficients

Consider first diffusion ($\tilde{n} > \sin\theta$) and treat the continuity conditions for $U(z)$ and $U'(z)$ at the two interfaces ($z = 0, L$) independently. Let r_0 and t_0 be the coefficients at the $z = 0$ interface, then

$$r_0 = \left(\cos\theta - \sqrt{\tilde{n}^2 - \sin^2\theta}\right) \Big/ \left(\cos\theta + \sqrt{\tilde{n}^2 - \sin^2\theta}\right),$$
$$t_0 = 2\cos\theta \Big/ \left(\cos\theta + \sqrt{\tilde{n}^2 - \sin^2\theta}\right). \quad (6)$$

For a wave traveling from region II to region I (e.g. a wave reflected from the $z = L$ interface) the corresponding coefficients $\tilde{r}_0$ and $\tilde{t}_0$ are

$$\tilde{r}_0 = -r_0, \qquad \tilde{t}_0 = \sqrt{\tilde{n}^2 - \sin^2\theta}\, t_0/\cos\theta. \quad (7)$$

Note that for diffusion all the coefficients in equations (6) and (7) are real. At the $z = L$ interface, we need only consider waves impinging from the left since, for our choice of particular solution, there is no incoming wave from the right in region III. Thus, the only reflection and transmission coefficients are

$$r_L = \tilde{r}_0 \exp\left(2\mathrm{i}n_{\rm I}\sqrt{\tilde{n}^2 - \sin^2\theta}kL\right),$$
$$t_L = \tilde{t}_0 \exp\left[\mathrm{i}n_{\rm I}\left(\sqrt{\tilde{n}^2 - \sin^2\theta} - \cos\theta\right)kL\right]. \quad (8)$$

Now, we may calculate the R and T coefficients by summing individual *multiple* reflection contributions, e.g. the first contribution to R will be r_0, the second will be $t_0 r_L \tilde{t}_0$ and so forth:

$$R = r_0 + t_0 r_L \tilde{t}_0 + t_0 r_L \tilde{r}_0 r_L \tilde{t}_0 + \cdots + t_0 r_L (\tilde{r}_0 r_L)^n \tilde{t}_0 + \cdots. \quad (9)$$

The series converges because $0 < \theta < \pi/2$ implies that $|\tilde{r}_0 r_L| < 1$. Summing, we find

$$R = r_0 + t_0 r_L \tilde{t}_0 / (1 - \tilde{r}_0 r_L). \quad (10)$$

In the same way

$$T = t_0 t_L + t_0 \tilde{r}_0 r_L t_L + \cdots + t_0 (\tilde{r}_0 r_L)^n t_L + \cdots$$
$$= t_0 t_L / (1 - \tilde{r}_0 r_L). \quad (11)$$

Consequently, using the identities $\tilde{r}_0 = -r_0$ and $r_0^2 + t_0\tilde{t}_0 = 1$, we find

$$|R|^2 = |r_0 + r_L|^2/|1 + r_0 r_L|^2 \quad \text{and} \quad |T|^2 = |t_0 t_L|^2/|1 + r_0 r_L|^2. \quad (12)$$

It follows after a little algebra that

$$|R|^2 + |T|^2 = 1, \quad (13)$$

as anticipated. With the above expressions for R and T, the so-called *wave limit* or total coherence, identical to the quantum mechanics results, we reproduce the standard phenomena of *resonance* when $|T|^2 = 1$. This occurs when the phase in r_L is such that

$$r_L = -r_0 \Leftrightarrow n_1\sqrt{\tilde{n}^2 - \sin^2\theta}kL = n\pi.$$

There is, however, another way to interpret the series expansions for R and T. Let us introduce it by simply observing a numerical fact. If we (modulus) square the individual terms in the series and then add, we find a different $|R|^2$ and $|T|^2$:

$$\sum_n |R_n|^2 = r_0^2 + (t_0 r_0 \tilde{t}_0)^2/(1 - r_0^4) \neq |R|^2 \quad \text{and} \quad \sum_n |T_n|^2 = (t_0\tilde{t}_0)^2/(1 - r_0^4) \neq |T|^2, \quad (14)$$

with conservation of energy in the *particle limit*, where the interference between individual n amplitudes is null:

$$\sum_n \left(|R_n|^2 + T_n|^2\right) = 1. \quad (15)$$

These two *limits* are easily explained. The former *wave limit* occurs when all the contributions overlap as is the case of plane waves. The second *particle limit* occurs when no overlapping occurs (see section 3). In quantum mechanics this latter limit corresponds to wavepackets small compared to the barrier width (L). The details of how the wavepackets are created is not important. It is the limit when the time taken for a wavepacket to travel back and forth in region II is sufficient to separate the individual reflected and/or transmitted wavepackets. There are, of course, intermediate cases of partial overlap. Notice that in the particle limit there is *no* resonance phenomena. Similarly, for any given localized optical transverse electric beam, we can calculate $|T|^2$ for various L values (see section 3) and see the transition from typical oscillatory (resonance) shape to a constant (particle) limit.

3. Localized beams (numerical analysis)

In quantum mechanics the particle limit is obtained by considering narrow (compared to the barrier width) wavepackets [1, 2, 11, 12]. This is done by integrating the plane wave results with, say, a Gaussian function in particle momentum. The optical equivalent is to integrate over the incoming angle θ and again this can be done with a Gaussian in θ (see below). In situations in which tunneling may occur one should formally limit the allowed values of θ to either the diffusion or tunneling regions. In practice a strongly peaked dependence around a mean θ value, say $\theta = \theta_0$, is sufficient as long as θ_0 is sufficiently removed from the critical angle θ_c.

We shall use for our numerical calculations the following Gaussian function:

$$g(\alpha) = \frac{\sqrt{\delta}}{(2\pi)^{3/4}} \exp\left[-\frac{(\alpha-\alpha_0)^2\,\delta^2}{4}\right], \tag{16}$$

with $\alpha = \cos\theta$ ($\alpha_0 = \cos\theta_0$) and $\delta = n_I kd$.

The integration over angles about θ_0 produces a spatial localization in z and y (the analogue of a quantum mechanics wavepacket). We recall that for our optical study all results are time-independent (stationary). The localized distributions in z and y are for the incoming, reflected and transmitted beams given by

$$E_{1,\text{inc}}(y,z) = \int_0^1 \mathrm{d}\alpha\, g(\alpha) \exp(\mathrm{i}\alpha\delta z_d) \times \exp(\mathrm{i}\sqrt{1-\alpha^2}\,\delta y_d),$$

$$E_{1,\text{ref}}(y,z) = \int_0^1 \mathrm{d}\alpha\, R(\alpha) g(\alpha) \exp(-\mathrm{i}\alpha\delta z_d) \times \exp\left(\mathrm{i}\sqrt{1-\alpha^2}\,\delta y_d\right),$$

$$E_{1,\text{tra}}(y,z) = \int_0^1 \mathrm{d}\alpha\, T(\alpha) g(\alpha) \exp(\mathrm{i}\alpha\delta z_d) \times \exp\left(\mathrm{i}\sqrt{1-\alpha^2}\,\delta y_d\right),$$

with $y_d = y/d$, $z_d = z/d$ and $L_d = L/d$.

For diffusion phenomena, and when the beam localizations are smaller than the dimension L of region II, we obtain a zebra-like structure sketched in figure 1(b). The various reflected (transmitted) beams are separated in the y–z plane. No interference occurs between them and consequently no resonance phenomena exists. As in quantum mechanics, these multiple structures do not occur in tunneling phenomena [3, 4, 13], see figure 1(a). Indeed for tunneling the sum $\sum_n |R_n|^2$ diverges, as does $\sum_n |T_n|^2$. Thus the individual terms cannot be identified with physical probabilities. In tunneling only *one* reflected and transmitted wave exists. The calculation of R_n and T_n is then, at best, a technique for the determination of R and T.

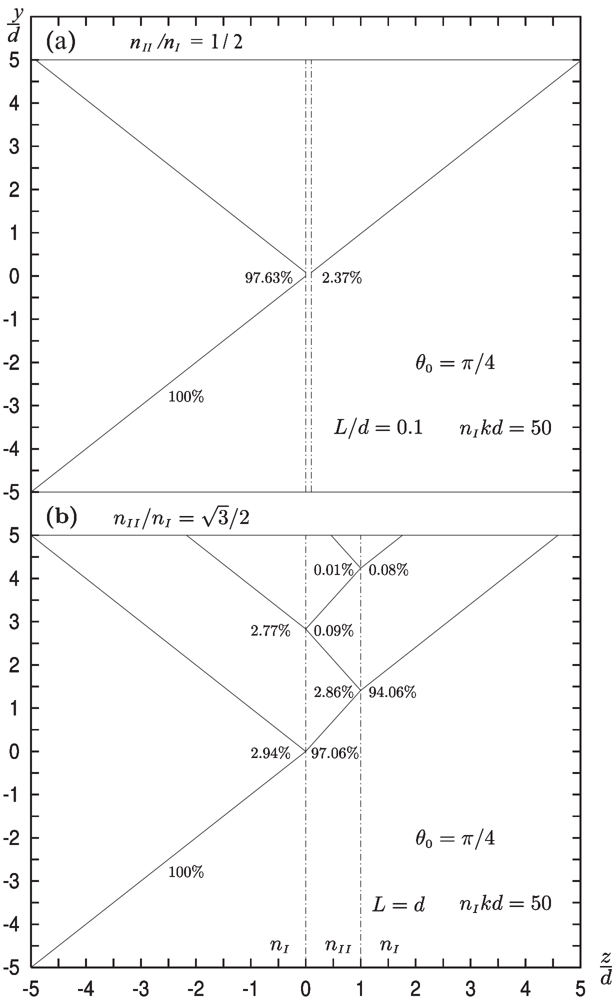


Figure 1. Tunneling (a) and diffusion (b) of a localized optical beam by a dielectric film. The localization is achieved by integrating over the incidence angle (narrow Gaussian distribution around the angle $\theta_0 = \pi/4$). For diffusion phenomena, when the beam localizations are smaller than or of the order of the dimension of region II, we obtain a zebra-like structure, i.e. a multiple diffusion.

We can exhibit these differences graphically by considering a narrow beam incident at an angle $\theta_0 = \pi/4$. Two cases for $\tilde{n}$ will be considered: $\tilde{n} = 1/2$ and $\tilde{n} = \sqrt{3}/2$. The choice of a narrow beam ($\delta = 50$) guarantees that the Gaussian angle distribution, centered in $\pi/4$, is practically zero for $\theta < \pi/6$ (the critical angle for $\tilde{n} = 1/2$) and for $\theta > \pi/3$ (the critical angle for $\tilde{n} = \sqrt{3}/2$). Consequently, for $\tilde{n} = 1/2$ we have 'tunneling' and for $\tilde{n} = \sqrt{3}/2$ we have multiple diffusion. In figures 2 and 3 we display, for the diffusion case, plots of $|E_1|^2$

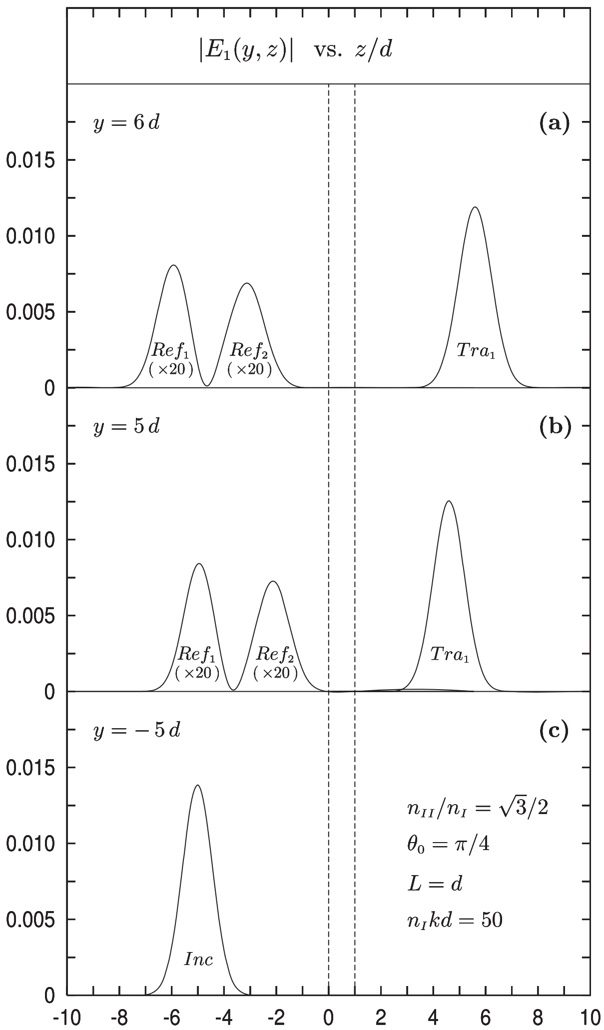


Figure 2. For the diffusion case, the plots of $|E_1|^2$ against z for fixed y show the multiple beams and the localization in the z axis.

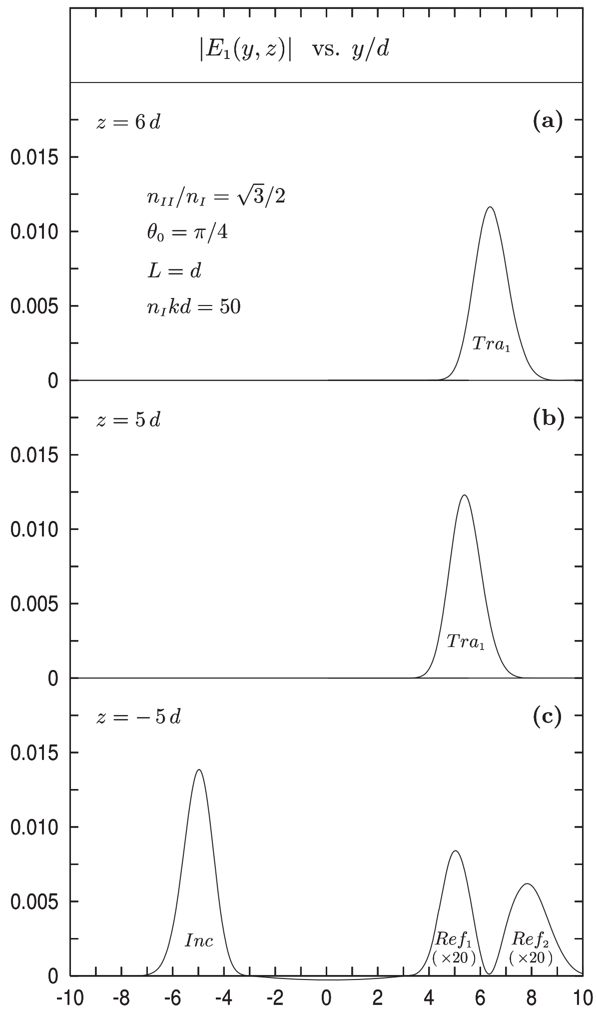


Figure 3. For the diffusion case, the plots of $|E_1|^2$ against y for fixed z show the multiple beams and the localization in the y axis.

against z for fixed y and against y for fixed z. We readily see the multiple beams. Comparison of figures 2(a) and (b) shows that the reflected beams remain separated in z.

4. Conclusions

In this paper, we have studied the behavior of a localized beam in a stratified medium. *The localization is achieved by integrating over the incidence angle.* Depending on the value of $\tilde{n}$, we have two phenomena. One is the formation of multiple beams, the other occurring for tunneling yields a single reflected and transmitted beam. We have shown some examples of these phenomena. In both cases resonance effects are absent. If we substitute the y axis with the time axis, we replicate the results of multiple diffusion and/or tunneling in non-relativistic quantum mechanics. There are, of course, some significant differences. Foremost are the absence of $\hbar$ and the interpretation of $|R|^2$ and $|T|^2$ in terms of energy probabilities. The wavepackets in quantum mechanics move in time. In our optical model all results are stationary. This is a significantly useful feature, since time measurements are all but impractical in quantum mechanics [4].

This analogy allows us to anticipate some further consequences for localized optical beams. For example, while the resonance phenomenon in tunneling is absent for a single barrier, it surprisingly reappears [2] for twin or even multiple identical barriers (always in the tunneling regime). Again this is a consequence of interference. However, if the size of the barriers, including the inter-barrier distances, is much larger than the incoming beam, resonance will not occur and multiple beams, caused by reflections between barriers, will appear. The ephemeral nature of probability densities makes an optical analogy simpler to create and study. It is even conceivable that questions related to *tunneling times* for which there are diverse definitions [4] and the related Hartman effect [3] could be studied experimentally. Another potential source of study is the effect of localization (wavepackets) on theoretical predictions almost always based upon a plane wave analysis.

The results of this paper, with the wave and particle limits, clearly demonstrate that these effects can be significant. In particle physics the effect of wavepackets upon phenomena such as neutrino oscillations, and oscillation phenomena in general, have little or no possibility of experimental testing. Perhaps through optics, we may investigate some of these questions.

We have exhibited in our numerical analysis, and shown in our graphs, the absence of a particle limit in the case of tunneling. The theoretical method of calculation we have used, based upon the sum of individual contributions, still works but only if interpreted as an analytic continuation of the diffusion case. The infinite series in tunneling formally diverges. This is most simply seen by the fact that in this case r_0 is complex with *unitary modulus*.

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