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Dirac spinors and flavor oscillations

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Abstract. In the standard treatment of particle oscillations the mass eigenstates are implicitly assumed to be *scalars* and, consequently, the spinorial form of the neutrino wave functions is *not* included in the calculations. To analyze this additional effect, we discuss the oscillation probability formula obtained by using the Dirac equation as evolution equation for the neutrino mass eigenstates. The initial localization of the spinor state also implies an interference between positive and negative energy components of mass eigenstate wave packets which modifies the standard oscillation probability.

1 Introduction

Since a long time ago, particle mixing [1] and oscillations [2,3] continue to stimulate interesting and sometimes fascinating discussions on the many subtleties of the quantum mechanics involved in oscillation phenomena. Measurements of various features of the fluxes of atmospheric [4] and solar [5,6] neutrinos have provided, in the last years, evidence for neutrino oscillations and therefore for neutrino masses and mixing. In particular, it renewed the interest in understanding the derivation of the flavor conversion probability formula and in overcoming the main difficulties hidden in the standard theoretical approaches. In particular, an increasing number of theoretical papers have recently questioned the validity of the standard plane wave treatment of oscillations by resorting to *intermediate* [7–9] and *external* [10,11] wave packet frameworks.

The standard plane wave treatment [12,13] is certainly the simplest and probably the most intuitive way to introduce the oscillation length and to immediately obtain an expression for the oscillation probability. In such a formalism, a plane wave is associated with each mass eigenstate. For the two-flavor case, the mass eigenstate phase difference is

$$\Delta\Phi = \Delta\,(E\,T - p\,L)\,. \tag{1}$$

Thus, an initially *pure* flavor eigenstate will be modified with time and distance. The probability for a flavor transition is usually expressed in terms of the mixing angle θ and of the relative phase $\Delta\Phi$ by

$$P(\boldsymbol{\nu_\alpha} \to \boldsymbol{\nu_\beta}) = \sin^2[2\theta]\,\sin^2\left[\frac{\Delta\Phi}{2}\right]. \tag{2}$$

The Lorentz invariant difference of the phase $\Delta\Phi$ is then conventionally evaluated by setting $\Delta T = \Delta L = 0$ and considering, for ultra-relativistic particles, $T \approx L$ and $p_{1,2} \approx E_{1,2}$, i.e.

$$\Delta\Phi = T\,\Delta E - L\,\Delta p \approx L\,(\Delta E - \Delta p) \approx \frac{\Delta m^2}{2\bar{p}}\,L\,. \tag{3}$$

By using such an approximation, one gets the well-known expression [13]

$$P(\boldsymbol{\nu_\alpha} \to \boldsymbol{\nu_\beta}; L) = \sin^2[2\theta]\,\sin^2\left[\frac{\Delta m^2}{4\bar{p}}\,L\right]. \tag{4}$$

In the plane wave formalism, the most controversial point is certainly represented by the derivation of formulas containing extra factors in the oscillation length [14–18]. The use of wave packets allows us to understand the origin of these extra factors. In the plane wave approach, it is implicitly assumed that at creation the flavor eigenstate is unique even up to the phase at all points and times of creation. In the wave packet treatment, at time T and at a fixed position in the overlapping region, one experiences the interference between space points whose separation at creation is given by $\Delta v\,T$ and this implies that an additional initial phase is automatically included in the wave packet formalism [7,9]. The final result contains the difference of phase given in (3). We do not intend here to re-discuss the many controversies in the plane wave derivations of the oscillation probability formula. We only remark that a plane wave approach leads to conceptual difficulties and fails to explain fundamental aspects of particle oscillations (i.e. localization and coherence length). Wave packets eliminate some of these problems [19]. In fact, the use of wave packets for propagating mass eigenstates (*intermediate* wave packet model) guarantees the existence of a coherence length, avoids the ambiguous approximations in the plane wave derivation

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of the phase difference and, under particular conditions of minimal *slippage* recovers the oscillation probability given in (4). Unfortunately, it is not easy to determine the size of the wave packets at creation and it is not clear whether it makes sense to consider a unique time of creation [9,20]. A common argument against the *intermediate* wave packet formalism is that oscillating neutrinos are neither prepared nor observed. Consequently, it would be more convenient to write a transition probability between the observable particles involved in the production and detection process. This point of view characterizes the so-called *external* wave packet approach [10,11]. The oscillating particle, described as an internal line of a Feynman diagram by a relativistic mixed scalar propagator, propagates between the source and target (*external*) particles represented by wave packets. The function which represents the overlap of the incoming and outgoing wave packets in the *external* wave packet model corresponds to the wave function of the propagating mass eigenstate in the *intermediate* wave packet formalism. Remarkably, it could be shown that the probability densities for ultra-relativistic stable oscillating particles in both frameworks are mathematically equivalent [11]. However, the *intermediate* wave packet picture brings up a problem, as the overlap function takes into account not only the properties of the source, but also of the detector. This is unusual for a wave packet interpretation and not satisfying for causality [11]. This point was clarified by Giunti [10] who solved this problem by proposing an improved version of the *intermediate* wave packet model where the wave packet of the oscillating particle is explicitly computed with field-theoretical methods in terms of *external* wave packets. Despite not being applied in a completely free way, the (*intermediate*) wave packet treatment commonly simplifies the discussion of some physical aspects coming with the oscillation phenomena [9, 17]. Thus, it makes sense, as a preliminary investigation, to consider a wave packet associated with the propagating particle.

In this paper, we aim to investigate how the oscillation formula is modified by using *fermionic* instead of *scalar* particles. To do so, we shall use the Dirac equation as the evolution equation for the mass eigenstates. Before introducing the Dirac formalism, in Sect. 2, we briefly review the *intermediate* wave packet model for scalar particles [19]. In this section, by choosing a *gaussian* wave packet to describe the localization of our initial flavor state, we obtain an analytical expression for the flavor conversion probability. This allows us to identify the wave packet *slippage* and *spreading* effects. In Sect. 3, we introduce the Dirac formalism and show that a superposition of both positive and negative frequency solutions of the Dirac equation is often a necessary condition to correctly describe the time evolution of the mass eigenstate wave packets. We give, for strictly peaked momentum distributions and ultra-relativistic particles, an analytic expression for the Dirac flavor conversion probability. The results obtained in the context of a wave packet treatment of oscillation phenomena are (briefly) compared with quantum field theory calculations [11,21,22]. This allows one to understand how our analysis could be included within the *external* wave packets formalism. We draw our conclusions in Sect. 4.

2 Gaussian wave packets

The main aspects of oscillation phenomena can be understood by studying the two-flavor problem. In addition, a substantial mathematical simplification results from the assumption that the space dependence of the wave functions is one-dimensional (z-axis). Therefore, we shall use these simplifications to calculate the oscillation probabilities. In this context, the time evolution of the flavor wave packets can be described by

$$\begin{aligned}\Phi(z,t) &= \phi_1(z,t)\cos\theta\,\boldsymbol{\nu_1} + \phi_2(z,t)\sin\theta\,\boldsymbol{\nu_2} \\ &= \left[\phi_1(z,t)\cos^2\theta + \phi_2(z,t)\sin^2\theta\right]\boldsymbol{\nu_\alpha} \\ &\quad + \left[\phi_1(z,t) - \phi_2(z,t)\right]\cos\theta\sin\theta\,\boldsymbol{\nu_\beta} \\ &= \phi_\alpha(z,t;\theta)\,\boldsymbol{\nu_\alpha} + \phi_\beta(z,t;\theta)\,\boldsymbol{\nu_\beta}\,, \end{aligned} \tag{5}$$

where $\boldsymbol{\nu_\alpha}$ and $\boldsymbol{\nu_\beta}$ are flavor eigenstates and $\boldsymbol{\nu_1}$ and $\boldsymbol{\nu_2}$ are mass eigenstates. The probability of finding a flavor state $\boldsymbol{\nu_\beta}$ at the instant t is equal to the integrand squared modulus of the $\boldsymbol{\nu_\beta}$ coefficient

$$\begin{aligned} P_{\rm scalar}(\boldsymbol{\nu_\alpha}\to\boldsymbol{\nu_\beta};t) &= \int_{-\infty}^{+\infty} {\rm d}z\ |\phi_\beta(z,t;\theta)|^2 \\ &= \frac{\sin^2[2\theta]}{2}\left\{1 - {\rm INT}_{\rm scalar}(t)\right\}, \end{aligned} \tag{6}$$

where ${\rm INT}_{\rm scalar}(t)$ represents the interference oscillating term between the (scalar) mass eigenstate wave packets $\phi_1(z,t)$ and $\phi_2(z,t)$, i.e.

$${\rm INT}_{\rm scalar}(t) = {\rm Re}\left[\int_{-\infty}^{+\infty} {\rm d}z\,\phi_1^\dagger(z,t)\phi_2(z,t)\right]. \tag{7}$$

Let us consider mass eigenstate wave packets given at time $t = 0$ by

$$\phi_i(z,0) = \left(\frac{2}{\pi a^2}\right)^{\frac{1}{4}} \exp\left[-\frac{z^2}{a^2}\right]\exp\left[{\rm i}p_i\,z\right]. \tag{8}$$

The wave functions which describe their time evolution are

$$\begin{aligned}&\phi_i(z,t) \\ &= \int_{-\infty}^{+\infty}\frac{{\rm d}p_z}{2\pi}\,\varphi(p_z - p_i)\exp\left[-{\rm i}\,E_i(p_z)\,t + {\rm i}\,p_z\,z\right], \end{aligned} \tag{9}$$

where

$$E_i(p_z) = (p_z^2 + m_i^2)^{\frac{1}{2}}$$

and

$$\varphi(p_z - p_i) = (2\pi a^2)^{\frac{1}{4}}\exp\left[-\frac{(p_z - p_i)^2 a^2}{4}\right].$$

In order to obtain the oscillation probability, we must calculate the interference term ${\rm INT}_{\rm scalar}(t)$, i.e. we have to solve the following integral

$$\int_{-\infty}^{+\infty}\frac{{\rm d}p_z}{2\pi}\,\varphi(p_z - p_1)\varphi(p_z - p_2)\exp\left[-{\rm i}\,\Delta E(p_z)\,t\right]$$

$$= \exp\left[-\frac{(a\Delta p)^2}{8}\right] \times \int_{-\infty}^{+\infty} \frac{\mathrm{d}p_z}{2\pi}\,\varphi^2(p_z-\bar{p})\exp\left[-\mathrm{i}\Delta E(p_z)t\right], \tag{10}$$

where we have changed the z integration into a p_z integration and introduced the quantities $\Delta p = p_1 - p_2$, $\bar{p} = \frac{1}{2}(p_1+p_2)$ and $\Delta E(p_z) = E_1(p_z) - E_2(p_z)$. The oscillation term is bounded by the exponential function of $a\,\Delta p$ at any instant of time. Under this condition we could never observe a *pure* flavor eigenstate. Besides, oscillations are considerably suppressed if $a\,\Delta p > 1$. A necessary condition to observe oscillations is that $a\,\Delta p \ll 1$. This constraint can also be expressed by $\delta p \gg \Delta p$ where δp is the momentum uncertainty of the particle. The overlap between the momentum distributions is indeed relevant only for $\delta p \gg \Delta p$. Consequently, without loss of generality, we can assume

$$\mathrm{INT}_{\mathrm{scalar}}(t) = \mathrm{Re}\left[\int_{-\infty}^{+\infty} \frac{\mathrm{d}p_z}{2\pi}\,\varphi^2(p_z-\bar{p})\,\exp\left[-\mathrm{i}\,\Delta E_i(p_z)\,t\right]\right]. \tag{11}$$

In the literature, this equation is often obtained by assuming two mass eigenstate wave packets described by the "same" momentum distribution centered around the average momentum $\bar{p}$ $(= p_0)$. This simplifying hypothesis also guarantees *instantaneous* creation of a *pure* flavor eigenstate $\boldsymbol{\nu_\alpha}$ at $t = 0$. In fact, for $\phi_1(z,0) = \phi_2(z,0)$ we get from (5)

$$\phi_\alpha(z,0,\theta) = \left(\frac{2}{\pi a^2}\right)^{\frac{1}{4}} \exp\left[-\frac{z^2}{a^2}\right]\exp\left[\mathrm{i}\bar{p}\,z\right] \tag{12}$$

and

$$\phi_\beta(z,0,\theta) = 0\,. \tag{13}$$

To analytically solve the integral in (11), let us rewrite the energy $E_i(p_z)$ as follows:

$$E_i(p_z) = \bar{p}\left[1+\left(\frac{m_i}{\bar{p}}\right)^2 + 2\left(\frac{p_z-\bar{p}}{\bar{p}}\right) + \left(\frac{p_z-\bar{p}}{\bar{p}}\right)^2\right]^{\frac{1}{2}} = \bar{p}\,(1+\chi)\left[1+\frac{\zeta_i}{(1+\chi)^2}\right]^{\frac{1}{2}}, \tag{14}$$

where

$$\zeta_i = \left(\frac{m_i}{\bar{p}}\right)^2 \quad \text{and} \quad \chi = \frac{p_z-\bar{p}}{\bar{p}}\,. \tag{15}$$

In what follows, we shall consider ultra-relativistic particles and assume a sharply peaked momentum distribution, i.e.

$$m_i \ll \bar{p} \;\Rightarrow\; \zeta_i \ll 1 \quad \text{and} \quad \delta p \ll \bar{p} \;\Rightarrow\; \chi \ll 1\,.$$

Let us now expand the energy $E_i(p_z)$ in a power series of ζ_i and χ. We choose to cut off the power series terms of order ζ_i^2 $\left(\frac{m_i^4}{\bar{p}^4}\right)$, so that

$$E_i(p_z) \approx \bar{p}\left[1+\chi+\frac{\zeta_i}{2(1+\chi)}\right] = \bar{p}\left[1+\chi+\frac{\zeta_i}{2}\sum_{j=0}^{\infty}(-1)^j\chi^j\right]. \tag{17}$$

In this case, the energy difference becomes

$$\Delta E(p_z) \approx \bar{p}\,\frac{\Delta\zeta}{2}\sum_{j=0}^{\infty}(-1)^j\chi^j\,. \tag{18}$$

By considering only the first term in the χ expansion, we reproduce the plane wave result. Indeed,

$$\Delta E^{[0]}(p_z) = \bar{p}\,\frac{\Delta\zeta}{2}\,. \tag{19}$$

An approximation of order χ^k $(k \geq 1)$ in (18) requires some constraints on χ. Since we cut off terms of order $\Delta\zeta^2$ and we wish to consider terms up to $\chi^k\Delta\zeta$ in (18), it is necessary to satisfy the constraint $\chi^k\Delta\zeta > \frac{\Delta\zeta^2}{2}$ which implies $\chi > \bar{\zeta}^{\frac{1}{k}}$ $(\bar{\zeta} = \frac{\zeta_1+\zeta_2}{2})$. At the same time, for eliminating $\chi^{k+1}\Delta\zeta$, we have to impose $\chi^{k+1}\Delta\zeta \leq \frac{\Delta\zeta^2}{2}$ which can be rewritten as $\chi \leq \bar{\zeta}^{\frac{1}{1+k}}$. In this way, an approximation of order χ^k will be consistent in the range $\bar{\zeta}^{\frac{1}{k}} < \chi \leq \bar{\zeta}^{\frac{1}{k+1}}$. Meanwhile, the integral in (11) can be solved *analytically* only when $k \leq 2$. By taking into account terms up to the order χ^2, the energy difference becomes

$$\Delta E^{[2]}(p_z) = \bar{p}\,\frac{\Delta\zeta}{2}\,(1-\chi+\chi^2)\,. \tag{20}$$

If we substitute (20) in (11) we obtain

$$\begin{aligned}
&\mathrm{INT}_{\mathrm{scalar}}(t) \\
&\approx \mathrm{Re}\left\{\frac{a\bar{p}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathrm{d}\chi\;\exp\left[-\frac{(a\bar{p}\chi)^2}{2}\right]\right. \\
&\qquad \left.\times\exp\left[-\mathrm{i}\left(\mathcal{S}(t)+\frac{a\bar{p}\chi}{\sqrt{2}}\,\mathcal{Q}(t)+\frac{(a\bar{p}\chi)^2}{2}\mathcal{R}(t)\right)\right]\right\} \\
&= \mathrm{Re}\left\{\left(\frac{1}{1+\mathrm{i}\mathcal{R}(t)}\right)^{\frac{1}{2}}\right. \\
&\qquad \left.\times\exp\left[-\frac{\mathcal{Q}^2(t)}{4(1+\mathrm{i}\mathcal{R}(t))}-\mathrm{i}\mathcal{S}(t)\right]\right\},
\end{aligned} \tag{21}$$

where

$$\mathcal{S}(t) = \frac{\Delta m^2}{2\bar{p}}t, \quad \mathcal{Q}(t) = -\frac{\Delta m^2 t}{\sqrt{2}a\bar{p}^2} \quad \text{and} \quad \mathcal{R}(t) = \frac{\Delta m^2 t}{a^2\bar{p}^3}\,. \tag{22}$$

By suppressing the variable (t) dependence, we can rewrite $\mathrm{INT}_{\mathrm{scalar}}(t)$ as

$$\mathrm{INT}_{\mathrm{scalar}}(t) \approx \exp\left[-\frac{\mathcal{Q}^2}{4(1+\mathcal{R}^2)}\right] \times\left\{\sqrt{\frac{(1+\mathcal{R}^2)^{\frac{1}{2}}+1}{2(1+\mathcal{R}^2)}}\cos\left[\mathcal{S}-\frac{\mathcal{Q}^2\mathcal{R}}{4(1+\mathcal{R}^2)}\right]\right. \tag{23}$$

$$-\sqrt{\frac{(1+\mathcal{R}^2)^{\frac{1}{2}}-1}{2(1+\mathcal{R}^2)}}\sin\left[\mathcal{S}-\frac{\mathcal{Q}^2\mathcal{R}}{4(1+\mathcal{R}^2)}\right]\Bigg\}\,.$$

The above result deserves some comments. The wave packet *spreading* is parameterized by $\mathcal{R}(t)$. At the same time, the *slippage* effect between mass eigenstates is predominantly quantified by the $\mathcal{Q}(t)$ parameter. The *spreading* of wave packets is a secondary effect with respect to the *slippage* since from (22) we can write

$$\frac{\mathcal{R}(t)}{\mathcal{Q}(t)}\approx\frac{1}{a\bar{p}}\,. \tag{24}$$

Under *minimal spreading* conditions, i.e. when $\mathcal{R}\ll 1$, (23) becomes

$$\begin{aligned}\textsc{Int}_{\text{scalar}}(t)&\approx\exp\left[-\frac{\mathcal{Q}^2\,(1-\mathcal{R}^2)}{4}\right] \\ &\times\left\{\left(1-\frac{3\mathcal{R}^2}{8}\right)\cos\left[\mathcal{S}-\frac{\mathcal{Q}^2\mathcal{R}}{4}\right]-\frac{\mathcal{R}}{2}\sin\left[\mathcal{S}-\frac{\mathcal{Q}^2\mathcal{R}}{4}\right]\right\}\,,\end{aligned} \tag{25}$$

where the oscillating character is predominantly given by the cosine function behavior. The exponential term with $\mathcal{R}(t)$ extends the interference between the mass eigenstate wave packets, and consequently the oscillating character, for (a little) longer times. Taking into account terms up to the order χ in (18), we can write

$$\Delta E^{[1]}(p_z)=\bar{p}\,\frac{\Delta\zeta}{2}(1-\chi) \tag{26}$$

and compute the oscillation probability with the leading corrections due to the *slippage* effect,

$$\begin{aligned}&P_{\text{scalar}}(\boldsymbol{\nu_\alpha}\to\boldsymbol{\nu_\beta};t) \\ &\approx\frac{\sin^2[2\theta]}{2}\left\{1-\exp\left[-\frac{\mathcal{Q}^2(t)}{4}\right]\cos\left[\mathcal{S}(t)\right]\right\}\,,\end{aligned} \tag{27}$$

which corresponds to the same result obtained by [9]. Under *minimal slippage* conditions, i.e. when $\mathcal{Q}(t)\ll 1$, the (27) reproduces the plane wave formula (4).

$$\begin{aligned}&P_{\text{scalar}}(\boldsymbol{\nu_\alpha}\to\boldsymbol{\nu_\beta};t) \\ &\approx\frac{\sin^2[2\theta]}{2}\left\{1-\left(1-\frac{\mathcal{Q}^2(t)}{4}\right)\cos\left[\mathcal{S}t\right]\right\} \\ &\approx\frac{\sin^2[2\theta]}{2}\left\{1-\cos\left[\mathcal{S}(t)\right]\right\} \\ &=\sin^2[2\theta]\sin^2\left[\frac{\Delta m^2}{4\bar{p}}\,t\right]\,.\end{aligned} \tag{28}$$

3 Dirac formalism

The results in the previous section have been obtained by considering *scalar* mass eigenstates. Neutrinos are, however, *fermions*. The time evolution of a spin one-half particle has to be described by the Dirac equation. To introduce the *fermionic* character in the study of quantum oscillation phenomena, we shall use the Dirac equation as the evolution equation for the mass eigenstates. Equation (5) now becomes

$$\begin{aligned}\Psi(z,t)&=\psi_1(z,t)\cos\theta\,\boldsymbol{\nu_1}+\psi_2(z,t)\sin\theta\,\boldsymbol{\nu_2} \\ &=\left[\psi_1(z,t)\cos^2\theta+\psi_2(z,t)\sin^2\theta\right]\boldsymbol{\nu_\alpha} \\ &\qquad+\left[\psi_1(z,t)-\psi_2(z,t)\right]\cos\theta\sin\theta\,\boldsymbol{\nu_\beta} \\ &=\psi_\alpha(z,t;\theta)\,\boldsymbol{\nu_\alpha}+\psi_\beta(z,t;\theta)\,\boldsymbol{\nu_\beta}\,,\end{aligned} \tag{29}$$

where $\psi_i(z,t)$ satisfies the Dirac equation for a mass m_i. The natural extension of (13) reads

$$\psi_\alpha(z,0,\theta)=\phi_\alpha(z,0,\theta)w\,, \tag{30}$$

where w is a constant spinor which satisfies the normalization condition $w^\dagger w=1$.

3.1 Dirac wave packets and the oscillation formula

To describe the time evolution of mass eigenstate Dirac wave packets, we could be inclined to superpose only positive frequency solutions of the Dirac equation. This seems, at first glance, a reasonable choice. However, when the initial state has the form given in (30), it is necessary to superpose both positive and negative frequency solutions of the Dirac equation. Let us clear up this point. The flavor state $\psi_\alpha(z,t,\theta)$ is now expressed in terms of

$$\begin{aligned}\psi_i(z,t)&=\int_{-\infty}^{+\infty}\frac{\mathrm{d}p_z}{2\pi}\exp\left[\mathrm{i}p_z z\right] \\ &\times\sum_{s=1,2}\left\{b_i^s(p_z)u_i^s(p_z)\exp\left[-\mathrm{i}E_i(p_z)t\right]\right. \\ &\left.+d_i^{s*}(\text{-}p_z)v_i^s(\text{-}p_z)\exp\left[+\mathrm{i}E_i(p_z)t\right]\right\}\,.\end{aligned} \tag{31}$$

At time $t=0$ the mass eigenstate wave functions satisfy $\psi_1(z,0)=\psi_2(z,0)$ (this guarantees the *instantaneous* creation of a *pure* flavor eigenstate $\boldsymbol{\nu_\alpha}$ as we have pointed out in Sect. 2). The Fourier transform of $\psi_i(z,0)$ is

$$\sum_{s=1,2}\left[b_i^s(p_z)u_i^s(p_z)+d_i^{s*}(\text{-}p_z)v_i^s(\text{-}p_z)\right]\,. \tag{32}$$

By observing that the Fourier transform of $\phi_\alpha(z,0,\theta)$ is given by $\varphi(p_z-\bar{p})$ (see (13)), we immediately obtain the Fourier transform of $\psi_\alpha(z,0,\theta)$,

$$\varphi(p_z-\bar{p})w=\sum_{s=1,2}\left[b_i^s(p_z)u_i^s(p_z)+d_i^{s*}(\text{-}p_z)v_i^s(\text{-}p_z)\right]\,. \tag{33}$$

Using the orthogonality properties of Dirac spinors, we find [23]

$$b_i^s(p_z)=\varphi(p_z-\bar{p})u_i^{s\dagger}(p_z)w\,,$$

$$d_i^{s*}(\text{-}p_z) = \varphi(p_z - \bar{p}) v_i^{s\dagger}(\text{-}p_z) w\,. \tag{34}$$

These coefficients carry important physical information. For *any* initial state which has the form given in (30), the negative frequency solution coefficients $d_i^{s*}(\text{-}p_z)$ necessarily provides a non-null contribution to the time evolving wave packet. This obliges us to take the complete set of Dirac equation solutions to construct the wave packet. Only if we consider a momentum distribution given by a delta function (plane wave limit) and suppose an initial spinor w being a positive energy mass eigenstate with momentum $\bar{p}$, the contribution due to $d_i^{s*}(\text{-}p_z)$ will be null.

Having introduced the Dirac wave packet prescription, we are now in a position to calculate the flavor conversion formula. The following calculations do not depend on the gamma matrix representation. By substituting the coefficients given by (34) in (31) and using the well-known spinor properties [23]

$$\begin{aligned}\sum_{s=1,2} u_i^s(p_z)\overline{u}_i^s(p_z) &= \frac{\gamma^0 E_i(p_z) - \gamma^3 p_z + m_i}{2E_i(p_z)}\,,\\ \sum_{s=1,2} v_i^s(\text{-}p_z)\overline{v}_i^s(\text{-}p_z) &= \frac{\gamma^0 E_i(p_z) + \gamma^3 p_z - m_i}{2E_i(p_z)}\,,\end{aligned} \tag{35}$$

we obtain

$$\begin{aligned}\psi_i(z,t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p_z}{2\pi}\, \varphi(p_z - \bar{p}) \exp\left[\mathrm{i}p_z z\right] \\ \times \left\{\cos\left[E_i(p_z)t\right] - \frac{\mathrm{i}\gamma^0\left(\gamma^3 p_z + m_i\right)}{E_i(p_z)} \sin\left[E_i(p_z)t\right]\right\} w\,.\end{aligned} \tag{36}$$

By simple mathematical manipulations, the new interference oscillating term will be written as

$$\begin{aligned}\mathrm{INT}_{\mathrm{Dirac}}(t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p_z}{2\pi}\, \varphi^2(p_z - \bar{p}) \\ \times \left\{(1 - F(p_z))\cos\left[\Delta E(p_z)t\right] \right.\\ \left. + F(p_z)\cos\left[2\bar{E}(p_z)t\right]\right\}\,,\end{aligned} \tag{37}$$

where

$$\bar{E}(p_z) = \frac{E_1(p_z) + E_2(p_z)}{2}$$

and

$$F(p_z) = \frac{1}{2} - \frac{p_z^2 + m_1 m_2}{2E_1(p_z)E_2(p_z)}\,.$$

What is interesting about the result in (37) is that it was obtained without any assumption on the initial spinor w. Otherwise, the initial spinor carries some fundamental physical information about the created state. And this could be relevant in the study of chiral oscillations [28] where the initial state plays a fundamental role. With respect to the standard treatment of neutrino oscillations done by using *scalar* wave packets and leading to the interference term

$$\mathrm{INT}_{\mathrm{scalar}}(t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p_z}{2\pi}\, \varphi^2(p_z - \bar{p}) \cos\left[\Delta E(p_z)\, t\right], \tag{38}$$

we note in $\mathrm{INT}_{\mathrm{Dirac}}(t)$ two additional terms. In the first one, the *standard* oscillating term $\cos\left[\Delta E(p_z)\, t\right]$, which arises from the interference between mass eigenstate components of equal sign frequencies, is multiplied by a *new factor* obtained by the products

$$u_1^\dagger(p_z)\, u_2(p_z), \quad v_1^\dagger(\text{-}p_z)\, v_2(\text{-}p_z) \quad \text{and h.c.}$$

The second one, $\cos\left[2\bar{E}(p_z)t\right]$, is a *new* oscillating term which comes from the interference between mass eigenstate components of positive and negative frequencies. The factor multiplying such an additional oscillating term is obtained by the products

$$u_1^\dagger(p_z)\, v_2(\text{-}p_z), \quad v_1^\dagger(\text{-}p_z)\, u_2(p_z) \quad \text{and h.c.}$$

The new oscillations have very high frequencies. Such a peculiar oscillating behavior is similar to the phenomenon referred to as *Zitterbewegung*. In atomic physics, the electron exhibits this violent quantum fluctuation in the position and becomes sensitive to an effective potential which explains the Darwin term in the hydrogen atom [24]. We shall see later that, at the instant of creation, such rapid oscillations introduce a small modification in the oscillation formula.

We plot the function $F(p_z)$ in Fig. 1. We can readily observe that it goes rapidly to zero for $p_z \gg m_{1,2}$; it has a minimum at $p_z = 0$ and two maxima at $p_z = \pm\sqrt{m_1 m_2}$. The maximum value of $F(p_z)$ is

$$F_{\max}(p_z) = \frac{1}{2}\left(1 - \frac{\sqrt{m_1 m_2}}{m_1 + m_2}\right)\,, \tag{39}$$

which vanishes in the limit $m_1 = m_2$. As we can see in Fig. 1, the new effects are relevant only when $\Delta m \approx m_1 \gg m_2$.

3.2 The oscillation formula with spreading

To quantify the new effects exhibited in the oscillation probability formula let us calculate the integral of (37) by assuming an ultra-relativistic particle and following the localization condition given by (16). We use the same criteria adopted in Sect. 2 to expand the energy (14) in a power series of ζ_i and χ. A second order approximation in χ allows us to write the energy dependent terms as

$$\begin{aligned}\Delta E^{[2]}(p_z) &= \frac{\Delta m^2}{2\bar{p}}(1 - \chi + \chi^2), \\ \bar{E}^{[2]}(p_z) &\\ &= \bar{p}\left[\left(2 + \frac{m_1^2 + m_2^2}{2\bar{p}^2}\right) + \left(2 - \frac{m_1^2 + m_2^2}{2\bar{p}^2}\right)\chi\right.\end{aligned} \tag{40}$$

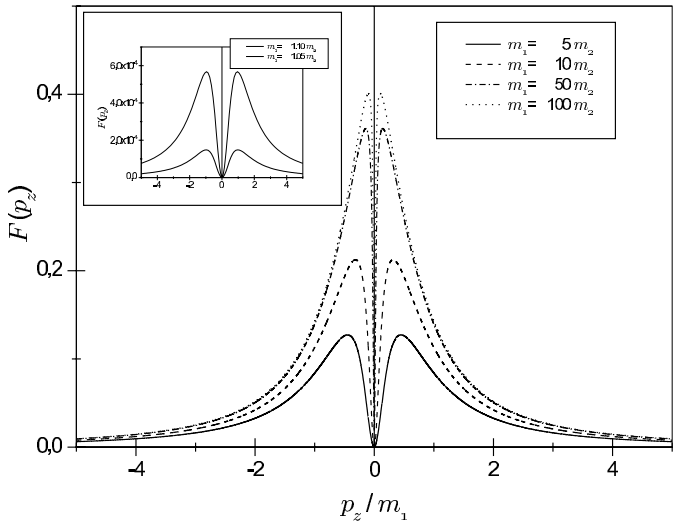

Fig. 1. The function $F(p_z)$ is plotted for different values of the ratio between m_1 and m_2. For a momentum distribution sharply peaked around $\bar{p} \gg m_{1,2}$, $F(p_z)$ does not play a significant role in the "modified" oscillation formula. In the case of $m_1 \approx m_2$, independently of the value of $\bar{p}$ and of the momentum distribution width, the maximum values of $F(p_z)$ are negligible and consequently $F(p_z)$ is practically suppressed in the calculation (see the amplification in the upper box)

$$+ \frac{m_1^2 + m_2^2}{2\bar{p}^2}\chi^2\Bigg] , \tag{41}$$

$$F^{[2]}(p_z) = \left(\frac{\Delta m}{2\bar{p}}\right)^2 (1 - 2\chi + 3\chi^2) . \tag{42}$$

Since we have approximated not only $\Delta E(p_z)$, but also $F(p_z)$ and $E(p_z)$, the range of validity for an analytical approximation of order χ^k is now given by $\left(\frac{\bar{\zeta}^2}{\Delta\zeta}\right)^{\frac{1}{k}} < \chi \leq \bar{\zeta}^{\frac{1}{k+1}}$. By substituting $\varphi(p_z - \bar{p})$ and the approximations (40)–(42) in (37), we obtain

$$\begin{aligned}
\mathrm{INT}_{\mathrm{Dirac}}(t) \approx{}& \frac{a\bar{p}}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \mathrm{d}\chi \, \exp\left[-\frac{(a\bar{p}\chi)^2}{2}\right] \\
&\times \Bigg\{ \left[1 - \left(\frac{\Delta m}{2\bar{p}}\right)^2 (1 - 2\chi + 3\chi^2)\right] \\
&\quad \times \cos\left[\frac{\Delta m^2\, t}{2\bar{p}}(1 - \chi + \chi^2)\right] \\
&+ \left(\frac{\Delta m}{2\bar{p}}\right)^2 (1 - 2\chi + 3\chi^2) \\
&\times \cos\left[\bar{p}t\left(2(1+\chi) + \frac{m_1^2 + m_2^2}{2\bar{p}^2}(1 - \chi + \chi^2)\right)\right]\Bigg\} .
\end{aligned} \tag{43}$$

A new integrating variable $\sigma = \frac{a\bar{p}\chi}{\sqrt{2}}$ and the coefficients

$$\mathcal{S}_-(t) = \frac{\Delta m^2}{2\bar{p}}t , \qquad \mathcal{S}_+(t) = \bar{p}t\left(2 + \frac{m_1^2 + m_2^2}{2\bar{p}^2}\right) ,$$

$$\mathcal{Q}_-(t) = -\frac{\Delta m^2}{\sqrt{2}a\bar{p}^2}t , \quad \mathcal{Q}_+(t) = \frac{\sqrt{2}\bar{p}t}{a\bar{p}}\left(2 - \frac{m_1^2 + m_2^2}{2\bar{p}^2}\right) ,$$

$$\mathcal{R}_-(t) = \frac{\Delta m^2}{a^2\bar{p}^3}t , \qquad \mathcal{R}_+(t) = \bar{p}t\left(\frac{m_1^2 + m_2^2}{a^2\bar{p}^4}\right) \tag{44}$$

enable us to write (43) in the form

$$\begin{aligned}
\mathrm{INT}_{\mathrm{Dirac}}(t) \approx{}& \int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma}{\sqrt{\pi}} \, \exp\left[-\sigma^2\right] \\
&\times \mathrm{Re}\Bigg\{ \left[1 - \left(\frac{\Delta m}{2\bar{p}}\right)^2 \left(1 - \frac{2\sqrt{2}}{a\bar{p}}\sigma + \frac{6}{(a\bar{p})^2}\sigma^2\right)\right] \\
&\quad \times \exp\left[-\mathrm{i}\mathcal{S}_-(t) - \mathrm{i}\mathcal{Q}_-(t)\sigma - \mathrm{i}\mathcal{R}_-(t)\sigma^2\right] \\
&+ \left(\frac{\Delta m}{2\bar{p}}\right)^2 \left(1 - \frac{2\sqrt{2}}{a\bar{p}}\sigma + \frac{6}{(a\bar{p})^2}\sigma^2\right) \\
&\quad \times \exp\left[-\mathrm{i}\mathcal{S}_+(t) - \mathrm{i}\mathcal{Q}_+(t)\sigma - \mathrm{i}\mathcal{R}_+(t)\sigma^2\right]\Bigg\} \\
={}& \mathrm{Re}\left[H_-(t)G_-(t) + H_+(t)G_+(t)\right] ,
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
&G_\pm(t) \\
&= \left(\frac{1}{1 + \mathrm{i}\mathcal{R}_\pm(t)}\right)^{\frac{1}{2}} \exp\left[-\frac{\mathcal{Q}_\pm^2(t)}{4(1 + \mathrm{i}\mathcal{R}_\pm(t))} - \mathrm{i}\mathcal{S}_\pm(t)\right]
\end{aligned} \tag{46}$$

are obtained in the same way as (27), and

$$\begin{aligned}
H_\pm(t) = \frac{1}{2} \mp \Bigg\{ &\frac{1}{2} - \left(\frac{\Delta m}{2\bar{p}}\right)^2 \Bigg[1 + \mathrm{i}\frac{\sqrt{2}}{a\bar{p}}\frac{\mathcal{Q}_\pm(t)}{1 + \mathrm{i}\mathcal{R}_\pm(t)} \\
&+ \frac{3}{(a\bar{p})^2}\left(\frac{1}{1 + \mathrm{i}\mathcal{R}_\pm(t)} - \frac{\mathcal{Q}_\pm^2(t)}{2(1 + \mathrm{i}\mathcal{R}_\pm(t))^2}\right)\Bigg]\Bigg\}
\end{aligned} \tag{47}$$

arises from the new coefficients which include $F(p_z)$.

3.3 The oscillation formula without spreading

A more satisfactory interpretation of the modifications introduced by the Dirac formalism is given when we restrict our study to a first order approximation in χ, i.e. without considering the wave packet *spreading*. In fact, we could take into account terms up to the order χ in (40)–(42) and obtain a simpler approximation,

$$\begin{aligned}
&\mathrm{INT}_{\mathrm{Dirac}}(t) \\
&\approx \int_{-\infty}^{+\infty} \frac{\mathrm{d}\sigma}{\sqrt{\pi}} \, \mathrm{Re}\Bigg\{ \left[1 - \left(\frac{\Delta m}{2\bar{p}}\right)^2 \left(1 - \frac{2\sqrt{2}}{a\bar{p}}\sigma\right)\right] \\
&\quad \times \exp\left[-\mathrm{i}\mathcal{S}_-(t) - \mathrm{i}\mathcal{Q}_-(t)\sigma - \sigma^2\right] \\
&+ \left(\frac{\Delta m}{2\bar{p}}\right)^2 \left(1 - \frac{2\sqrt{2}}{a\bar{p}}\sigma\right) \\
&\quad \times \exp\left[-\mathrm{i}\mathcal{S}_+(t) - \mathrm{i}\mathcal{Q}_+(t)\sigma - \sigma^2\right]\Bigg\}
\end{aligned}$$

$$= \mathrm{Re}\Bigg\{ \left[1 - \left(\frac{\Delta m}{2\bar{p}}\right)^2 \left(1 + \mathrm{i}\frac{\sqrt{2}}{a\bar{p}}\mathcal{Q}_-(t)\right)\right] \times \exp\left[-\frac{\mathcal{Q}_-^2(t)}{4} - \mathrm{i}\mathcal{S}_-(t)\right] + \left(\frac{\Delta m}{2\bar{p}}\right)^2 \left(1 + \mathrm{i}\frac{\sqrt{2}}{a\bar{p}}\mathcal{Q}_+(t)\right) \times \exp\left[-\frac{\mathcal{Q}_+^2(t)}{4} - \mathrm{i}\mathcal{S}_+(t)\right]\Bigg\}. \quad (48)$$

By using the explicit expressions for $\mathcal{Q}_\pm(t)$ and $\mathcal{S}_\pm(t)$ we get

$$\begin{aligned}\mathrm{INT}_{\mathrm{Dirac}}(t) \approx{}& \exp\left[-\left(\frac{\Delta m^2\, t}{2\sqrt{2}a\bar{p}^2}\right)^2\right] \times \left\{\left[1 - \left(\frac{\Delta m}{2\bar{p}}\right)^2\right]\cos\left[\frac{\Delta m^2}{2\bar{p}}t\right] + \left(\frac{\Delta m}{2\bar{p}}\right)^2 \frac{\Delta m^2}{a^2\bar{p}^3}t\sin\left[\frac{\Delta m^2}{2\bar{p}}t\right]\right\} \\ &+ \exp\left[-\frac{t^2}{2a^2}\left(2 - \frac{m_1^2+m_2^2}{2\bar{p}^2}\right)^2\right] \times \left(\frac{\Delta m}{2\bar{p}}\right)^2 \left\{\cos\left[\bar{p}t\left(2 + \frac{m_1^2+m_2^2}{2\bar{p}^2}\right)\right] + \frac{2\bar{p}t}{(a\bar{p})^2}\left(2 - \frac{m_1^2+m_2^2}{2\bar{p}^2}\right) \times \sin\left[\bar{p}t\left(2 + \frac{m_1^2+m_2^2}{2\bar{p}^2}\right)\right]\right\}. \end{aligned} \quad (49)$$

As we have already noticed, the oscillating functions going with the second exponential function in (49) arises from the interference between positive and negative frequency solutions of the Dirac equation. It produces very high frequency oscillations which is similar to the quoted phenomenon of *Zitterbewegung* [24]. The oscillation length which characterizes the very high frequency oscillations is given by $L_{\mathrm{Osc}}^{\mathrm{VHF}} \approx \frac{2\pi}{\bar{p}}$. Obviously, $L_{\mathrm{Osc}}^{\mathrm{VHF}}$ is much smaller than the standard oscillation length given by $L_{\mathrm{Osc}}^{\mathrm{Std}} = \frac{4\pi\bar{p}}{\Delta m^2}$. It means that the propagating particle exhibits a violent quantum fluctuation of its flavor quantum number around a flavor average value which oscillates with $L_{\mathrm{Osc}}^{\mathrm{Std}}$. Meanwhile, except at times $t \sim 0$, it provides a practically null contribution to the oscillation probability. To explain such a statement, let us suppose that an experimental measurement takes place after a time $T \approx L$ for ultra-relativistic particles. The observability conditions impose that the propagation distance L must be larger than the wave packet localization a. Since the (second) exponential function vanishes when $L \gg a$, for measurable distances, the effective flavor conversion formula will not contain such very high frequency oscillation terms and can be written as

$$\begin{aligned} P_{\mathrm{Dirac}}(\boldsymbol{\nu_\alpha} \to \boldsymbol{\nu_\beta}; L) \approx{}& \frac{\sin^2[2\theta]}{2} \times \left\{1 - \exp\left[-\left(\frac{\Delta m^2\, L}{2\sqrt{2}a\bar{p}^2}\right)^2\right]\right\} \\ &\times \left\{\left[1 - \left(\frac{\Delta m}{2\bar{p}}\right)^2\right]\cos\left[\frac{\Delta m^2}{2\bar{p}}L\right] + \left(\frac{\Delta m}{2\bar{p}}\right)^2 \frac{\Delta m^2}{a^2\bar{p}^3}L\sin\left[\frac{\Delta m^2}{2\bar{p}}L\right]\right\}\right\}. \end{aligned} \quad (50)$$

For distances which are restricted to the interval $a \ll L \ll a\frac{2\sqrt{2}\bar{p}^2}{\Delta m^2}$ we observe the *minimal slippage* between the wave packets. In this case, we could suddenly approximate the oscillation probability by

$$\begin{aligned} P_{\mathrm{Dirac}}(\boldsymbol{\nu_\alpha} \to \boldsymbol{\nu_\beta}; L) \approx{}& \frac{\sin^2[2\theta]}{2}\left\{1 - \left[1 - \left(\frac{\Delta m^2 L}{2\sqrt{2}a\bar{p}^2}\right)^2\right] \times \left[1 - \left(\frac{\Delta m}{2\bar{p}}\right)^2\right]\cos\left[\frac{\Delta m^2}{2\bar{p}}L\right]\right\}; \end{aligned} \quad (51)$$

however, we reemphasize that it is *not* valid for $T \approx L \sim 0$ when the rapid oscillations are still relevant ($L < a$). By comparing the result of (51) with the *scalar* oscillation probability of (27), we notice a deviation of the order $\left(\frac{\Delta m}{2\bar{p}}\right)^2$ that appears as an additional coefficient of the cosine function. It is not relevant in the ultra-relativistic limit as we have noticed after studying the function $F(p_z)$.

3.4 A brief extension to a quantum field treatment

To finalize our study, we try to establish a tenuous correspondence between our results and the QFT treatment. It was extensively demonstrated in the literature [10, 20, 21] that the oscillating particle cannot be treated in isolation. The oscillation process must be considered globally: the oscillating states become intermediate states, not directly observed, which propagate between a *source* and a *detector*. This idea can be implemented in QFT when the intermediate oscillating states are represented by internal lines of Feynman diagrams and the interacting particles at source/detector are described by *external* wave packets [11, 21]. In this context, let us consider the weak flavor-changing processes occurring through the intermediate propagation of a neutrino,

$$P_I \to P_F + \alpha + \nu_\alpha \ \text{(oscillation)} \ \nu_\beta + D_I \to \beta + D_F\,, \quad (52)$$

where P_I and P_F (D_I and D_F) are respectively the initial and final production (detection) particles. The amplitude for the process is represented by

$$\mathcal{A} = \left\langle P_F, D_F \left| \mathbf{T}\left(\exp\left[-\mathrm{i}\int \mathrm{d}x^4\, \mathcal{H}_{\mathrm{I}}\right]\right) - \mathbf{1}\right| P_I, D_I \right\rangle, \quad (53)$$

where $\mathcal{H}_{\mathrm{I}}$ is the interaction Hamiltonian for the intermediate particle and $\mathbf{T}$ is the time ordering operator. After

some mathematical manipulations [11], this amplitude can be represented by the integral

$$\mathcal{A} = \int \frac{\mathrm{d}E\,\mathrm{d}\mathbf{p}^3}{(2\pi)^4}\, F(E,\mathbf{p}) \times G(E,\mathbf{p},t_D,t_P)\, \exp\left[\mathrm{i}\,\mathbf{p}\cdot(\mathbf{x}_D-\mathbf{x}_P)\right], \tag{54}$$

where the function $F(E,\mathbf{p})$ represents the *overlap* of the incoming and outgoing wave packets, both at the source and at the detector, and the *Green* function in the momentum space, $G(E,\mathbf{p},t_D,t_P)$, represents the fermion propagator which carries the information of the oscillation process. The overlap function is independent of production and detection times and positions $(t_P,t_D,\mathbf{x}_P,\mathbf{x}_D)$ and depends on the directions of incoming and outgoing momenta. In a certain way, the physical conditions of source and detector, in terms of time and space intervals, are better defined in this framework than in the *intermediate* wave packet framework. Anyway, to understand the oscillation process we must turn back to the definition of mixing in quantum mechanics. It is similar in field theory, except that it applies to fields, not to physical states. This difference allows one to bypass the problems arising in the definition of flavor and mass bases [11]. In one-dimensional spatial coordinates, the mixing is illustrated by the unitary transformation

$$\psi_\sigma(z,t;\theta) = \mathcal{G}^{-1}(\theta;t)\,\psi_i(z,t)\,\mathcal{G}(\theta;t) \tag{55}$$

as the result of the non-coincidence of the flavor basis ($\sigma = \alpha,\,\beta$) and the mass basis ($i = 1,\,2$). Equation (55) gives (29) when the generator of the mixing transformations $\mathcal{G}(\theta;t)$ is given by

$$\mathcal{G}(\theta;t) = \exp\left[\theta\int \mathrm{d}z\,\psi_1(z,t)\psi_2(z,t) - \psi_2(z,t)\psi_1(z,t)\right]. \tag{56}$$

By taking the one-dimensional representation of (54), the propagator $G(E,p_z,t_D,t_P)$ can also be written in the flavor basis as

$$\begin{aligned} &G^{\alpha\beta}(\theta;E,p_z,T)\\ &= \mathcal{G}^{-1}(\theta;t)\,G(E,p_z,T)\,\mathcal{G}(\theta;t)\\ &= \mathcal{G}^{-1}(\theta;t)\,G(E,p_z,t_D,t_P)\,\mathcal{G}(\theta;t), \end{aligned} \tag{57}$$

with $T = t_D - t_P$.

In particular, by following the Blasone and Vitiello (BV) prescription [22,25], the definition of a Fock space of weak eigenstates becomes possible and a non-perturbative flavor oscillation amplitude can be derived. In this case, the complete Lagrangian (density) is split in a propagation Lagrangian,

$$\begin{aligned} \mathcal{L}_\mathrm{p}(z,t) &= \bar\psi_1(z,t)\,(\mathrm{i}\not\partial - m_1)\,\psi_1(z,t)\\ &\quad +\bar\psi_2(z,t)\,(\mathrm{i}\not\partial - m_2)\,\psi_2(z,t), \end{aligned} \tag{58}$$

and an interaction Lagrangian

$$\begin{aligned} &\mathcal{L}_\mathrm{i}(z,t)\\ &= \bar\psi_\alpha(z,t;\theta)\,(\mathrm{i}\not\partial - m_\alpha)\,\psi_\alpha(z,t;\theta)\\ &\quad +\bar\psi_\beta(z,t;\theta)\,(\mathrm{i}\not\partial - m_\beta)\,\psi_\beta(z,t;\theta)\\ &\quad -m_{\alpha\beta}\left(\bar\psi_\alpha(z,t;\theta)\psi_\beta(z,t;\theta)\bar\psi_\beta(z,t;\theta)\psi_\alpha(z,t;\theta)\right), \end{aligned} \tag{59}$$

where

$$m_{\alpha(\beta)} = m_{1(2)}\,\cos^2\theta + m_{2(1)}\,\sin^2\theta$$

and

$$m_{\alpha\beta} = (m_1 - m_2)\,\cos\theta\sin\theta\,.$$

In general, the two subsets of the Lagrangian can be distinguished if there is a flavor transformation which is a symmetry of $\mathcal{L}_\mathrm{i}(z,t)$ but not of $\mathcal{L}_\mathrm{p}(z,t)$. Particle mixing occurs if the propagator built from $\mathcal{L}_\mathrm{p}(z,t)$, and representing the creation of a particle of flavor α at point z and the annihilation of a particle of flavor β at point z', is not diagonal, i.e. not zero for $\beta = \alpha$. The free fields $\psi_i(z,t)$ can be quantized in the usual way by rewriting the momentum distributions $b_i^s(p_z)$ and $d_i^{s*}(\text{-}p_z)$ in (31) as creation and annihilation operators $\mathrm{B}_i^s(p_z)$ and $\mathrm{D}_i^{s\dagger}(\text{-}p_z)$. The interacting fields are then given by

$$\begin{aligned} \psi_\sigma(z,t) &= \int_{-\infty}^{+\infty}\frac{\mathrm{d}p_z}{2\pi}\exp\left[\mathrm{i}p_z z\right]\\ &\quad\times\sum_{s=1,2}\left\{\mathrm{B}_\sigma^s(p_z;t)\,u_\sigma^s(p_z;t) + \mathrm{D}_\sigma^{s*}(\text{-}p_z;t)\,v_\sigma^s(\text{-}p_z;t)\right\}, \end{aligned} \tag{60}$$

where the new flavor creation and annihilation operators which satisfy canonical anticommutation relations are defined by means of Bogoliubov transformations [25] as

$$\mathrm{B}_\sigma^s(p_z;t) = \mathcal{G}^{-1}(\theta;t)\,\mathrm{B}_i^s(p_z)\,\mathcal{G}(\theta;t)$$

and

$$\mathrm{D}_\sigma^s(\text{-}p_z;t) = \mathcal{G}^{-1}(\theta;t)\,\mathrm{D}_i^s(\text{-}p_z)\,\mathcal{G}(\theta;t)\,.$$

By following the BV prescription [22], which takes into account the above definitions, it was demonstrated [27] that the flavor conversion formula can be written as

$$\begin{aligned} &P(\boldsymbol{\nu_\alpha}\to\boldsymbol{\nu_\beta};t)\\ &= \left|\left\{\mathrm{B}_\beta^s(\bar p;t),\,\mathrm{B}_\alpha^s(\bar p;t)\right\}\right|^2 + \left|\left\{\mathrm{D}_\beta^s(\text{-}\bar p;t),\,\mathrm{B}_\alpha^s(\bar p;t)\right\}\right|^2, \end{aligned} \tag{61}$$

which is calculated without considering the localization conditions imposed by wave packets, i.e. by assuming that $p_z \approx \bar p$. When the explicit forms of the flavor annihilation and creation operators are substituted in (61), it was also demonstrated [25] that the flavor oscillation formula becomes

$$\begin{aligned} P(\boldsymbol{\nu_\alpha}\to\boldsymbol{\nu_\beta};t) &= \frac{\sin^2[2\theta]}{2}\\ &\quad\times\left\{(1-F(\bar p))\cos\left[\Delta E(\bar p)t\right] + F(\bar p)\cos\left[2\bar E(\bar p)t\right]\right\}\\ &\approx \sin^2[2\theta]\left\{\left[1-\left(\frac{\Delta m}{2\bar p}\right)^2\right]\sin^2\left[\frac{\Delta m^2}{4\bar p}t\right]\right. \end{aligned} \tag{62}$$

$$+\left(\frac{\Delta m}{2\bar{p}}\right)^2 \sin^2\left[\bar{p}t\left(1+\frac{m_1^2+m_2^2}{4\bar{p}^2}\right)\right]\Bigg\}\,,$$

where the last approximation is in the relativistic limit $\bar{p} \gg \sqrt{m_1 m_2}$. After some simple mathematical manipulations, (62) gives exactly the oscillation probability $P_{\rm Dirac}(\boldsymbol{\nu_\alpha} \to \boldsymbol{\nu_\beta}; L)$ calculated from (49) when it is assumed that the wave packet width a tends to infinity and $t \sim L$.

This new oscillation formula tends to the standard one (4) in the ultra-relativistic limit. If the mass eigenstates were nearly degenerate, we could have focused on the case of a non-relativistic oscillating particle having *very* distinct mass eigenstates. Under these conditions, the quantum theory of measurement says that interference vanishes. Therefore, as we have already pointed out, the effects are, under realistic conditions, far from observable. Besides, in spite of working on a QFT framework, the lack of observability conditions must be overcome by implementing *external* wave packets, i.e. by calculating the explicit form of (54) for fermions. Such a procedure was applied by Beuthe for scalar particles [11] and, in a very particular analysis, based on the BV calculations and on our *intermediate* wave packet results, it could be extended to the fermionic case.

4 Conclusions

In this paper we have computed the modifications to the flavor conversion probability caused by the introduction of the spinorial form of neutrino wave functions. To describe the time evolution of the mass eigenstates, we have introduced wave packets constructed by superposing the Dirac equation solutions. By following an analytical study with *gaussian* wave packets we have computed the new effects that can be observed in the flavor conversion probability formula. Our study leads to the conclusion that the *fermionic* nature of the particles and the interference between positive and negative frequency components of mass eigenstate wave packets modify the standard oscillation probability which is obtained by implicitly assuming a *scalar* nature of the mass eigenstates. Nevertheless, under particular assumptions, i.e. ultra-relativistic particles and sharply peaked momentum distributions, these modifications introduce correction factors proportional to $\left(\frac{\Delta m}{2\bar{p}}\right)^2$ which are negligible in the ultra-relativistic limit.

We know, however, that the most rigorous treatment of oscillations might be done in the quantum field theory framework. Meanwhile, the prescription of oscillating neutrinos as Dirac spinors had not yet been completely and accurately described in a quantum field formalism. The BV model [22, 26] to neutrino/particle mixing and oscillations is the most preeminent model for trying to reach this aim. The authors of these references have attempted to define a Fock space of weak eigenstates and to derive a non-perturbative oscillation formula. Flavor creation and annihilation operators, satisfying canonical (anti-) commutation relations, are defined by means of Bogoliubov transformations. As a result, new oscillation formulas are obtained for fermions and bosons, with the oscillation frequency depending not only on the difference but also on the sum of the energies of the different mass eigenstates.

By using Dirac wave packets, we have reproduced an oscillation probability formula with the same mathematical structure as those obtained in the BV model [22, 26] in a QFT framework. The study with Dirac wave packets enables us to quantify separately each new effect present in the oscillation formula. Imposing the initial constraint where we have a *pure* flavor eigenstate at the time of creation $t = 0$ for any constant spinor w, we could calculate the contribution of new effects to the oscillation probability. Particularly, we have noticed that a term of very high oscillation frequency depending on the sum of energies introduces a very small modification in the characteristic of the oscillation phenomena. In addition, the spinorial form of the wave functions subtly modifies the coefficients of the oscillating terms in the flavor conversion formula.

To conclude, we emphasize one more conceptual aspect arising from the Dirac formalism. Dirac wave packets enable us to develop a study of chiral oscillations [28]. In the standard model of flavor-changing interactions, neutrinos with positive chirality are decoupled from the neutrino absorbing charged weak currents. In the ultra-relativistic limit, a state with *left-handed* helicity is practically a state with negative chirality. If the interactions at the source and detector are chiral, only the component with negative chirality contributes to the propagation. Therefore, the possibility of chiral oscillations can subtly modify the oscillation formula. In this context, the study of chiral and flavor oscillations could also deserve some further specific studies.

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