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Quaternionic Analyticity

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Abstract—Complex analyticity is generalized to hypercomplex functions, quaternion or octonion, in such a manner that it includes the standard complex definition and does not reduce analytic functions to a trivial class. A brief comparison with other definitions is presented.

Keywords—Quaternions, Analytic functions, Differential operators.

1. INTRODUCTION

In this letter, we shall propose a new definition for the generalization of the condition for complex analyticity for complex functions to quaternionic or octonionic functions of the corresponding variables. However, in the following unless explicitly mentioned, we will limit the discussion for clarity to quaternionic functions of quaternionic variables.

Since the discovery of quaternions by Hamilton [1] in the last century, a recurring question has been the best way to extend complex analyticity to quaternionic functions of quaternionic variables. The most immediate idea is to extend the concept of differentiability from the two-dimensional complex plain to the four-dimensional quaternionic space. This can easily be done and involves the imposition of three quaternionic equations. If one defines a general quaternion by

$$q = x_0 + ix_1 + jx_2 + kx_3, \tag{1}$$

where $x_{0,1,2,3} \in \mathcal{R}$ and i, j, and k are the noncommuting imaginary units such that

$$i^2 = j^2 = k^2 = -1 \qquad \text{and} \qquad [i, j] = 2k \quad \text{(cyclic)},$$

then the condition for differentiability of the quaternionic function $f(q)$ yields the three equations

$$\partial_{x_0} f(q) = -i\partial_{x_1} f(q) = -j\partial_{x_2} f(q) = -k\partial_{x_3} f(q). \tag{2}$$

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The result of these equations is the restriction of the class of differentiable functions to merely linear functions in q.

$$f(q) = c_1 + qc_2, \tag{3}$$

where $c_{1,2}$ are arbitrary quaternionic coefficients. Actually, the position of the constant depends upon whether the inverse of the increment in the derivative is placed upon the left or the right of the increment of the function $f(q)$. The above form assumes the left choice. If both options are imposed simultaneously, then the holomorphic class of functions reduces to linear polynomials of q with c_2 real. In any case, this generalization is so restrictive that it loses all practical interest. We also note that it excludes (except for the trivial constant functions) the class of analytic/holomorphic functions of standard complex analysis, characterized by Taylor series in the corresponding complex variable.

A more sophisticated attempt was made in the 1930s by Fueter and collaborators [2], who defined analyticity by means of a single quaternionic partial differential equation which includes the standard complex analyticity equation for complex functions of the corresponding single complex variable. He showed that this definition led to close analogues of Cauchy's theorem, Cauchy's integral formula, and the Laurent expansion. A complete bibliography of Fueter's work is contained in [3] and a simple account of the elementary parts of the theory has been given by Deavours [4]. A further extension of Fueter involves a third-order differential equation [5] which we mention only briefly in this work. However, if either of these is the best choice is still to be demonstrated, and from time to time variations on the theme appear in the literature [5]. The situation is enriched by the noncommutative nature of hypercomplex numbers which permits the definition of so-called left/right derivative operators according to the position of the imaginary units with respect to the function. Even combinations or admixtures of these alternative derivatives may be used. This "complication" is significant in certain other physical and mathematical applications [6–8].

The proposal in this work is much simpler. It is to define a "local" derivative operator that depends upon the four-dimensional point at which the derivative is to be made. Each nonreal quaternion point together with the real axis defines a unique complex plain and it is the complex variable of this plain that we use to define analyticity, in complete analogy to complex analysis. As a consequence, the class of analytic functions are generalized to include all polynomial functions of a single quaternionic variable with (right acting) quaternionic coefficients. This approach has the nontrivial virtue of being directly generalizable to octonionic functions of octonionic variables.

2. GLOBAL ANALYTICITY

The complex derivative operator, ∂_z, is defined for $z = x + iy$, by

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right), \tag{4}$$

such that its action upon a monomial of z is simply

$$\partial_z z^n = nz^{n-1} \tag{5}$$

while it gives zero if applied to a polynomial of $\bar{z} = x - iy$. Similarly, the derivative operator for $\bar{z}$ can be defined by the operator

$$\partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right). \tag{6}$$

The conditions for a regular function $f(z, \bar{z})$, the Cauchy-Riemann equations, can then be expressed by

$$\partial_{\bar{z}} f(z, \bar{z}) = 0. \tag{7}$$

The well-known solutions of this equation are polynomial functions of z without any dependence upon $\bar{z}$. At the level of complex numbers this definition of analyticity coincides with the existence of a unique derivative.

The natural generalization of the above complex derivative operator to a quaternionic derivative, acting from the left, is the operator

$$\partial_q = \frac{1}{2}\left(\partial_{x_0} - \frac{i\partial_{x_1} + j\partial_{x_2} + k\partial_{x_3}}{3}\right). \tag{8}$$

We have chosen the normalization factors in ∂_q so that it gives the expected result when applied to linear functions of the variable q, i.e.,

$$\partial_q(c_1 + qc_2) = c_2, \tag{9}$$

with c_1 and c_2 quaternionic constants as before. This operator also annihilates

$$\bar{q} = x_0 - ix_1 - jx_2 - kx_3.$$

However, it does not act in a simple way upon higher powers of q or $\bar{q}$. More importantly, it does not reduce to the complex derivative operator when applied to functions independent of, say, x_2 and x_3.

One of the possible alternatives, at this point, is to define $\partial_{\bar{q}}$ so that it yields the desired complex limit. This limit corresponds to functions of only the real and one imaginary variable and involving only one imaginary unit (a limitation upon the constants). The proposal of Fueter is to define this operator by

$$\partial_{\bar{q}} = \partial_{x_0} + i\partial x_1 + j\partial_{x_2} + k\partial_{x_3}. \tag{10}$$

This yields the so-called Cauchy-Riemann-Fueter (CRF) equation,

$$\partial_{\bar{q}} f = 0, \tag{11}$$

where $\bar{q}$ is the conjugate of q and $\partial_{\bar{q}}$ the conjugate of ∂_q. Not even linear functions in q satisfy this equation. We seem to have reduced analytic functions to mere constants. But one quaternionic equation is surely less restrictive than the three of holomorphy first quoted, equation (2). In fact, the CRF equation has many (but not q polynomial) solutions particularly in what we shall call the *canonical complex variable limit* (see below). In complex analysis, there is no analogy for the appearance of new solutions when one reduces the number of variables. There, if a function depends upon, e.g., only the real variable, the analyticity conditions (Cauchy-Riemann equations) reduce the function to a constant which is already included in the general class of polynomial functions in z.

The canonical complex limit refers to polynomial functions of complex variables involving *only* one of the *basic* imaginary units, i.e., with as variable z either $x_0 + ix_1$ or $x_0 + jx_2$ or $x_0 + kx_3$. These are regular in the above Fueter sense. Nevertheless, since we can multiply each term in the Taylor expansion of the analytic function *from the right* by independent quaternionic coefficients and still maintain analyticity, $f(z)$ need not lie in the same complex plain as its variable z, nor indeed be restricted to lie in a particular plain in quaternionic space. Nevertheless, with the above restriction to *canonical complex variables*, the standard complex analytic functions with complex coefficients are indeed CRF quaternionic analytic. On the other hand, a complex $(1, \iota)$ polynomial of a complex $\zeta = x_0 + \iota x$, where ι is a linear (not necessarily canonical) normalized combination of ix_1, jx_2, and kx_3, is *not* in general a regular function, in the Fueter sense. We also observe that the above-mentioned functions by no means exhaust the class of quaternionic analytic functions, as can easily be seen by considering the set of functions independent of only one of the real variables such as x_3. This definition of regular functions by Fueter permits a quaternionic version of Cauchy's theorem and of Cauchy's integral formula. We would like to define an analyticity condition which both includes arbitrary complex functions of corresponding complex variables projected from q, and even more important, extends the class of regular functions to polynomial functions in q. To the best of our knowledge, the method we describe below has not been proposed previously. The *trick* is obvious once seen. It, however, requires the extension of the concept of a derivative operator from those with constant (even quaternionic) coefficients, to one with variable coefficients dependent upon the point of application. In physical terms, we would say (in analogy with gauge transformations) that the derivative operator passes from a *global* form to a *local* form.

3. LOCAL ANALYTICITY

As already mentioned in the introduction, a point $q \notin \mathcal{R}$ together with the real axis $q = x_0$ specifies a unique complex plain with imaginary axis given by the unit

$$\iota = \frac{(ix_1 + jx_2 + kx_3)}{|\vec{x}|}, \tag{12}$$

where $|\vec{x}|^2 = x_1^2 + x_2^2 + x_3^2$. With ζ already defined as $x_0 + \iota x$, and $x = |\vec{x}|$, the analyticity condition we propose reads

$$\partial_{\bar{\zeta}} f(\zeta) = \frac{\partial_{x_0} + \iota \partial_x}{2} f(q) = 0, \tag{13}$$

or explicitly, ignoring the constant factor of 1/2,

$$\left[\partial_{x_0} + \frac{ix_1 + jx_2 + kx_3}{|\vec{x}|^2} \left(x_1 \partial_{x_1} + x_2 \partial_{x_2} + x_3 \partial_{x_3} \right) \right] f(q) = 0. \tag{14}$$

This formally reproduces to the Cauchy-Riemann equations in the corresponding complex plain, which include, as special cases, the three canonical plains. However, it involves a quaternionic derivative and includes amongst the class of regular functions *all* polynomial functions of q with right-acting quaternionic coefficients. That is, since

$$[\partial_{\bar{\zeta}},\, q] = 0, \tag{15}$$

a class of solutions of the analyticity conditions is the Taylor series

$$f(q) = \sum_{n \geq 0} q^n c_n, \tag{16}$$

where the c_n are arbitrary quaternionic constants. This is the same class of functions obtained by Fueter by means of a *third-order* analyticity condition. It should be obvious that a similar definition of a regular function for octonions reproduces a similar result. The nonassociative nature of octonions plays no role in this since our analyticity condition is of first order.

In conclusion, we have described, from what we hope is an original viewpoint, the quest for an appropriate quaternionic analyticity condition. We have introduced a local derivative operator and used it in a first-order analyticity condition which includes and generalizes to hypercomplex functions the equations of Cauchy-Riemann. The result is a rich and we believe significant class of analytic functions obtained previously but in a much more laborious manner by Fueter.

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