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Tunnelling through two barriers

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Abstract

Recent theoretical studies of the tunnelling through two opaque barriers claim that the transit time is independent of the barrier widths and of the separation distance between the barriers. Such a result is based on the use of the stationary phase method and the hypothesis of a single transmitted wave packet. In this Letter, we propose an alternative treatment based on a multiple peaks decomposition of the transmitted wave. We observe, that if multiple reflections are allowed for correctly (infinite peaks) the transit time between the barriers appears exactly as expected.

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It is a well-known result that the tunnelling time, calculated by using the stationary phase method (SPM) approximation in the limit of an opaque barrier, is independent of the barrier width [1]. Such a phenomenon, called the *Hartman effect* [2] implies arbitrarily large velocities inside the barrier. When this is obtained with the use of the nonrelativistic Schrödinger equation, which can be considered exact only when the velocity of light is infinite, there is no paradox involved, only perhaps some doubt about the relevance. One can in particular question, in this context, the use of the terminology "super-luminal velocities". However, since similar results have more recently been obtained with the Dirac equation [3,4], there is good reason to be perplexed and to invoke further analysis.

We address in this Letter a variant of this subject which connects closely to a recent paper upon above-barrier diffusion [5]. In recent years, the Hartman analysis has been extended to a potential model with two successive barriers separated by a free propagation region. Again in the opaque limit for both barriers, it has been observed that, far from resonances, the tunnelling phase time depends neither upon the barrier widths nor upon the distance between the barriers [6,7]. Thus, this result predicts, contrary to common sense, unbounded group velocities even

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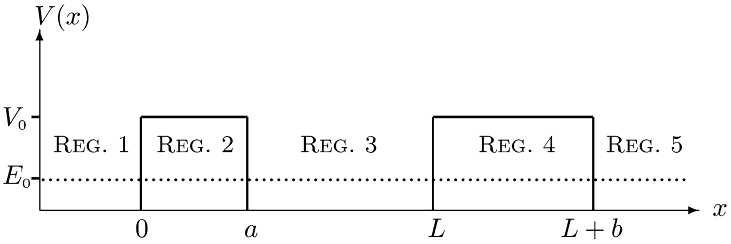

$$V(x)=\begin{cases}0, & x<0,\\ V_0, & 0<x<a,\\ 0, & a<x<L,\\ V_0, & L<x<L+b,\\ 0, & L+b<x.\end{cases}$$

Fig. 1. The function $\varphi(t)$ vs t for the indicated values of V_0 and ω. The five curves are for different values of E_0. For each of the curves the cross-over points are indicated by the large dots.

in the free region between the two barriers. This phenomenon, also valid for an arbitrary number of barriers [8,9], is known as the *generalized Hartman effect*. We shall demonstrate that a different analysis, which allows for multiple scattering between the barriers and, consequently, the existence of multiple peaks, alters this result. Indeed the generalized Hartman effect represents an example of an ambiguity in the use of the stationary phase method. The SPM tunnelling time, also know as Bohm–Wigner or phase time, is one of the four tunnelling times introduced in literature. For an interesting unified derivation of these times we refer the reader to the recent excellent paper by Yamada [10], where the Larmor time, the Büttiker–Landauer time, the Bohm–Wigner time, and the Pollak–Miller time are derived in a unified manner from the Gell-Mann–Hartle decoherence functional for resident (Larmor and Büttiker–Landauer) and passage (Bohm–Wigner and Pollak–Miller) time.

The starting point of the analysis is the one-dimensional Schrödinger equation

$$\Psi_{xx}(k,x)=\frac{2m}{\hbar^2}\big[V(x)-E\big]\Psi(k,x), \tag{1}$$

for a particle of mass m in a double barrier potential, see Fig. 1.

We have maintained different barrier widths because we shall later discuss the case of an opaque limit only for the second barrier but, for simplicity, we have given to the two barriers the same height. It is easy to generalize our formulas to the case of different barrier heights. It is important to note in advance that the use of barriers of different widths is *not* essential to our conclusions. It merely allowed us to express some formulas in a more general form than had previously appeared in the literature. It does not affect our conclusions. The essential difference in our work with respect to those of previous authors [7] who concluded that there existed a generalized Hartman effect, is the decomposition of the through-going amplitude before applying the stationary phase method. In other words, we shall show that there exist multiple peaks (due to multiple reflections in the enclosed potential-free region) in contrast with the conventional assumption of a single outgoing peak [9].

The standard procedure for finding the stationary solutions can now be applied. The general solution for $\Psi(k,x)$ and any $E<V_0$ in the five regions are

$$\Psi(k,x)=\begin{cases}\text{Region 1:} & e^{ikx}+A_{1R}e^{-ikx} \qquad \big[k=\sqrt{2mE/\hbar^2}\,\big],\\ \text{Region 2:} & \alpha_1e^{-\chi x}+\beta_1e^{\chi x} \qquad \big[\chi=\sqrt{2m(V_0-E)/\hbar^2}\,\big],\\ \text{Region 3:} & A_{1T}(e^{ikx}+A_{2R}e^{-ikx}),\\ \text{Region 4:} & A_{1T}[\alpha_2e^{-\chi(x-L)}+\beta_2e^{\chi(x-L)}],\\ \text{Region 5:} & A_{1T}A_{2T}e^{ikx}.\end{cases} \tag{2}$$

The requirement of continuity of Ψ and Ψ_x at $r=0,a,L$ and $L+b$ give the matching conditions from which, after some algebraic manipulations, we find

$$
\begin{aligned}
A_{1R} &= \frac{\sinh(\chi a)}{\sinh(\chi a + 2i\varphi)}\left[1 - \frac{\sinh(\chi b)\sinh(\chi a - 2i\varphi)}{\sinh(\chi a)\sinh(\chi b + 2i\varphi)} e^{2ik(L-a)}\right] \Big/ \mathcal{D}, \\
A_{1T} &= \frac{2i\chi k}{w^2} \frac{e^{-ika}}{\sinh(\chi a + 2i\varphi)} \Big/ \mathcal{D}, \\
A_{1T}A_{2R} &= \frac{2i\chi k}{w^2} \frac{\sinh(\chi b) e^{ik(2L-a)}}{\sinh(\chi a + 2i\varphi)\sinh(\chi b + 2i\varphi)} \Big/ \mathcal{D}, \\
A_{1T}A_{2T} &= \left(\frac{2i\chi k}{w^2}\right)^2 \frac{e^{-ik(a+b)}}{\sinh(\chi a + 2i\varphi)\sinh(\chi b + 2i\varphi)} \Big/ \mathcal{D},
\end{aligned} \tag{3}
$$

where

$$
\mathcal{D} = \left[1 - \frac{\sinh(\chi a)\sinh(\chi b)}{\sinh(\chi a + 2i\varphi)\sinh(\chi b + 2i\varphi)} e^{2ik(L-a)}\right], \qquad w = \sqrt{2mV_0/\hbar^2} \quad \text{and} \quad \varphi = \arctan[\chi/k].
$$

If we consider the (double) opaque limit, χa and $\chi b \gg 1$, we reproduce (when $a = b$) the coefficients which have been used to obtain the generalized Hartman effect [6,7,9],

$$
\begin{aligned}
A_{1R} &\approx \exp[-2i\varphi], \\
A_{1T} &\approx \frac{2\chi k}{w^2} \exp[-\chi a - ikL] / \sin\bigl[2\varphi - k(L-a)\bigr], \\
A_{1T}A_{2R} &\approx \frac{2\chi k}{w^2} \exp[-\chi a + ikL - 2i\varphi] / \sin\bigl[2\varphi - k(L-a)\bigr], \\
A_{1T}A_{2T} &\approx \frac{8i\chi^2 k^2}{w^4} \exp\bigl[-\chi(a+b) - ik(L+b) - 2i\varphi\bigr] / \sin\bigl[2\varphi - k(L-a)\bigr].
\end{aligned} \tag{4}
$$

We recall that all the above amplitudes are to be multiplied by the plane-wave phase factor and the chosen modulation function assumed peaked at $k = k_0$. Without repeating all the arguments leading to the generalized Hartman effect (including the SPM itself), it suffices to note that, in the expression for $A_{1T}A_{2T}$ of Eqs. (4), the $L + b$ dependence in the phase will be canceled by the plane wave phase calculated at $x = L + b$. There remains an $L - a$ dependence in the modulus but that does not affect the time for the appearance of the maximum in region 5, under the hypothesis of a *single peak*.

Previous analysis upon the generalized Hartman effect can be improved on several grounds. The first point is the standard use in the through-going phase of the peak momentum k_0 of the incoming wave. The Hartman time for transit through a single barrier then reads as $\tau_0 = 2\varphi'(E_0)$, where the prime stands for differentiation with respect to E. What we like to call the "filter effect" of a barrier preferentially allows the higher momentum components to transit. So, in the opaque limit, only the highest incoming momentum passes through. This point is clearly discussed in the classical paper of Hartman [2]. For example, the group velocity in region 5, $\tilde{k}_0/m$, is always higher than the incoming group velocity k_0/m. A more precise expression for the Hartman time is thus $\tilde{\tau}_0 = 2\varphi'(\tilde{E}_0)$.

Another principal assumption is the implicit use of a single reflected and transmitted wave packet. A natural alternative exists which still uses the SPM but involves multiple reflections with the "expected" transit times (see below). This new treatment is suggested by the momentum distribution of the transmitted wave which clearly displays multiple momentum peaks and by the existence in Eq. (2) of the two exponentials with positive and negative signs implies the existence of reflection in region 3. Actually, this is true if and only if the coefficient A_{2R} is nonzero which is always the case as Eqs. (4) proves. Now, it is quite strange to admit multiple reflections in region 3 without admitting multiple outgoing wave-packets in region 5.

Now for the alternative approach. Before going to the opaque limit, the denominator factor $\mathcal{D}$ in Eqs. (3), can be legitimately expanded as a series in the numerator. In particular, in region 5, we obtain the following transmitted amplitude

$$A_{1T}A_{2T} = \left(\frac{2i\chi k}{w^2}\right)^2 \frac{e^{-ik(a+b)}}{\sinh(\chi a + 2i\varphi)\sinh(\chi b + 2i\varphi)} \times \sum_{n=1}^{\infty}\left(\frac{\sinh(\chi a)\sinh(\chi b)}{\sinh(\chi a + 2i\varphi)\sinh(\chi b + 2i\varphi)} e^{2ik(L-a)}\right)^{n-1}. \tag{5}$$

An (infinite) sum of terms each of which can be analyzed with the SPM to determine the position of their maxima. This is just the procedure used in Ref. [5]. The phase of the first transmitted peak ($n = 1$) is given by

$$\Phi_1 = kx - Et - k(a+b) - \arctan\big[\tan(2\varphi)\coth(\chi a)\big] - \arctan\big[\tan(2\varphi)\coth(\chi b)\big]. \tag{6}$$

Consequently, the first transmitted peak appears at $x = L + b$ in perfect accord with the expected ($L - a$ dependent) transit time. The dependence of the subsequent phases ($n = 2, 3, \ldots$),

$$\Phi_n = \Phi_1 + (n-1)\big\{2k(L-a) - \arctan\big[\tan(2\varphi)\coth(\chi a)\big] - \arctan\big[\tan(2\varphi)\coth(\chi b)\big]\big\}, \tag{7}$$

contain extra multiples of $2k(L - a)$. In fact, in the opaque limit

$$\Phi_n \approx kx - Et - k(a+b) - 4\varphi + (n-1)\big[2k(L-a) - 4\varphi\big], \tag{8}$$

the filter effect of the first barrier guarantees that the group velocity in region 3 is the same as that in region 5. Thus, the successive exit times for the maxima predicted by the SPM are proportional to multiples of the back/foreward travel times in the inter-barrier region plus multiples of twice the Hartman time,

$$t_n = (m/\tilde{k}_0)\big[(2n-1)(L-a)\big] + 2n\tilde{\tau}_0. \tag{9}$$

The multiples of the Hartman time that appear above are a consequence of the multiple double-reflections. The so-called "delay-time" in reflection in region 3 coincides with the Hartman transit time in the opaque limit since the relevant momentum used in the SPM for both is the upper limit $\tilde{k}_0$.

In conclusion, we have shown that the generalized Hartman effect is the consequence of treating the outgoing wave as a single wave packet. As an alternative approach, we have shown that a treatment based on a multiple peaks decomposition of the transmitted wave packet (region 5) reproduces the expected transit time between the barriers (region 3).

In the case of broad wave packets, we have a strongly interfering transmitted waves and the result of their interference remains a completely open question. The method presented in this Letter could be not very useful under these conditions and probably the superposition of the overlapping wave packets could reproduce a reshaping of the pulse. In the case of narrow outgoing wave packets there is a little overlap between the transmitted waves and the multiple peaks treatment is certainly the more appropriate. The transition between these cases is surely a complex problem which depend on the details of the pulse and deserves further investigations.

For technical reasons it is nontrivial to resolve numerically the twin barrier problems in the opaque limit in which formally the transmitted probability goes to zero. However, if, for example, only the second barrier is made opaque and the first is kept narrow, it is very easy to perform numerical calculations and see the multiple peaks in the reflected amplitude. Similarly for the foreward/backward flowing peaks in the intermediate region 3. A detailed numerical analysis of the twin opaque barriers could be very important to clarify under which circumstances the multiple-peaks in the transmitted region 5 are clearly seen and when the filter effect could be ignored (*universality* property [11]).

To determine the size of the outgoing wave packets, we employ the Heisenberg uncertainty principle. In the opaque limit, the dominating part of the modulus of each of the outgoing wave-packets is given by the filter term: $\exp[-\chi(a+b)]$. This means that $\Delta\chi = 1/(a+b)$. Consequently, $\Delta k = \langle\chi/k\rangle\Delta\chi \ll 1/(a+b)$. It follows that $\Delta x \gg (a+b)$. Thus, in the opaque limit, for any fixed value of the inter-barrier separation $L - a$, the wave packet widths will eventually exceed the separation between successive peaks. Now the successive wave peaks of the outgoing amplitude are, in the opaque limit, almost equal in magnitude. For example in the case when interference does not shift these maxima (low interference) a resulting “dragon-like” structure emerges. The highest first maximum appears, as given in Eq. (6), at the transit time in region 3 plus the *two* Hartman times (because of two barriers). However, one must ask what physical significance can be given to this maximum. The only definition of position that is reasonable (for broad wave packets) is the mean position of the wave packet and this appears at a very different time from the maximum (first peak). It is the asymmetry predicted by our interpretation in the transmitted wave train that makes this question of mean vs. maximum relevant, and distinguishes our procedure from a single wave analysis. This last discussion is at the moment a theoretical speculation. Many questions in relation to the parameters involved in the twin opaque barrier problem remain completely open and only a detailed numerical analysis [12] could be definitively clarify this interesting quantum-mechanical puzzle. Finally, we hope that the analysis presented in this Letter be seen as a starting point for further debates on this intriguing research topic and not as a criticism to previous results based on a single peak treatment. Our main objective was and is to shed new light on the appropriate use of the stationary phase method in quantum mechanics.

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