Barrier paradox in the Klein zone

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We study the solutions for a one-dimensional electrostatic potential in the Dirac equation when the incoming wave packet exhibits the Klein paradox (pair production). With a barrier potential we demonstrate the existence of multiple reflections (and transmissions). The antiparticle solutions which are necessarily localized within the barrier region create new pairs with each reflection at the potential walls. Consequently we encounter a new "paradox" for the barrier because successive outgoing wave amplitudes grow geometrically.

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I. INTRODUCTION

In this paper, we consider two related one-dimensional (electrostatic) square potentials: the step and barrier potential within the Dirac equation. We shall analyze them for the case when the incoming energy (energies for wave packets) are in what we shall call the *Klein zone*, i.e., when $E < V_0 - m$, where V_0 is the step/barrier height and m is the particle mass. This is the situation in which only oscillatory solutions exist throughout and where the so called Klein paradox reigns for the step [1].

They will be analyzed with use of the stationary plane wave method [2–4]. However, when appropriate, we shall also use wave packet arguments and terminology such as group velocity, arrival times, etc. For brevity, we will not recall here the underlying formalism of convolution integrals or stationary phase methods. We hope that these switches from time independent to time dependent viewpoints, even if in the same sentence, will not lead to any confusion. Actually, this is quite a common practice and occurs, for example, whenever two contributing plane wave solutions are referred to individually as "incoming" and/or "outgoing."

We start with the step potential by recalling in the next section the arguments which lead to the Klein paradox, in which the reflection probability is higher than the incoming probability. This is really no longer a paradox since it is universally interpreted as due to the creation of a particle-antiparticle pair at the potential discontinuity [5–9]. For an electrostatic potential the antiparticle "sees" a potential dip where the particle sees a potential rise and vice versa [10,11]. This explains why oscillatory solutions appear in the plane wave analysis both for free regions (zero potential) and non. The below potential oscillatory solutions are thus identified with *physical* (above potential) antiparticles. The antiparticles will be shown to have energy $-E$ over a potential of $-V_0$, so that with respect to the potential free region they also lie in a Klein zone, i.e., $-E < 0 - m$ since necessarily the relativistic energy satisfied $E > m$. Even the alternative candidate step solution in which the reflection probability is less than the incoming probability can only be understood by *pair annihilation* at the step. However, this solution necessarily implies the existence of an incoming antiparticle in addition to the incoming particle and thus is rejected because it violates the initial conditions assumed.

As for the barrier, a plane wave analysis yields only *one* solution, that with the sum of the reflected and transmitted probabilities (both positive) lower than the incoming probability [12,13]. Not only does this solution appear inconsistent with the step result (Klein paradox) but it seems, to us, not interpretable in terms of pair production and/or annihilation. To try and understand the situation, we shall then apply a procedure we have called the two step approach to the barrier. We previously applied it to the above barrier diffusion [14]. In that case it exemplified the presence of multiple wave packets [15], due to multiple reflections, which only in the limit of *complete overlap* reproduced the plane wave barrier result and consequent resonance effects.

While the two step analysis guarantees consistency between the step and barrier because it uses the former for the calculation of the latter, it will lead us to a new kind of paradox. Any antiparticle created at either of the steps that form the barrier will necessarily lie entrapped in the barrier region (which it sees as a well). It will bounce back and forth indefinitely. Since it also satisfies the Klein condition, it will have a nonzero probability of creating antiparticle-particle pairs at each reflection. This means that the density of antiparticles in the well will grow at each reflection. A corresponding geometric growth of the multiple reflected and transmitted particle probabilities also occurs.

In the Appendix, we collect the relevant Dirac solutions and define our conventions [16]. This is done in three dimensions, although for our calculations we consider only one-dimensional potentials along the z axis. We shall also assume that the incoming spin is polarized along the same axis (i.e., a positive helicity eigenstate). Since all spin-flip terms can readily be shown to be absent, we shall neglect them and this will simplify our continuity equations. The Appendix also contains the Dirac solutions in the presence of a nonzero, but constant, potential which we will indeed use.

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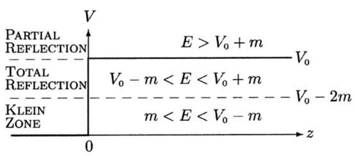
In the next section we describe the solutions to the step potential and in particular the Klein paradox. These demonstrations can be found in numerous articles [5–9] and textbooks [10,11,16], but with an important proviso. The traditional approach uses "positive energy" (above-potential) spinors under the step. We, on the contrary, employ the below-potential solutions there. Nevertheless, the results for *probabilities* turn out to be the same. In Sec. III, we present our plane wave analysis of the barrier. We describe the difficulties of the interpretation of this solution in terms of pair creation/annihilation. Section IV repeats this analysis but with the two step method. An important revaluation of the results of the previous section can then be made. We conclude in Sec. V with a resume of our results and a discussion of the predicted unlimited growth of antiparticle and particle wave packets densities.

II. THE SINGLE STEP

Our potential model is one-dimensional with the z axis chosen as the privileged space direction. The potential is chosen as



$$V(z) = \begin{cases} 0\,, & z<0\,, \quad \text{REGION I}\,, \\ V_0\,, & z>0\,, \quad \text{REGION II}\,. \end{cases}$$

For the purpose of this paper, the energy of each plane wave lies by assumption in the Klein zone $E<V_0-m$. We also assume that the incoming wave is a positive helicity positive energy solution. Ignoring, for simplicity, the spin flip terms which can easily be shown not to exist [16], the reflected wave is consequently a negative helicity (because of the change in direction) outgoing particle. The solutions in the free zone can thus be written as

$$\Psi_{\text{I}}(z,t) = \{u^{(1)}(p;0)\exp[ipz] + Ru^{(1)}(-p;0) \times\exp[-ipz]\}\exp[-iEt], \qquad (1)$$

with

$$u^{(1)}(p;0) = \begin{pmatrix} 1 \\ 0 \\ \dfrac{p}{E+m} \\ 0 \end{pmatrix},$$

and $p=\sqrt{E^2-m^2}$. By choice we use unnormalized spinors which are equivalent to the convention of absorbing any normalization factors into the coefficients 1, R, and T. The reflection coefficient is R. The reflection probability is consequently $|R|^2$. It should be warned that some other authors use instead the letter R for the reflection probability.

The solution *under* the step (see Appendix) can initially be taken as either of two forms differing in the sign of the momentum, not *both* because we expect there to be only one outgoing wave in the region with $z>0$ in analogy with the above step case. We shall consider them one at a time since it is by no means obvious *which* of them represents a physical outgoing (right moving) object. Consider first the $u^{(3)}\times(q;V_0)\exp[iqz]$ solution, with transmission coefficient T,

$$\Psi_{\text{II}}(z,t) = Tu^{(3)}(q;V_0)\exp[iqz]\exp[-iEt], \qquad (2)$$

where

$$u^{(3)}(q;V_0) = \begin{pmatrix} -\dfrac{q}{|E-V_0|+m} \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

and $q=\sqrt{(E-V_0)^2-m^2}$.

Continuity at $z=0$ yields

$$1+R = -\frac{q}{|E-V_0|+m}T,$$

$$1-R = \frac{E+m}{p}T.$$

We note that with our choice of spinors we have just two equations in two unknowns and the fact that a solution to these equations exists is confirmation, *a posteriori*, of the absence of spin-flip. The solution is

$$R = \frac{\alpha+1}{\alpha-1} \quad \text{and} \quad T = \sqrt{\frac{E-m}{E+m}}\frac{2}{1-\alpha}, \qquad (3)$$

with

$$\alpha = \sqrt{(|E-V_0|-m)(E-m)}/\sqrt{(|E-V_0|+m)(E+m)}.$$

Since we are in the Klein zone $|E-V_0|>m$ and hence $1>\alpha>0$. This means that $R<-1$ and that the reflection probability is $|R|^2>1$, i.e., exceeds the incoming probability (Klein paradox). The direct calculation of the transmitted probability is not $|T|^2$ because we are in a region with a different value of the potential to the incoming wave. The transmitted probability is given by

$$\begin{aligned} &\frac{q}{|E-V_0|}|Tu^{(3)}(q;V_0)|^2/\frac{p}{E}|u^{(1)}(p;0)|^2 \\ &= \frac{q}{|E-V_0|}\frac{E-m}{E+m}\frac{4}{(1-\alpha)^2}\frac{2|E-V_0|}{|E-V_0|+m}\frac{E}{p}\frac{E+m}{2E} \\ &= \frac{4\alpha}{(1-\alpha)^2} = |R|^2-1. \end{aligned} \qquad (4)$$

The same result is obtained by using the traditional spinor $u^{(1)}$ under the step [16]. This latter choice of spinor is formally incorrect and has the unpleasant feature of containing a denominator $(E-V_0)+m$ which tends to zero in the (antiparticle) rest frame limit (limit of the Klein zone) when $[-E-(-V_0)]\to m$. However, it is just this vanishing denominator which inverts the "small component" with the "large component" in the nonrelativistic situation, and consequently yields the same results as that above.

Now, let us interpret the Klein paradox in *physical* terms.

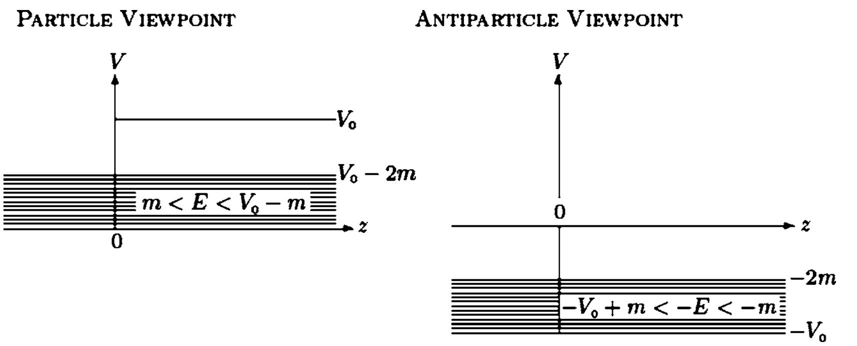


FIG. 1. Particle and antiparticle viewpoints. The energy of the "particle" is that of the incoming particle, i.e., E. Since the antiparticle sees a potential of $-V_0$ its energy, say E_a, must satisfy $[E_a-(-V_0)]^2=(E-V_0)^2$. Consequently, $E_a=-E$.

Since the incoming particle is assumed charged (recall that the potential is of an electrostatic nature) this means that the reflected wave packet carries more charge than the incoming one. Extra particle charge has been created at reflection. Since charge is a conserved quantum number the wave packet in the positive z region must be of opposite charge, i.e., represent an antiparticle wave packet traveling above a potential trough. It is a basic axiom of our interpretation that the below barrier solutions *cannot* be particles, they must represent antiparticles or "holes." Hence, *pair creation* has occurred, or more precisely has a probability of occurring. What is more, the antiparticle wave packet *must move to the right* since it must exist for all future times to guarantee charge conservation [5–9].

Figure 1 shows the particle and antiparticle viewpoint. The potential flips sign because the antiparticle's charge is opposite to that of the particle. The energy of the "particle" represented by $u^{(3)}$ is, by fiat, that of the incoming particle, i.e., E. The "particle" lies under the potential V_0. The modulus of its momentum is the expression for q given above. After reinterpretation as a physical antiparticle the magnitude of its momentum must still be given by q. Since the antiparticle sees a potential of $-V_0$ its energy, say E_a must satisfy $[E_a-(-V_0)]^2=(E-V_0)^2$, from which we conclude therefore that $E_a=-E$ (see Fig. 1). This means that *energy has been conserved in the pair creation*. Because this has occurred at a potential boundary there is no threshold energy needed. A similar interpretation occurs with the Klein-Gordon equation [11]. However, there is a difficulty with the above interpretation. A direct calculation of the group velocity of the below potential "particle" solution yields,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}q} = \frac{q}{E-V_0}.$$

This is *negative* because $E < V_0$. We can only reconcile this result with a right-moving antiparticle by invoking the Feynmann-Stuckelburg rule that below-barrier solutions travel backward in time, whence $\mathrm{d}t < 0$ and consequently $\mathrm{d}z > 0$. The corresponding antiparticle wave function has $\mathrm{d}t > 0$ as have all physical particles and hence must exhibit a positive group velocity.

We take this opportunity to observe that the conventional charge conjugate solution differs in its space-time structure from that of the "particle" by an overall complex conjugation. This has *no effect* upon the calculation of the group velocity. Thus the charge conjugate wave function cannot represent correctly the antiparticle state (it would yield the wrong sign for the group velocity) [17]. Another important observation, even if obvious, is that to satisfy the continuity equations we need at least three "touching" plane waves. Pair creation in the absence of an incoming wave is not a solution.

Finally we consider, briefly, the alternative plane wave solution under the step,

$$\Psi'_{\mathrm{II}}(z,t) = T' u^{(3)}(-q;V_0)\exp[-iqz]\exp[-iEt]. \qquad (5)$$

This yields the following solution to the continuity equations:

$$R' = \frac{\alpha-1}{\alpha+1} \quad \text{and} \quad T' = \sqrt{\frac{E-m}{E+m}}\frac{2}{\alpha+1} \qquad (6)$$

with α given as above. Now $|R'| < 1$ and part of the incoming charge has been *annihilated*. Thus this below-barrier solution *must* involve an incoming antiparticle from the right. It is therefore inconsistent with the assumed initial boundary conditions and is consequently rejected. It is the Klein paradox solution which is generally considered valid for the step.

III. THE PLANE WAVE BARRIER ANALYSIS

For the barrier,

$$V(z) = \begin{cases} 0, & z < 0, \quad \text{region I}, \\ V_0, & 0 < z < l, \quad \text{region II}, \\ 0, & z > l, \quad \text{region III}. \end{cases}$$

We now derive the plane wave solution. In the left free region the incoming and reflected solutions yield the combined wave function,

$$\Psi_{\mathrm{I}}(z,t) = \{u^{(1)}(p;0)\exp[ipz] + R_B u^{(1)}(-p;0)\exp[-ipz]\} \times \exp[-iEt], \qquad (7)$$

with R_B the barrier reflection coefficient. The solutions "under" the potential give

$$\Psi_{\mathrm{II}}(z,t) = \{A_B u^{(3)}(q;V_0)\exp[iqz] + B_B u^{(3)}(-q;V_0)\exp[-iqz]\}\exp[-iEt]. \qquad (8)$$

Such solutions represent right (A_B) and left (B_B) moving antiparticles. The latter are now allowed as a consequence of reflection at the second potential discontinuity. The solution in the free region beyond the barrier is given by a single outgoing plane wave,

$$\Psi_{\text{III}}(z,t) = T_B u^{(1)}(p;0)\exp[ipz]\exp[-iEt], \tag{9}$$

with T_B the transmission coefficient.

The continuity conditions, best displayed in matrix form are thus,

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ R_B \end{pmatrix} = \sqrt{\frac{E+m}{E-m}}\begin{pmatrix} -\alpha & \alpha \\ 1 & 1 \end{pmatrix}\begin{pmatrix} A_B \\ B_B \end{pmatrix},$$

$$\begin{pmatrix} -e^{iql} & e^{-iql} \\ e^{iql} & e^{-iql} \end{pmatrix}\begin{pmatrix} A_B \\ B_B \end{pmatrix} = \sqrt{\frac{|E-V_0|+m}{|E-V_0|-m}}\begin{pmatrix} 1 \\ \alpha \end{pmatrix} T_B e^{ipl},$$

where α is as defined in the previous section. The solutions of these equations are straightforward,

$$R_B = i\frac{\alpha^2-1}{2\alpha}\sin(ql)T_B e^{ipl}$$

and

$$T_B = e^{-ipl}\Big/\left[\cos(ql) + i\frac{\alpha^2+1}{2\alpha}\sin(ql)\right]. \tag{10}$$

Consequently, both reflected and transmitted probabilities are positive and less than one (the incoming probability). Probability is conserved,

$$|R_B|^2 + |T_B|^2 = 1. \tag{11}$$

At first sight this solution seems perfectly acceptable and even conventional as long as one does not try to interpret it in physical terms, i.e., as long as one does not look into region II. Notice that it is characterized by resonance phenomena when the sine term in the denominator of T_B vanishes, in which cases the transmission probability becomes unity. However, how can this solution be compatible with the step analysis, and specifically with the Klein paradox. If the barrier is extended indefinitely we should in some way tend to the step solution and pair production. This is what occurs for above barrier diffusion, albeit in a nontrivial manner. In the above barrier case, *multiple reflections* occur for the barrier whereas only a single reflection occurs for the step potential [14]. It is the *first* reflected wave packet of the barrier that coincides with the step result (including its instantaneous reflection). One could say that the subsequent wave packets for the barrier exist but are delayed indefinitely as the barrier length grows without limit. A long barrier (with respect to the incoming packet width) thus reproduces the step diffusion result for a finite time after impact of the incoming particle.

Let us try to interpret the results of this section. Since the reflection probability is less than the incoming probability particle charge in region I has decreased. Pair annihilation has occurred at the origin discontinuity. The annihilating antiparticle can only have come from pair creation at the second discontinuity at a time previous to its annihilation. Therefore the outgoing particle in region III had also to be created at this earlier time. However, here we already have a problem since the creation of a pair without an incoming contribution is incompatible with the step continuity equations as already noted in the previous section. Is the (first) antiparticle produced at the first or second discontinuity? If it were at first, coincident with the incoming particle, we would have no problem with continuity, but we would have a problem with charge conservation since we are not in accordance with the Klein paradox. If at the second discontinuity we face the aforementioned violation of continuity, there is no incoming particle in region III.

In the following section we shall approach the barrier as a two step process and consequently reinterpret the above results.

TABLE I. The three reflection scenarios in the two step approach. The first line simply lists the step results. In brackets in columns 3 and 4, we have indicated the regions in which the reflected and transmitted waves travel.

Discontinuity point	Incoming wave	Reflected coefficient	Transmitted coefficient
$z=0$	I $\rightarrow$ II	(I) $\frac{\alpha+1}{\alpha-1}$	(II) $\sqrt{\frac{E-m}{E+m}}\frac{2}{1-\alpha}$
$z=l$	II $\rightarrow$ III	(II) $\frac{\alpha+1}{\alpha-1}e^{2iql}$	(III) $\sqrt{\frac{E+m}{E-m}}\frac{2\alpha}{\alpha-1}e^{i(q-p)l}$
$z=0$	I $\leftarrow$ II	(II) $\frac{\alpha+1}{\alpha-1}$	(I) $\sqrt{\frac{E+m}{E-m}}\frac{2\alpha}{1-\alpha}$

IV. THE TWO STEP APPROACH

The two step approach to the barrier actually involves the calculation and the use of three step potentials. One is that already calculated in Sec. II at $z=0$, for a left incoming particle, characterized by $|R|>1$. Another is for a left incoming antiparticle reflected from the potential discontinuity at $z=l$. The third is for the reflected antiparticle impinging upon the first potential discontinuity at the origin and hence arriving from the right. Since these antiparticles are themselves in the Klein zone for the potential well in which they are entrapped, they also must exhibit $|R|>1$. At each reflection (be it of a particle or antiparticle) pair creation occurs. Again it would be more correct to say that a nonzero probability for pair creation occurs. We have performed the calculation with the below potential solution (instead of the antiparticle solution) and a potential barrier throughout but the physical interpretation is as we have just described. It is important to note that pair creation occurs at each reflection and an outgoing wave packet is associated with each pair production; at least as long as the wave packet width is much smaller than the barrier (well) width.

We quote in Table I, without demonstration, the three reflection scenarios just described. The build up of the total reflected and transmitted coefficients is then straightforward. The first few contributions to the reflection coefficient are (R_1 is just the Klein paradox)

$$R_1 = \frac{\alpha+1}{\alpha-1},$$

$$R_2 = \frac{4\alpha(\alpha+1)}{(\alpha-1)^3} e^{2iql},$$

$$R_3 = \frac{4\alpha(\alpha+1)^3}{(\alpha-1)^5} e^{4iql},$$

$$\vdots$$

The subsequent terms are obtained by multiplying by the loop factor

$$\left(\frac{\alpha+1}{\alpha-1}\right)^2 e^{2iql}. \qquad (12)$$

This loop factor is simply the product of the antiparticle reflection coefficient at $z=0$ by that at $z=l$ as can be seen from the table. The increasing phase factor is essential for yielding the exit times (calculated with the stationary phase method) of the various wave packets (more about this later in this section).

The transmitted coefficients are even simpler since all terms are related by the loop factor (12). The first few terms are

$$T_1 = -\frac{4\alpha}{(\alpha-1)^2} e^{i(q-p)l},$$

$$T_2 = -\frac{4\alpha(\alpha+1)^2}{(\alpha-1)^4} e^{i(3q-p)l},$$

$$T_3 = -\frac{4\alpha(\alpha+1)^4}{(\alpha-1)^6} e^{i(5q-p)l},$$

$$\vdots$$

The loop factor has a modulus greater than one, so that with the exception of the first reflected wave the probabilities of subsequent successive wave packets grow geometrically by the factor

$$\left(\frac{\alpha+1}{\alpha-1}\right)^4. \qquad (13)$$

This growth factor in probability is just the fourth power of the single step reflection coefficient (which is real and greater than one). Both of the above series if summed diverge. However, it is interesting to note that if one does formally sum them, then one exactly finds the R_B and T_B coefficients of Sec. III,

$$R_B = \frac{\alpha+1}{\alpha-1}\left\{1 + \frac{4\alpha}{(\alpha-1)^2} e^{2iql} \sum_{n=0}^{\infty} \left[\left(\frac{\alpha+1}{\alpha-1}\right)^2 e^{2iql}\right]^n\right\}$$
$$= i\frac{\alpha^2-1}{2\alpha}\sin(ql) \Big/ \left[\cos(ql) + i\frac{\alpha^2+1}{2\alpha}\sin(ql)\right],$$

$$T_B = -\frac{4\alpha}{(\alpha-1)^2} e^{i(q-p)l} \sum_{n=0}^{\infty} \left[\left(\frac{\alpha+1}{\alpha-1}\right)^2 e^{2iql}\right]^n$$
$$= e^{-ipl} \Big/ \left[\cos(ql) + i\frac{\alpha^2+1}{2\alpha}\sin(ql)\right].$$

Since the reflected and transmitted series do not converge, we must conclude that the expressions for R_B and T_B are *not* physical. At most they encrypt the physical multiwave-packets series. This is in contrast to what happens for the above-barrier diffusion where $|R|$ for the step is less than one. Then for plane waves (infinite width wave packets) the barrier results represent a physical limit and do indeed exhibit resonance phenomena.

There is a problem with the above results. All exit times except for the first reflected wave are negative. This is again connected to time flow under the barrier, or alternatively because the antiparticle energy is $-E$ and not E. In either case the time dependent phase factor differs in region II from that in the free regions. Strictly speaking the stationary plane wave analysis breaks down because of this. Phase factors due to the time dependent plane-wave phase must be taken into account.

Fortunately, we do not need to repeat all our analysis because we know the antiparticle group velocity $\pm q_0/E_0$ and hence all exit times. These are of course positive with successive times spaced by $2lE_0/q_0$. This is exactly what is obtained if we take $q \to -q$ in all the amplitudes. The probability predictions do not change.

V. CONCLUSIONS

We have presented in this work the Klein zone analysis for the step and barrier. In our derivation, we have preferred to use the below-potential "particle" solutions, i.e., the spinor $u^{(3)}$. This is not the traditional choice, where the $u^{(1)}$ spinor is adopted both in the free and potential regions [5–8]. As we have already noted in Sec. II, this traditional choice has the unpleasant feature of containing, in the spinor $u^{(1)}$, a denominator which vanishes when $E-V_0=-m$ (i.e., at the Klein zone limit where the antiparticle is at rest). However, since this vanishing denominator, in practice, inverts the small and large nonrelativistic components, the two approaches are compatible and give the same probabilities. We have also observed that the barrier region solutions (specifically in their space-time structure) cannot be the correct antiparticle wave solutions because they have the wrong group velocity. What is however even more surprising is the fact that by simple charge conjugation of these wave functions this problem is not resolved, so even the charge conjugate solution cannot be the correct antiparticle wave function. A similar observation, but based upon different arguments, has been given by Sakurai [17]. Fortunately, this is not an essential question for our calculations and we have had no difficulty or confusion when talking about the antiparticles and their motion, because this is uniquely determined by charge conservation. However, it is a question which merits further study.

The Klein paradox is characterized by pair production. As others have already noted this can be viewed (*a posteriori*) as an anticipation of field theory [9]. The barrier plane wave solutions seem, at first sight, not to exhibit the Klein paradox. Thus, of concern to us was the fact that there appeared an inconsistency between the standard barrier result and a two step calculation. It was also impossible to interpret the plane wave barrier results in terms of pair creation. One of our principal objectives in this work has been to confront these two approaches. The two step (or more in general multiple step approach) is not a new idea. In previous applications it has been lauded as a effective calculational technique [12,15]. We, on the other hand, have emphasized both here and in a previous paper [14] its interpretation in terms of multiple wave packets. In our previous work upon above potential barrier diffusion, confirmation was also obtained with the help of numerical calculations.

With the Klein paradox, our series solution, each term of which represents a wave packet, is nonconvergent. The barrier solution represents the formal sum of these series and it is therefore nonphysical. We have also observed that the correct amplitudes are the complex conjugate of our listed results, however this has no effect upon any predicted probabilities. If the barrier (well) width is much greater than the wave packet widths, then each term in the series can be studied separately. Each term yields a separate wave packet and specific exit times. The first reflected wave (R_1) is instantaneously reflected. The first transmitted amplitude ($T_1[q\to -q]$) exits at time lE_0/q_0. Subsequent reflected wave packets emerge with intervals of $2lE_0/q_0$ and the same for the transmitted wave packets. If we do not make a distinction between reflected and transmitted waves, then a wave packet emerges, in either direction, at intervals of lE_0/q_0, the antiparticle transit time across its potential well.

The most interesting features of our analysis are:

(1) Pair production occurs at zero energy cost.

(2) The antiparticles produced via pair production are *permanently* trapped within the potential well (barrier) region. They lie in a Klein zone of their own.

(3) These localized antiparticles have a *continuous* energy spectrum. They are technically not "bound states" because their existence requires a dynamical process—pair product—at each reflection. With the Dirac equation, we must distinguish between static bound state solutions and dynamic localized solutions. The former are present in the tunneling zone.

(4) The correct amplitudes, with the correct exit times, are those usually calculated by stationary plane wave analysis with the substitution $q\to -q$.

The barrier results ($R_B[q\to -q]$ and $T_B[q\to -q]$) are nonphysical since they imply the sum of a divergent series. However, it is to be noted that the continual growth of antiparticles within the barrier/well region is also a mathematical abstraction. We expect that as the localized antiparticle density increases, a corresponding decrease in the barrier potential height V_0 occurs, i.e., an attractive space charge effect should accrue. Such an effect was confirmed quantitatively some time ago, within the Thomas Fermi approximation, in the three-dimensional case, by Müller and Rafelski [18].

The Dirac equation has had an incredible success rate when its predictions are correctly interpreted. Its negative energy solutions lead to the prediction of antiparticles. Zitterbewegung is connected to the proven existence of the Darwin term in atomic physics. The Klein paradox predicts the phenomenon of pair production. We can therefore hope that *dynamic localized energy spectra*, connected to the geometric growth of periodic particle emissions, will be confirmed experimentally.

APPENDIX

The Dirac equation in presence of an *electrostatic potential* $A_\mu=(A_0,\mathbf{0})$ reads [16]

$$(i\gamma^\mu\partial_\mu - e\gamma^0 A_0 - m)\Psi(\boldsymbol{r},t)=0. \tag{A1}$$

This equation can be rewritten as follows:

$$i\partial_t\Psi(\boldsymbol{r},t)=(H_0+V_0)\Psi(\boldsymbol{r},t), \tag{A2}$$

where

$$H_0=-i\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m\gamma^0 \quad \text{and} \quad V_0=eA_0.$$

Here e denotes the charge of the particle ($e=-|e|$ for the electron). For a stationary solution $\Psi(\boldsymbol{r},t)\propto\exp[-iEt]$, we obtain

$$H_0\Psi(\boldsymbol{r},t)=(E-V_0)\Psi(\boldsymbol{r},t). \tag{A3}$$

Using the Pauli-Dirac set of gamma matrices

$$\gamma^0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad \text{and} \quad \boldsymbol{\gamma}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ -\boldsymbol{\sigma} & 0\end{pmatrix},$$

the covariant normalized spinorial solutions u_N are

$$u_N^{(1,2)}(\boldsymbol{q};V_0)=\sqrt{(E-V_0+m)}\begin{pmatrix}\chi^{(1,2)}\\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{q}}{E-V_0+m}\chi^{(1,2)}\end{pmatrix}$$

$$\text{for } E-V_0>m,$$

$$u_N^{(3,4)}(\boldsymbol{q};V_0)=\sqrt{(|E-V_0|+m)}\begin{pmatrix}-\dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{q}}{|E-V_0|+m}\chi^{(1,2)}\\ \chi^{(1,2)}\end{pmatrix}$$

$$\text{for } E-V_0<-m, \tag{A4}$$

where

$$(E-V_0)^2-\boldsymbol{q}^2=m^2, \quad \chi^{(1)}=\begin{pmatrix}1\\0\end{pmatrix}, \quad \text{and} \quad \chi^{(2)}=\begin{pmatrix}0\\1\end{pmatrix}.$$

The two sets of solutions $u_N^{(1,2)}$ and $u_N^{(3,4)}$, sometimes referred to, imprecisely, as "positive" and "negative" solutions, are in fact not determined by the sign of E, but by whether $E>V_0+m$ or $E<V_0-m$. Hence, E may be fixed but the solutions depend upon whether in any given region the energy is above or below the potential. For each of the previous spinors there are in general two separate solutions corresponding to opposite momentum. In the text of this paper, we employ the one-dimensional unnormalized spinors u,

$$u^{(1,2)}(q;V_0) = \begin{pmatrix} \chi^{(1,2)} \\ \dfrac{q}{E-V_0+m}\chi^{(1,2)} \end{pmatrix} \quad \text{for } E-V_0 > m,$$

$$u^{(3,4)}(q;V_0) = \begin{pmatrix} -\dfrac{q}{|E-V_0|+m}\chi^{(1,2)} \\ \chi^{(1,2)} \end{pmatrix} \quad \text{for } E-V_0 < -m. \tag{A5}$$

In free space, $V_0=0$, the above spinors become

$$u^{(1,2)}(p;0) = \begin{pmatrix} \chi^{(1,2)} \\ \dfrac{p}{E+m}\chi^{(1,2)} \end{pmatrix} \quad \text{for } E > m,$$

$$u^{(3,4)}(p;0) = \begin{pmatrix} -\dfrac{p}{|E|+m}\chi^{(1,2)} \\ \chi^{(1,2)} \end{pmatrix} \quad \text{for } E < -m. \tag{A6}$$

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