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# Real Linear Quaternionic Differential Operators

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**Abstract**—The renewed interest in searching for quaternionic deviations of standard (complex) quantum mechanics resulted, in the last years, in a better understanding of the quaternionic mathematical tools needed to solve quantum mechanical problems. In particular, a relevant progress has been achieved in solving eigenvalue problems and differential equations for quaternionic operators. The practical methods recently proposed to solve quaternionic and complex linear second-order differential equations with constant coefficients represent a fundamental starting point to discuss quaternionic potentials in quantum mechanics and study possible violations from complex theories. Nevertheless, only for a restricted class of real linear quaternionic differential operators (namely, symmetric operators) the solution of differential problems was given. In this paper, we study real linear quaternionic differential equations. The proposed resolution's method is based on the Jordan canonical form of (real linear) quaternionic matrices. © 2004 Elsevier Ltd. All rights reserved.

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## 1. INTRODUCTION

Before going into the details of the discussion of  $\mathbb{R}$  linear quaternionic differential operators, we briefly recall the technique used to solve  $\mathbb{H}$  and  $\mathbb{C}$  linear differential equations with constant quaternionic coefficients and show the difficulties in extending the method of resolution to the  $\mathbb{R}$ -linear case. We do not attempt a formal discussion of quaternionic theory of differential equations. Instead, we take an operational and intuitive approach. For the convenience of the reader and to make our exposition as self-contained as possible, we follow the mathematical notation and

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terminology used in our previous paper [2]. In particular, the operators

$$L_\mu = (1, L_i, L_j, L_k) \quad \text{and} \quad R_\mu = (1, R_i, R_j, R_k), \quad \mu = 0, 1, 2, 3,$$

will denote the left and right action on quaternionic functions of real variable,  $\varphi(x)$ , of the imaginary units  $i, j$ , and  $k$ . To shorten notation, we shall use the upper-script  $\mathbb{X} = \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$  to indicate the  $\mathbb{R}, \mathbb{C}$ , and  $\mathbb{H}$  linearity (from the right) of  $n$ -order quaternionic differential operators,  $\mathcal{D}_n^{\mathbb{X}}$ , and  $n$ -dimensional quaternionic matrices,  $M_n^{\mathbb{X}}$ .

The general solution of  $n$ -order  $\mathbb{C}$  linear homogeneous ordinary differential equations

$$\mathcal{D}_n^{\mathbb{C}} \varphi(x) = 0, \tag{1}$$

where

$$\mathcal{D}_n^{\mathbb{C}} = \frac{d^n}{dx^n} - \sum_{p=0}^{n-1} a_{\mathbb{C}}^{(p)} \frac{d^p}{dx^p} = \frac{d^n}{dx^n} - \sum_{p=0}^{n-1} \left[ \sum_{\mu=0}^3 \sum_{\nu=0}^1 a_{\mu\nu}^{(p)} L_\mu R_\nu \right] \frac{d^p}{dx^p}, \quad a_{\mu\nu}^{(p)} \in \mathbb{R},$$

has the form

$$\varphi(x) = \sum_{m=1}^{2n} \varphi_m(x) c_m, \tag{2}$$

where  $\{\varphi_1(x), \dots, \varphi_{2n}(x)\}$  represent  $2n$  quaternionic particular solutions linearly independent over  $\mathbb{C}(1, i)$  and  $c_m$  are complex constants determined by the initial values of the function  $\varphi(x)$  and its derivatives

$$\varphi(x_0) = \varphi_0, \quad \frac{d\varphi}{dx}(x_0) = \varphi_1, \dots, \frac{d^{n-1}\varphi}{dx^{n-1}}(x_0) = \varphi_{n-1} \in \mathbb{H}. \tag{3}$$

The solution of equation (1) is given by

$$\varphi(x) = \sum_{p=1}^n \left\{ \exp [M_n^{\mathbb{C}}(x - x_0)] \right\}_{1p} \varphi_{p-1} = \sum_{p=1}^n \left\{ S_n^{\mathbb{C}} \exp [J_n^{\mathbb{C}}(x - x_0)] (S_n^{\mathbb{C}})^{-1} \right\}_{1p} \varphi_{p-1}, \tag{4}$$

where

$$M_n^{\mathbb{C}} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ a_{\mathbb{C}}^{(0)} & a_{\mathbb{C}}^{(1)} & a_{\mathbb{C}}^{(2)} & \dots & a_{\mathbb{C}}^{(n-1)} \end{pmatrix} \tag{5}$$

and  $J_n^{\mathbb{C}}$  represents the Jordan canonical form of  $M_n^{\mathbb{C}}$ . Note that  $J_n^{\mathbb{C}} = D_n^{\mathbb{C}} + N_n^{\mathbb{C}}$ , where  $D_n^{\mathbb{C}}$  is diagonal and  $N_n^{\mathbb{C}}$  is nilpotent. The Jordan matrix  $J_n^{\mathbb{C}}$  can be determined by solving the right complex eigenvalue problem [3-5] for  $M_n^{\mathbb{C}}$ .

It is worth pointing out that from equation (4) we can recover the form of the particular solutions  $\varphi_m(x)$ . For example, in the case of a null nilpotent matrix, the general solution (2) can be rewritten in terms of left acting quaternionic coefficients ( $u_p$  and  $v_p$ ), complex exponentials ( $\exp[z_p x]$  and  $\exp[w_p x]$ ), and right acting complex constants ( $c_p$  and  $\tilde{c}_p$ ) determined by the initial conditions. Explicitly, we find

$$\varphi(x) = \sum_{p=1}^n \{u_p \exp[z_p x] c_p + v_p \exp[w_p x] \tilde{c}_p\}, \tag{6}$$

where the complex coefficients  $\{z_1, w_1, \dots, z_n, w_n\}$  represent the right eigenvalues of the quaternionic matrix  $M_n^C$ . In the case of equal complex eigenvalues and equal quaternionic coefficients, the remaining particular solutions are determined by using the nilpotent matrix  $N_n^C$ .

For  $\mathbb{H}$  linear quaternionic matrices,  $M_n^{\mathbb{H}}$ , it is possible to show that  $v_p = u_p j$  and  $w_p = \bar{z}_p$ . Consequently, for  $\mathbb{H}$  linear quaternionic differential operators, the general solution (6) reduces to

$$\varphi(x) = \sum_{p=1}^n u_p \exp[z_p x] (c_p + j\bar{c}_p) = \sum_{p=1}^n \exp[q_p x] h_p, \tag{7}$$

where  $q_p = u_p z_p u_p^{-1}$ . The initial conditions (3) shall fix the  $n$  quaternionic constants  $h_p$ .

Due to the  $\mathbb{R}$  linearity, the general solution of  $n$ -order homogeneous ordinary differential equations with quaternionic constant coefficients which appear on the left and on the right has the form

$$\varphi(x) = \sum_{s=1}^{4n} \varphi_s(x) r_s, \tag{8}$$

where  $\{\varphi_1(x), \dots, \varphi_{4n}(x)\}$  represent  $4n$  quaternionic particular solutions, linearly independent over  $\mathbb{R}$ , and  $r_s$  are real constants fixed by equations (3). The question still unanswered is how to determine the particular solutions  $\varphi_s(x)$ . The natural choice of left acting quaternionic coefficients and real exponentials

$$\varphi_s(x) = u_s \exp[\lambda_s x],$$

does not represent a satisfactory answer. In fact, such particular solutions are at most valid for the real part of the eigenvalue spectrum of the matrix  $M_n^{\mathbb{R}}$ .

## 2. CONSTANT COEFFICIENTS

Let us now consider the second order  $\mathbb{R}$  linear quaternionic differential equation

$$\mathcal{D}_2^{\mathbb{R}} \varphi(x) = 0, \tag{9}$$

where

$$\mathcal{D}_2^{\mathbb{R}} = \frac{d^2}{dx^2} - a_{\mathbb{R}}^{(1)} \frac{d}{dx} - a_{\mathbb{R}}^{(0)} = \frac{d^2}{dx^2} - \sum_{\mu, \nu=0}^3 a_{\mu\nu}^{(1)} L_{\mu} R_{\nu} \frac{d}{dx} - \sum_{\mu, \nu=0}^3 a_{\mu\nu}^{(0)} L_{\mu} R_{\nu}, \quad a_{\mu\nu}^{(0,1)} \in \mathbb{R}.$$

By introducing the  $\mathbb{R}$  linear quaternionic matrix

$$M_2^{\mathbb{R}} = \begin{pmatrix} 0 & 1 \\ a_{\mathbb{R}}^{(0)} & a_{\mathbb{R}}^{(1)} \end{pmatrix}$$

and the quaternionic column vector

$$\Phi = \begin{bmatrix} \varphi \\ \frac{d}{dx} \varphi \end{bmatrix},$$

we can rewrite equation (9) in matrix form

$$\frac{d}{dx} \Phi(x) = M_2^{\mathbb{R}} \Phi(x). \tag{10}$$

The real matrix counterpart of the quaternionic operator  $M_2^{\mathbb{R}}$ , from now on denoted by  $M_8[\mathbb{R}]$ , has an eight-dimensional eigenvalue spectrum characterized by real numbers and/or complex

conjugate pairs (the translation tables are given in the appendix). Let  $J_8[\mathbb{C}]$  be the complex Jordan form of  $M_8[\mathbb{R}]$ ,

$$J_8[\mathbb{C}] = R_{2m} \oplus Z_n \oplus \bar{Z}_n, \quad m + n = 4, \tag{11}$$

where  $R_{2m}$  and  $Z_n$  represent the matrix Jordan blocks containing, respectively, the real and complex eigenvalues of  $M_8[\mathbb{R}]$ . By using an appropriate similarity matrix

$$S_8[\mathbb{C}] = \begin{pmatrix} S_{2m \times 2m}^{(1)}[\mathbb{R}] & S_{2m \times n}^{(4)}[\mathbb{C}] & \bar{S}_{2m \times n}^{(4)}[\mathbb{C}] \\ S_{n \times 2m}^{(2)}[\mathbb{R}] & S_{n \times n}^{(5)}[\mathbb{C}] & \bar{S}_{n \times n}^{(5)}[\mathbb{C}] \\ S_{n \times 2m}^{(3)}[\mathbb{R}] & S_{n \times n}^{(6)}[\mathbb{C}] & \bar{S}_{n \times n}^{(6)}[\mathbb{C}] \end{pmatrix}, \tag{12}$$

we can rewrite  $M_8[\mathbb{R}]$  as product of three complex matrices, that is  $S_8[\mathbb{C}]J_8[\mathbb{C}]S_8^{-1}[\mathbb{C}]$ . The problem is now represented by the impossibility to translate the single terms of the previous matrix product by  $\mathbb{R}$  linear quaternionic matrices. To overcome this difficulty we introduce the complex matrix

$$W_8[\mathbb{C}] = \mathbf{1}_{2m} \oplus \left[ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \otimes \mathbf{1}_n \right]. \tag{13}$$

An easy algebraic calculation shows that

$$M_8[\mathbb{R}] = T_8[\mathbb{R}] J_8[\mathbb{R}] T_8^{-1}[\mathbb{R}],$$

where

$$T_8[\mathbb{R}] = S_8[\mathbb{C}] W_8[\mathbb{C}] = \begin{pmatrix} S_{2m \times 2m}^{(1)}[\mathbb{R}] & \sqrt{2} \operatorname{Re} \left\{ S_{2m \times n}^{(4)}[\mathbb{C}] \right\} & -\sqrt{2} \operatorname{Im} \left\{ S_{2m \times n}^{(4)}[\mathbb{C}] \right\} \\ S_{n \times 2m}^{(2)}[\mathbb{R}] & \sqrt{2} \operatorname{Re} \left\{ S_{n \times n}^{(5)}[\mathbb{C}] \right\} & -\sqrt{2} \operatorname{Im} \left\{ S_{n \times n}^{(5)}[\mathbb{C}] \right\} \\ S_{n \times 2m}^{(3)}[\mathbb{R}] & \sqrt{2} \operatorname{Re} \left\{ S_{n \times n}^{(6)}[\mathbb{C}] \right\} & -\sqrt{2} \operatorname{Im} \left\{ S_{n \times n}^{(6)}[\mathbb{C}] \right\} \end{pmatrix} \tag{14}$$

and

$$J_8[\mathbb{R}] = W_8^{-1}[\mathbb{C}] J_8[\mathbb{C}] W_8[\mathbb{C}] = R_{2m} \oplus \begin{pmatrix} \operatorname{Re}[Z_n] & -\operatorname{Im}[Z_n] \\ \operatorname{Im}[Z_n] & \operatorname{Re}[Z_n] \end{pmatrix}. \tag{15}$$

The real Jordan canonical form  $J_8[\mathbb{R}]$  can be decomposed into the sum of three commuting real matrices, that is the diagonal matrix

$$D_8[\mathbb{R}] = \operatorname{Diag} \left\{ \lambda_1, \dots, \lambda_m, \bar{\lambda}_1, \dots, \bar{\lambda}_m, \operatorname{Re}[z_1], \dots, \operatorname{Re}[z_n], \operatorname{Re}[z_1], \dots, \operatorname{Re}[z_n] \right\},$$

the anti-symmetric matrix

$$A_8[\mathbb{R}] = \mathbf{0}_{2m} \oplus \left[ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \operatorname{Diag} \left\{ \operatorname{Im}[z_1], \dots, \operatorname{Im}[z_n] \right\} \right],$$

and the nilpotent matrix  $N_8[\mathbb{R}]$  (a lower triangular matrix whose only nonzero elements are ones which appear in some of the subdiagonal positions). The real Jordan form  $J_8[\mathbb{R}] = D_8[\mathbb{R}] + A_8[\mathbb{R}] + N_8[\mathbb{R}]$  and the similarity transformation  $T_8[\mathbb{R}]$  can be now translated in their real linear quaternionic counterparts  $J_2^{\mathbb{R}} = D_2^{\mathbb{R}} + A_2^{\mathbb{R}} + N_2^{\mathbb{R}}$  and  $T_2^{\mathbb{R}}$ . The matrix solution of equation (10) then reads

$$\begin{aligned} \Phi(x) &= \exp \left[ M_2^{\mathbb{R}}(x - x_0) \right] \Phi(x_0) \\ &= T_2^{\mathbb{R}} \exp \left[ J_2^{\mathbb{R}}(x - x_0) \right] (T_2^{\mathbb{R}})^{-1} \Phi(x_0) \\ &= T_{\mathbb{R}} \exp \left[ D_{\mathbb{R}}(x - x_0) \right] \exp \left[ A_{\mathbb{R}}(x - x_0) \right] \exp \left[ N_{\mathbb{R}}(x - x_0) \right] (T_2^{\mathbb{R}})^{-1} \Phi(x_0). \end{aligned} \tag{16}$$

In the case of a null nilpotent matrix (explicit examples are given in Section A), it can be verified by direct calculations that the general solution of real linear quaternionic differential equations with constant coefficients is

$$\begin{aligned} \varphi(x) = & \sum_{r=1}^m \left\{ u_r \exp[\lambda_r x] \alpha_r + \tilde{u}_r \exp[\tilde{\lambda}_r x] \beta_r \right\} \\ & + \sum_{p=1}^n \{ v_p \cos[b_p x] - \tilde{v}_p \sin[b_p x] \} \exp[a_p x] \gamma_p \\ & + \sum_{p=1}^n \{ \tilde{v}_p \cos[b_p x] + v_p \sin[b_p x] \} \exp[a_p x] \delta_p, \quad m + n = 4, \end{aligned} \tag{17}$$

where  $a_p = \text{Re}[z_p]$ ,  $b_p = \text{Im}[z_p]$ , and  $u_r, \tilde{u}_r, v_p, \tilde{v}_p \in \mathbb{H}$ . The eight real constants  $\alpha_r, \beta_r, \gamma_p$ , and  $\delta_p$  are fixed by the quaternionic initial conditions  $\varphi(x_0) = \varphi_0$  and  $\frac{d\varphi}{dx}(x_0) = \varphi_1$ . The important point to note here is that the particular solutions corresponding to the complex eigenvalue  $z = a + ib$  are given by a quaternionic combination of  $\cos[bx] \exp[ax]$  and  $\sin[bx] \exp[ax]$ , namely

$$\{v \cos[bx] - \tilde{v} \sin[bx]\} \exp[ax] \quad \text{and} \quad \{\tilde{v} \cos[bx] + v \sin[bx]\} \exp[ax].$$

For  $\mathbb{C}$  linear quaternionic differential operators (an explicit example is given in Section B), the general solution (17) reduces to

$$\varphi(x) = \sum_{r=1}^m u_r \exp[\lambda_r x] c_r + \sum_{p=1}^n v_p \exp[z_p x] d_p, \quad c_r, d_p \in \mathbb{C}. \tag{18}$$

**2.A. Null Nilpotent Matrix**

Let us solve the second-order  $\mathbb{R}$  linear quaternionic differential equation

$$\left[ \frac{d^2}{dx^2} - L_i R_j \frac{d}{dx} - L_j R_i \right] \varphi(x) = 0, \tag{19}$$

with initial conditions  $\varphi(0) = j$  and  $\frac{d\varphi}{dx}(0) = k$ . The matrix operator corresponding to this equation is

$$M_2^{\mathbb{R}} = \begin{pmatrix} 0 & 1 \\ L_j R_i & L_i R_j \end{pmatrix}. \tag{20}$$

By using the matrix  $T_2^{\mathbb{R}}$

$$\begin{aligned} \{T_2^{\mathbb{R}}\}_{11} &= \frac{1 + \sqrt{5}}{4} (L_i + L_k R_j - L_i R_k - L_k R_i) - \frac{1 - \sqrt{5}}{4} (1 - L_j R_j + L_j R_i - R_k), \\ \{T_2^{\mathbb{R}}\}_{12} &= \frac{\sqrt{3}}{2\sqrt{2}} (1 + L_i R_i - L_j R_i - L_k) - \frac{1}{2\sqrt{2}} (L_i + R_i + L_j + L_k R_i), \\ \{T_2^{\mathbb{R}}\}_{21} &= \frac{1}{2} (L_i R_i + L_k R_k - L_j R_k + R_i - L_j + R_j + L_k + L_i R_j), \\ \{T_2^{\mathbb{R}}\}_{22} &= \frac{1}{2\sqrt{2}} (L_j R_k - L_k R_j - L_i R_k + R_j), \end{aligned}$$

and its inverse

$$\begin{aligned} \{(T_2^{\mathbb{R}})^{-1}\}_{11} &= \frac{1}{4\sqrt{5}} (1 - L_j R_j + L_k R_j - L_i - L_k R_i - L_i R_k + L_j R_i + R_k), \\ \{(T_2^{\mathbb{R}})^{-1}\}_{12} &= \frac{5 + \sqrt{5}}{40} (L_i R_i + L_k R_k + L_i R_j - L_k) + \frac{5 - \sqrt{5}}{40} (L_j - R_j - L_j R_k - R_i), \\ \{(T_2^{\mathbb{R}})^{-1}\}_{21} &= \frac{1}{2\sqrt{6}} (1 + L_i R_i - L_j R_i + L_k), \\ \{(T_2^{\mathbb{R}})^{-1}\}_{22} &= \frac{1}{4\sqrt{6}} (L_j R_j - L_k R_k + L_i R_j + R_k) + \frac{1}{4\sqrt{2}} (L_k R_j - L_j R_k + L_i R_k + R_j), \end{aligned}$$

we can rewrite  $M_2^{\mathbb{R}}$  in terms of

$$D_2^{\mathbb{R}} = \frac{1}{2} \begin{pmatrix} L_i R_i + \sqrt{5} L_k R_k & 0 \\ 0 & L_j R_j \end{pmatrix} \quad \text{and} \quad A_2^{\mathbb{R}} = \frac{\sqrt{3}}{2} \begin{pmatrix} 0 & 0 \\ 0 & R_j \end{pmatrix}.$$

From equation (16) we obtain the following solution:

$$\begin{aligned} \varphi(x) &= \{T_2^{\mathbb{R}}\}_{11} \exp\left[\frac{L_i R_i + \sqrt{5} L_k R_k}{2} x\right] \left[\{(T_2^{\mathbb{R}})^{-1}\}_{11} j + \{(T_2^{\mathbb{R}})^{-1}\}_{12} k\right] \\ &+ \{T_2^{\mathbb{R}}\}_{12} \exp\left[\frac{L_j R_j + \sqrt{3} R_j}{2} x\right] \left[\{(T_2^{\mathbb{R}})^{-1}\}_{21} j + \{(T_2^{\mathbb{R}})^{-1}\}_{22} k\right] \\ &= \left\{ \frac{i-j}{2\sqrt{5}} \left( \sinh\left[\frac{\sqrt{5}}{2} x\right] - \sqrt{5} \cosh\left[\frac{\sqrt{5}}{2} x\right] \right) + \frac{1+k}{\sqrt{3}} \sin\left[\frac{\sqrt{3}}{2} x\right] \right\} \exp[x/2] \\ &+ \left\{ \frac{k-1}{\sqrt{5}} \sinh\left[\frac{\sqrt{5}}{2} x\right] + \frac{i+j}{2} \left( \cos\left[\frac{\sqrt{3}}{2} x\right] + \frac{1}{\sqrt{3}} \sin\left[\frac{\sqrt{3}}{2} x\right] \right) \right\} \exp[-x/2]. \end{aligned} \tag{21}$$

In the case of a null nilpotent matrix, the general solution of real linear quaternionic differential equations can be written in terms of the eigenvalue spectrum of  $M_8[\mathbb{R}]$ , real matrix counterpart of  $M_2^{\mathbb{R}}$ . In this particular case, the eigenvalue spectrum is

$$\left\{ \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, -\frac{1+\sqrt{5}}{2}, -\frac{1-\sqrt{5}}{2}, \frac{1+i\sqrt{3}}{2}, -\frac{1-i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}, -\frac{1+i\sqrt{3}}{2} \right\}.$$

Consequently, equation (17) becomes

$$\begin{aligned} \varphi(x) &= u_1 \exp\left[\frac{1+\sqrt{5}}{2} x\right] \alpha_1 + u_2 \exp\left[\frac{1-\sqrt{5}}{2} x\right] \alpha_2 + \tilde{u}_1 \exp\left[-\frac{1+\sqrt{5}}{2} x\right] \beta_1 \\ &+ \tilde{u}_2 \exp\left[-\frac{1-\sqrt{5}}{2} x\right] \beta_2 + \left\{ v_1 \cos\left[\frac{\sqrt{3}}{2} x\right] - \tilde{v}_1 \sin\left[\frac{\sqrt{3}}{2} x\right] \right\} \exp[x/2] \gamma_1 \\ &+ \left\{ \tilde{v}_1 \cos\left[\frac{\sqrt{3}}{2} x\right] + v_1 \sin\left[\frac{\sqrt{3}}{2} x\right] \right\} \exp[x/2] \delta_1 \\ &+ \left\{ v_2 \cos\left[\frac{\sqrt{3}}{2} x\right] - \tilde{v}_2 \sin\left[\frac{\sqrt{3}}{2} x\right] \right\} \exp[-x/2] \gamma_2 \\ &+ \left\{ \tilde{v}_2 \cos\left[\frac{\sqrt{3}}{2} x\right] + v_2 \sin\left[\frac{\sqrt{3}}{2} x\right] \right\} \exp[-x/2] \delta_2. \end{aligned}$$

By direct calculations, we can determine the quaternionic coefficients  $u_{1,2}$ ,  $\tilde{u}_{1,2}$ ,  $v_{1,2}$ , and  $\tilde{v}_{1,2}$ . We find

$$\begin{aligned} u_1 &= i - j, & \tilde{u}_1 &= 1 - k, \\ u_2 &= i - j, & \tilde{u}_2 &= 1 - k, \\ v_1 &= 1 + k, & \tilde{v}_1 &= 0, \\ v_2 &= i + j, & \tilde{v}_2 &= 0. \end{aligned}$$

The particular solutions corresponding to the complex eigenvalues of  $M_8[\mathbb{R}]$  are thus given by

$$z_1 = \frac{1+i\sqrt{3}}{2}, \quad \bar{z}_1 : (1+k) \exp[x/2] \begin{pmatrix} \cos\left[\frac{\sqrt{3}}{2} x\right] \\ \sin\left[\frac{\sqrt{3}}{2} x\right] \end{pmatrix},$$

and

$$z_2 = -\frac{1 - i\sqrt{3}}{2}, \quad \bar{z}_2 : (i + j) \exp[-x/2] \begin{matrix} \cos \left[ \frac{\sqrt{3}}{2} \right] \\ \sin \left[ \frac{\sqrt{3}}{2} \right] \end{matrix},$$

The initial conditions  $\varphi(0) = j$  and  $\frac{d\varphi}{dx}(0) = k$  make the solution completely determined,

$$\begin{aligned} \alpha_1 &= \frac{1 - \sqrt{5}}{4\sqrt{5}}, & \beta_1 &= \frac{1}{2\sqrt{5}}, & \gamma_1 &= 0, & \delta_1 &= \frac{1}{\sqrt{3}}, \\ \alpha_2 &= -\frac{1 + \sqrt{5}}{4\sqrt{5}}, & \beta_2 &= -\frac{1}{2\sqrt{5}}, & \gamma_2 &= \frac{1}{2}, & \delta_2 &= \frac{1}{2\sqrt{3}}. \end{aligned}$$

Let us now consider the following second-order  $\mathbb{R}$  linear quaternionic differential equation:

$$\left[ \frac{d^2}{dx^2} - (L_i R_i + L_j R_j) \frac{d}{dx} + R_k - L_k \right] \varphi(x) = 0. \tag{22}$$

The eigenvalue spectrum of the real matrix counterpart of

$$M_2^{\mathbb{R}} = \begin{pmatrix} 0 & 1 \\ Lk - R_k & L_i R_i + L_j R_j \end{pmatrix} \tag{23}$$

is

$$\{2, -2, 0, 0, 1 + i, -(1 - i), 1 - i, -(1 + i)\}.$$

In this case,  $v_1 \cos x \exp[x]$  and  $v_2 \cos x \exp[-x]$  does not represent (nontrivial) particular solutions. In fact, in order to satisfy equation (22) we have to impose the following constraints on the quaternionic coefficients  $v_{1,2}$ :

$$\begin{aligned} [L_i R_i + L_j R_j - 2] v_1 &= 0 \Rightarrow v_1 = i\beta + j\gamma, \\ [L_i R_i + L_j R_j + L_k - R_k] v_1 &= 0 \Rightarrow v_1 = 0, \\ [L_i R_i + L_j R_j + 2] v_2 &= 0 \Rightarrow v_2 = \alpha + j\delta, \\ [L_i R_i + L_j R_j - L_k + R_k] v_2 &= 0 \Rightarrow v_2 = 0. \end{aligned}$$

The particular solutions corresponding to the complex eigenvalues  $z_1 = 1 + i$  and  $z_2 = -(1 + i)$  are

$$z_1, \bar{z}_1 : \exp[x] \begin{matrix} j \cos[x] - i \sin[x] \\ i \cos[x] + j \sin[x] \end{matrix},$$

and

$$z_2, \bar{z}_2 : \exp[-x] \begin{matrix} j \cos[x] + i \sin[x] \\ i \cos[x] - j \sin[x] \end{matrix}.$$

The general solution is given by

$$\begin{aligned} \varphi(x) &= \exp[-2x] \alpha_1 + k \exp[2x] \alpha_2 + \beta_1 + k \beta_2 \\ &+ \{j \cos[x] - i \sin[x]\} \exp[x] \gamma_1 + \{i \cos[x] + j \sin[x]\} \exp[x] \delta_1 \\ &+ \{j \cos[x] + i \sin[x]\} \exp[-x] \gamma_2 + \{i \cos[x] - j \sin[x]\} \exp[-x] \delta_2. \end{aligned}$$

**2.B. Complex Linear Case**

We now determine the solution of the second-order  $\mathbb{C}$  linear quaternionic differential equation

$$\left[ \frac{d^2}{dx^2} - L_j R_i \right] \varphi(x) = 0. \tag{24}$$

The eigenvalue spectrum of the real matrix counterpart of

$$M_2^{\mathbb{R}} = \begin{pmatrix} 0 & 1 \\ L_j R_i & 0 \end{pmatrix} \tag{25}$$

is

$$\{1, -1, 1, -1, i, i, -i, -i\}.$$

The general solution is then given by

$$\begin{aligned} \varphi(x) = & u_1 \exp[x] \alpha_1 + u_2 \exp[-x] \alpha_2 + \tilde{u}_1 \exp[x] \beta_1 + \tilde{u}_2 \exp[-x] \beta_2 \\ & + \{v_1 \cos[x] - \tilde{v}_1 \sin[x]\} \gamma_1 + \{\tilde{v}_1 \cos[x] + v_1 \sin[x]\} \delta_1 \\ & + \{v_2 \cos[x] - \tilde{v}_2 \sin[x]\} \gamma_2 + \{\tilde{v}_2 \cos[x] + v_2 \sin[x]\} \delta_2. \end{aligned} \tag{26}$$

By direct calculations, we can determine the quaternionic coefficients  $u_{1,2}$ ,  $\tilde{u}_{1,2}$ ,  $v_{1,2}$ , and  $\tilde{v}_{1,2}$ . We find

$$\begin{aligned} u_1 &= k - 1, & \tilde{u}_1 &= j - i, \\ u_2 &= k - 1, & \tilde{u}_2 &= j - i, \\ v_1 &= 1 + k, & \tilde{v}_1 &= 0, \\ v_2 &= i + j, & \tilde{v}_2 &= 0. \end{aligned}$$

Consequently, equation (26) becomes

$$\varphi(x) = (k - 1) [\exp[x] c_1 + \exp[-x] c_2] + (1 + k) \{\exp[ix] d_1 + \exp[-ix] d_2\}. \tag{27}$$

The initial conditions  $\varphi(0) = j$  and  $\frac{d\varphi}{dx}(0) = k$  fix the following solution:

$$\varphi(x) = \{(j - i) \cosh x + (k - 1) \sinh x + (i + j) \exp[-ix]\}/2. \tag{28}$$

**2.C. Non-Null Nilpotent Matrix**

As a last example we consider the differential equation

$$\left[ \frac{d^2}{dx^2} - (L_i R_j + L_j R_i) \right] \varphi(x) = 0. \tag{29}$$

The eigenvalue spectrum of the real matrix counterpart of

$$M_2^{\mathbb{R}} = \begin{pmatrix} 0 & 1 \\ L_i R_j + L_j R_i & 0 \end{pmatrix} \tag{30}$$

is

$$\{0, 0, 0, 0, \sqrt{2}, -\sqrt{2}, i\sqrt{2}, -i\sqrt{2}\}.$$

By using an appropriate matrix  $T_2^{\mathbb{R}}$ , we can rewrite  $M_2^{\mathbb{R}}$  in terms of

$$\begin{aligned} D_2^{\mathbb{R}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 0 & L_j R_j + L_k R_k \end{pmatrix}, \\ A_2^{\mathbb{R}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 0 & L_i - R_i \end{pmatrix}, \quad \text{and} \quad N_2^{\mathbb{R}} = \frac{1}{2} \begin{pmatrix} L_i - L_k R_j & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$



$M_2^{\mathbb{R}} = T_2^{\mathbb{R}}(D_2^{\mathbb{R}} + A_2^{\mathbb{R}} + N_2^{\mathbb{R}})(T_2^{\mathbb{R}})^{-1}$ . Notwithstanding its diagonal form,  $N_2^{\mathbb{R}}$  represents a nilpotent matrix. In fact, an easy algebraic calculation shows that  $(N_2^{\mathbb{R}})^2 = 0$ . This implies in the general solution the explicit presence of the real variable  $x$ . Explicitly, we have

$$\begin{aligned} \varphi(x) = & \alpha_1 + k \alpha_2 + (\tilde{\alpha}_1 + k \tilde{\alpha}_2) x \\ & + (i - j) \left\{ \exp[\sqrt{2} x] \beta_1 + \exp[-\sqrt{2} x] \beta_2 \right\} \\ & + (i + j) \left\{ \cos[\sqrt{2} x] \gamma_1 + \sin[\sqrt{2} x] \delta_1 \right\}. \end{aligned}$$

To make this solution determined, let us specify certain supplementary constraints,  $\varphi(0) = j$  and  $\frac{d\varphi}{dx}(0) = k$ ,

$$\varphi(x) = kx + \frac{j - i}{2} \cosh[\sqrt{2} x] + \frac{i + j}{2} \cos[\sqrt{2} x]. \tag{31}$$

By changing the initial conditions,  $\varphi(0) = k$  and  $\frac{d\varphi}{dx}(0) = j$ , we find

$$\varphi(x) = k + \frac{j - i}{2\sqrt{2}} \sinh[\sqrt{2} x] + \frac{i + j}{2\sqrt{2}} \sin[\sqrt{2} x]. \tag{32}$$

### 3. CONCLUSIONS

In this paper, we have extended the resolution method of  $\mathbb{H}$  and  $\mathbb{C}$  linear ordinary differential equations with quaternionic constant coefficients [2] to  $\mathbb{R}$  linear differential operators. By a matrix approach, we have shown that particular solutions of differential equations with constant quaternionic coefficients which appear both on the left and on the right can be given in terms of the eigenvalues of the matrix  $M_{4n}[\mathbb{R}]$ , representing the real counterpart of the quaternionic operators,  $M_n^{\mathbb{R}}$ , associated to the  $\mathbb{R}$  linear quaternionic differential equation. In correspondence to the eigenvalue  $z = a + ib$ , we have found the following particular solutions:

$$\{q \cos[bx] - p \sin[bx]\} \exp[ax] \quad \text{and} \quad \{p \cos[bx] + q \sin[bx]\} \exp[ax], \quad q, p \in \mathbb{H}. \tag{33}$$

When  $q = p$  or  $p = 0$  these solutions reduce to

$$q \cos[bx] \exp[ax] \quad \text{and} \quad q \sin[bx] \exp[ax].$$

For  $\mathbb{C}$  linear differential operators, the particular solutions (33) couple to give

$$q \exp[(a + ib)x].$$

Our discussion can be viewed as a preliminary step towards a full understanding of the role that quaternions could play in analysis, linear algebra and, consequently, in physical applications. A theory of quaternionic differential operators is at present far from being conclusive and possible applications of these operators in physics, like that quaternionic perturbation potentials in quantum mechanics [6,7] and quaternionic (boson) gauge particles in quantum field theory [8,9], deserve further investigations.

### REAL TRANSLATION TABLES

EXAMPLE.

$$\begin{aligned} L_i - 2R_j - 3L_iR_k & \rightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} - 2 \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} - 3 \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ & \rightarrow \begin{pmatrix} 0 & -1 & 5 & 0 \\ 1 & 0 & 0 & 5 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \end{aligned}$$

Table 1.

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$
1	$R_i$	$R_j$	$R_k$
$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$
$L_i$	$L_i R_i$	$L_i R_j$	$L_i R_k$
$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$
$L_j$	$L_j R_i$	$L_j R_j$	$L_j R_k$
$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$
$L_k$	$L_k R_i$	$L_k R_j$	$L_k R_k$

Table 2.

1/4	1	$L_i R_i$	$L_j R_j$	$L_k R_k$	1/4	$L_j$	$R_j$	$L_i R_k$	$L_k R_i$
$A_{00}$	+	-	-	-	$A_{02}$	-	-	-	+
$A_{11}$	+	-	+	+	$A_{20}$	+	+	-	+
$A_{22}$	+	+	-	+	$A_{13}$	+	-	-	-
$A_{33}$	+	+	+	-	$A_{31}$	-	+	-	-
1/4	$L_i$	$R_i$	$L_j R_k$	$L_k R_j$	1/4	$L_k$	$R_k$	$L_i R_j$	$L_j R_i$
$A_{01}$	-	-	+	-	$A_{03}$	-	-	+	-
$A_{10}$	+	+	+	-	$A_{30}$	+	+	+	-
$A_{23}$	-	+	-	-	$A_{12}$	-	+	-	-
$A_{32}$	+	-	-	-	$A_{21}$	+	-	-	-

EXAMPLE.

$$A := \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 5 & 0 \\ 1 & 0 & 0 & 5 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

$$\begin{aligned} A \rightarrow & (A_{01}/4) [-L_i - R_i + L_j R_k - L_k R_j] \\ & + (A_{02}/4) [-L_j - R_j - L_i R_k + L_k R_i] \\ & + (A_{10}/4) [+L_i + R_i + L_j R_k - L_k R_j] \\ & + (A_{13}/4) [+L_j - R_j - L_i R_k - L_k R_i] \\ & + (A_{20}/4) [+L_j + R_j - L_i R_k + L_k R_i] \\ & + (A_{23}/4) [-L_i + R_i - L_j R_k - L_k R_j] \\ & + (A_{31}/4) [-L_j + R_j - L_i R_k - L_k R_i] \\ & + (A_{32}/4) [+L_i - R_i - L_j R_k - L_k R_j] \\ \rightarrow & L_i - 2 R_j - 3 L_i R_k. \end{aligned}$$

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