

Quaternionic eigenvalue problem

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We discuss the (right) eigenvalue equation for \mathbb{H} , \mathbb{C} and \mathbb{R} linear quaternionic operators. The possibility to introduce an isomorphism between these operators and real/complex matrices allows us to translate the quaternionic problem into an *equivalent* real or complex counterpart. Interesting applications are found in solving differential equations within quaternionic formulations of quantum mechanics. © 2002 American Institute of Physics. [DOI: 10.1063/1.1511789]

I. INTRODUCTION

The full understanding of the subtleties of the quaternionic eigenvalue problem still represents an intriguing challenge for mathematicians and physicists. The recent study of the eigenvalue problem for complex linear quaternionic operators¹ played a fundamental role in solving quaternionic differential equations.² In the last few years, interesting applications of quaternionic analysis and linear algebra were investigated in quantum mechanics.³ In particular, the solution of the Schrödinger equation in the presence of quaternionic perturbations was explicitly given for constant potentials and deviations from standard (complex) quantum mechanics discussed.⁴ In this article, we aim to complete the study begun in Ref. 1, where preliminary steps in solving the eigenvalue problem for complex linear quaternionic operators were traced. In order to extend to the \mathbb{R} -linear case the results obtained for the \mathbb{H} - and \mathbb{C} -linear quaternionic matrices, we have to introduce a system of *coupled* equations which represents the *new* eigenvalue problem for \mathbb{R} linear quaternionic operators. It is important to observe that no attempt to develop a complete theory of the quaternionic eigenvalue problem has been made here; this exceeds the scope of our article. A satisfactory discussion of the eigenvalue problem for quaternionic operators is at present far from being given. We could have directly investigated the eigenvalue equation in the quaternionic space, but we have preferred a more practical approach and chosen to handle the problem by finding a more familiar real or complex space isomorphic to the quaternionic one. We shall show that the isomorphism between \mathbb{H} -, \mathbb{C} -, and \mathbb{R} -linear quaternionic operators and real/complex matrices immediately allows us to translate the quaternionic (right) eigenvalue problem in a corresponding real or complex counterpart. The study of the new translated problem gives important information about the quaternionic solution. The results obtained are very useful in solving polynomial and differential equations. This could represent a fundamental step in understanding the potentiality of using quaternions in formulating quantum mechanics (by investigating *quaternionic* deviations from the standard theory^{3,4}) and gauge theory (by suggesting *new* unification groups⁵).

Throughout the article we shall denote by \mathbb{R} , \mathbb{C} , and \mathbb{H} the sets of real, complex, and quaternionic numbers, $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$, and by $V[n, X]$ and $M[n, X]$, respectively, the n -tuples and the $n \times n$ matrices over X . Linear quaternionic operators will be distinguished by their linearity from the

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right. In what follows, the notation \mathcal{O}_X stands for quaternionic operators linear (from the right) over the field X .

II. QUATERNIONIC ALGEBRA AND LINEAR OPERATORS

We now introduce the quaternionic algebra and some useful properties of H-, C-, and R-linear operators. The (real) quaternionic skew-field \mathbb{H} is an associative (division) algebra of rank 4 over \mathbb{R} ,

$$q = q_0 + iq_1 + jq_2 + kq_3, \quad q_{0,1,2,3} \in \mathbb{R}, \tag{1}$$

where

$$i^2 = j^2 = k^2 = ijk = -1, \tag{2}$$

endowed with an involutory anti-automorphism (*conjugation*)

$$q \rightarrow \bar{q} = q_0 - iq_1 - jq_2 - kq_3.$$

Due to the noncommutative nature of quaternions, we must distinguish between the left and right action of the quaternionic imaginary units i, j , and k . To do it, we introduce the operators

$$L_\mu = (1, L_i, L_j, L_k) \quad \text{and} \quad R_\mu = (1, R_i, R_j, R_k), \quad \mu = 0, 1, 2, 3,$$

which act on quaternionic vectors $\psi \in V[n, \mathbb{H}]$ in the following way:

$$L_\mu \psi = h_\mu \psi \quad \text{and} \quad R_\mu \psi = \psi h_\mu, \quad h_\mu = (1, i, j, k).$$

These operators satisfy

$$L_i^2 = L_j^2 = L_k^2 = L_i L_j L_k = R_i^2 = R_j^2 = R_k^2 = R_k R_j R_i = -1, \tag{3}$$

and

$$[L_\mu, R_\nu] = 0, \quad \mu, \nu = 0, 1, 2, 3. \tag{4}$$

Note that \mathbb{H} -linear quaternionic operators acting on a finite n -dimensional quaternionic vector space,

$$\mathcal{O}_\mathbb{H}(\psi_1 q_1 + \psi_2 q_2) = (\mathcal{O}_\mathbb{H} \psi_1) q_1 + (\mathcal{O}_\mathbb{H} \psi_2) q_2, \quad q_{1,2} \in \mathbb{H}, \quad \psi_{1,2} \in V[n, \mathbb{H}],$$

are in one-to-one correspondence with $n \times n$ quaternionic matrices:

$$\mathcal{O}_\mathbb{H} \leftrightarrow M_\mathbb{H} \in M[n, \mathbb{H}]. \tag{5}$$

Consequently, R- and C-linear quaternionic operators,⁶

$$\mathcal{O}_\mathbb{R}(\psi_1 r_1 + \psi_2 r_2) = (\mathcal{O}_\mathbb{R} \psi_1) r_1 + (\mathcal{O}_\mathbb{R} \psi_2) r_2, \quad r_{1,2} \in \mathbb{R}, \quad \psi_{1,2} \in V[n, \mathbb{H}],$$

$$\mathcal{O}_\mathbb{C}(\psi_1 c_1 + \psi_2 c_2) = (\mathcal{O}_\mathbb{C} \psi_1) c_1 + (\mathcal{O}_\mathbb{C} \psi_2) c_2, \quad c_{1,2} \in \mathbb{C}, \quad \psi_{1,2} \in V[n, \mathbb{H}],$$

can be represented by $n \times n$ quaternionic matrices $M_\mathbb{H}$ and right acting operators R_μ as follows:

$$\mathcal{O}_\mathbb{R} \leftrightarrow M_\mathbb{R} = \sum_{\mu=0}^3 M_{\mu, \mathbb{H}} R_\mu \quad \text{and} \quad \mathcal{O}_\mathbb{C} \leftrightarrow M_\mathbb{C} = \sum_{s=0}^1 M_{s, \mathbb{H}} R_s. \tag{6}$$

Thus, R-linear quaternionic operators consist of right multiplication by quaternionic numbers ($\mu = 0, 1, 2, 3$) whereas C-linear quaternionic operators are restricted to right multiplication by complex numbers ($s = 0, 1$).

III. THE EIGENVALUE PROBLEM

In this section, we briefly discuss the left and right eigenvalue equation for H-, C-, and R-linear quaternionic operators. As explicitly shown below, the conceptual difficulties which characterize the left eigenvalue problem readily disappear by resorting to right eigenvalues. The need to apply similarity transformations on H-, C-, and R-linear quaternionic matrices introduces *complex* or *real* constraints on the right eigenvalues. The choice of complex or real (right) eigenvalues will be extremely useful in finding a practical method of resolution and manipulating quaternionic matrices.

A. Left eigenvalues

The left eigenvalue problem for H-linear quaternionic operators reads

$$\mathcal{O}_H \psi = q \psi, \quad \psi \in V[n, \mathbb{H}], \quad q \in \mathbb{H}. \tag{7}$$

This problem has been recently studied in the mathematical literature.^{7,8} Nevertheless, no systematic way to approach the problem has been given. We point out some difficulties which appear in solving the left eigenvalue equation.

1. Similarity transformations

In finding the solution of Eq. (7), a first difficulty is represented by the impossibility to apply similarity transformations, $S_H \in M[n, \mathbb{H}]$, without losing the formal structure of the left eigenvalue equation. In fact, by observing that $S_H q \neq q S_H$, the quaternionic matrices

$$M_H \quad \text{and} \quad S_H M_H S_H^{-1} \tag{8}$$

do *not* necessarily satisfy the same eigenvalue equation. Consequently, we can have quaternionic matrices with the same left eigenvalue spectrum, but *no* similarity transformation relating them. Explicit examples are found in Ref. 1.

2. Hermitian operators

Let \mathcal{O}_H be a Hermitian quaternionic operator and ψ be the eigenvector corresponding to the eigenvalue q . By using Eq. (7) and denoting by $\langle \varphi | \psi \rangle$ the inner product in $V[n, \mathbb{H}]$, we obtain

$$0 = \langle \mathcal{O}_H \psi | \psi \rangle - \langle \psi | \mathcal{O}_H \psi \rangle \Rightarrow 0 = \langle q \psi | \psi \rangle - \langle \psi | q \psi \rangle \neq (\bar{q} - q) \langle \psi | \psi \rangle.$$

Consequently, the left eigenvalue problem for Hermitian operators could admit *quaternionic* solutions.⁹

3. Square operators and eigenvalues

As a last difficulty in the use of left eigenvalues, we observe that if ψ is an \mathcal{O}_H eigenvector with eigenvalue q , it will not necessarily be an \mathcal{O}_H^2 eigenvector with eigenvalue q^2 . In fact,

$$\mathcal{O}_H^2 \psi = \mathcal{O}_H q \psi \neq q \mathcal{O}_H \psi = q^2 \psi.$$

B. Right eigenvalues

The right eigenvalue equation for H-linear transformations reads

$$\mathcal{O}_H \psi = \psi q, \quad \psi \in V[n, \mathbb{H}], \quad q \in \mathbb{H}. \tag{9}$$

Such an equation can be reduced to a right complex eigenvalue equation rephrasing the quaternionic eigenvalues by unitary quaternions u ,

$$\mathcal{O}_H \psi u = \psi u \bar{u} q u = \psi u z, \quad z \in \mathbb{C}. \tag{10}$$

This trick obviously fails for complex and real linear transformations. In fact, due to the presence of the operators R_i in \mathcal{O}_C and $\mathbf{R} \equiv (R_i, R_j, R_k)$ in \mathcal{O}_R , we cannot apply unitary transformations from the right. Observe that

$$(\mathcal{O}_{R,C} \psi) u \neq \mathcal{O}_{R,C}(\psi u), \quad u \in \mathbb{H}. \tag{11}$$

The failure of the *associativity* in Eq. (11) suggests that we should consider *complex* eigenvalue equations for \mathbb{C} -linear quaternionic operators,

$$\mathcal{O}_C \psi = \psi z, \quad z \in \mathbb{C}, \tag{12}$$

and *real* eigenvalues for \mathbb{R} -linear quaternionic operators,

$$\mathcal{O}_R \psi = \psi r, \quad r \in \mathbb{R}. \tag{13}$$

These equations are formally invariant under \mathbb{C} - and \mathbb{R} -linear similarity transformations. Moreover, it can easily be proved that

$$\mathcal{O}_H^n \psi = \psi q^n, \quad \mathcal{O}_C^n \psi = \psi z^n, \quad \mathcal{O}_R^n \psi = \psi r^n.$$

It is important to note here that \mathbb{R} -linear quaternionic operators admit real eigenvalues only in particular cases. Thus, Eq. (13) has to be generalized. As shown later, a satisfactory discussion of the eigenvalue problem for \mathbb{R} -linear quaternionic operators will require the use of a system of *coupled* equations.

IV. CANONICAL FORMS

In this section, following the procedure introduced in the paper of Ref. 1, we discuss the canonical forms for \mathbb{H} -, \mathbb{C} -, and \mathbb{R} -linear quaternionic matrices. The results we will establish find an immediate application in the theory of quaternionic differential operators. In fact, by using the canonical form J_X of a given matrix M_X we can readily obtain the exponential

$$\exp[M_X x] = S_X \exp[J_X x] S_X^{-1}$$

and, consequently, avoiding tedious calculations, solve quaternionic differential equations with constant coefficients.²

A. \mathbb{H} -linear matrices

While matrices over commutative rings have gained much attention, the literature on matrices with quaternionic entries is often fragmentary. The main difficulty is that, due to the noncommutative nature of quaternions, the standard method of resolution breaks down. Consequently, finding eigenvalues and canonical forms represents a more delicate problem. The recent renewed interest in quaternionic matrix theory^{1,10} and its applications² shed new light on this intriguing research field. To facilitate access to the individual topics, we recall the main properties of \mathbb{H} -linear quaternionic matrices¹¹⁻¹⁴ and repeat the relevant theorems from Refs. 1, 10, 15, and 16 without proofs, thus making our exposition self-contained.

In approaching the problem of diagonalization we have to consider a right quaternionic eigenvalue equation. In fact, from

$$M_H \psi_k = \psi_k q_k, \quad k = 1, \dots, n, \quad \psi_k \in V[n, \mathbb{H}], \tag{14}$$

in the case $M_{\mathbb{H}}$ is diagonalizable, we immediately get the following matrix equation

$$M_{\mathbb{H}}S_{\mathbb{H}}[\psi_1, \psi_2, \dots, \psi_n] = S_{\mathbb{H}}[\psi_1, \psi_2, \dots, \psi_n]D_{\mathbb{H}},$$

where

$$D_{\mathbb{H}} = \text{diag}[q_1, q_2, \dots, q_n]$$

and $S_{\mathbb{H}} = S_{\mathbb{H}}[\psi_1, \psi_2, \dots, \psi_n]$ is defined by $\text{Col}_k(S_{\mathbb{H}}) = \psi_k$.

Consequently, the diagonalization of the matrix $M_{\mathbb{H}}$,

$$M_{\mathbb{H}} = S_{\mathbb{H}}D_{\mathbb{H}}S_{\mathbb{H}}^{-1},$$

is obtained by solving the corresponding right eigenvalue problem. It is important to note here that we have *infinite* ways to diagonalize a quaternionic matrix $M_{\mathbb{H}}$,

$$\text{diag}[u_1, u_2, \dots, u_n] \text{diag}[q_1, q_2, \dots, q_n] \text{diag}[\bar{u}_1, \bar{u}_2, \dots, \bar{u}_n].$$

Geometrically speaking this means that $\text{Im}[q]$ can arbitrarily be fixed on the sphere of ray $|\text{Im}[q]|$. By a particular choice of the unitary matrix

$$U_{\mathbb{H}} = \text{diag}[u_1, u_2, \dots, u_n]$$

we can set a preferred space direction, for example the positive i axis, and consequently a *complex* (positive) eigenvalue spectrum.

Let us now briefly recall some properties of the eigenvalue spectrum of \mathbb{H} -linear quaternionic matrices. By using the symplectic decomposition of the matrix $M_{\mathbb{H}}$,

$$M_{\mathbb{H}} = M_1 + jM_2, \quad M_{1,2} \in M[n, \mathbb{C}],$$

and the symplectic decomposition of the vector ψ ,

$$\psi = \psi_1 + j\psi_2, \quad \psi_{1,2} \in V[n, \mathbb{C}],$$

we can rewrite Eq. (10) in the following (complex) form:

$$\tilde{M}_{\mathbb{H}}\tilde{\psi} = z\tilde{\psi}, \tag{15}$$

where

$$\tilde{M}_{\mathbb{H}} = \begin{pmatrix} M_1 & -M_2^* \\ M_2 & M_1^* \end{pmatrix} \in M[2n, \mathbb{C}] \quad \text{and} \quad \tilde{\psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in V[2n, \mathbb{C}]. \tag{16}$$

The mapping

$$f: M_{\mathbb{H}} \mapsto \tilde{M}_{\mathbb{H}} \tag{17}$$

is an isomorphism of the ring of quaternionic matrices $M_{\mathbb{H}}$ into the ring of the corresponding complex counterparts $\tilde{M}_{\mathbb{H}}$. It is important to observe that this isomorphism do not preserve the inner product of eigenvectors.⁹ Nevertheless, the choice of a *complex projection* of quaternionic inner products¹⁷ opens the door to interesting applications in relativistic quantum mechanics.¹⁸ The *complex* orthogonality of quaternionic eigenvectors (instead of a quaternionic orthogonality) implies a doubling of solution in the two-dimensional quaternionic Dirac equation.¹⁷ The *four* (complex) orthogonal quaternionic solutions describe particle/antiparticle with spin up/down. The use of *complex* inner products is also a fundamental ingredient in the formulation of gauge theories by geometric algebras.⁵

The next theorem states the main property of the eigenvalue spectrum of \mathbb{H} -linear quaternionic matrices. For a detailed discussion, see Refs. 14–16.

Theorem 1: *Let $\tilde{M}_{\mathbb{H}}$ be the matrix given in Eq. (16). Then, its eigenvalues appear in complex conjugate pairs.*

By using the result of Theorem 1 and the Gram–Schmidt method, we can readily obtain the triangular form for \mathbb{H} -linear quaternionic matrices.

Theorem 2: *Every $M_{\mathbb{H}}$ is unitarily similar to an upper triangular matrix.*

Moreover, a Jordan form can be given for every \mathbb{H} -linear quaternionic matrix.

Theorem 3: *Every $n \times n$ matrix with real quaternion elements is similar under a matrix transformation with real quaternion elements to a matrix in (complex) Jordan normal form with diagonal elements in the complex field.*

To prove Theorem 3, we can use the isomorphism defined in (17). To any $\tilde{M}_{\mathbb{H}}$ corresponds a $2n \times 2n$ matrix B in the (complex) Jordan form $B = B_{\mathbb{H}} \oplus B_{\mathbb{H}}^*$, such that $\tilde{M}_{\mathbb{H}}P = PB$, where the (nonsingular) matrix P has the form

$$P = \begin{pmatrix} P_1 & -P_2^* \\ P_2 & P_1^* \end{pmatrix}.$$

This implies that

$$(P_1 + jP_2)^{-1} M_{\mathbb{H}}(P_1 + jP_2) = B_{\mathbb{H}} = D_{\mathbb{H}} + N_{\mathbb{H}}, \tag{18}$$

where $D_{\mathbb{H}}$ and $N_{\mathbb{H}}$ respectively denote the diagonal and the nilpotent parts of $B_{\mathbb{H}}$.

B. \mathbb{C} -linear matrices

Let us now consider \mathbb{C} -linear transformations. We can associate to any $n \times n$ \mathbb{C} -linear quaternionic matrix a $2n$ -dimensional complex matrix by the following mapping.

$$M_{\mathbb{C}} = M_{\mathbb{H}} + M'_{\mathbb{H}}R_i \leftrightarrow \tilde{M}_{\mathbb{C}} = f(M_{\mathbb{H}}) + if(M'_{\mathbb{H}}) = \begin{pmatrix} M_1 & -M_2^* \\ M_2 & M_1^* \end{pmatrix} + i \begin{pmatrix} M'_1 & -M'_2{}^* \\ M'_2 & M'_1{}^* \end{pmatrix}, \tag{19}$$

where f denotes the isomorphism defined in (17). Then the following proposition holds:

Proposition 1: Let $M_{\mathbb{C}}$ be a \mathbb{C} -linear quaternionic matrix and $\tilde{M}_{\mathbb{C}}$ its complex counterpart [see Eq. (19)]. The mapping

$$g: M_{\mathbb{C}} \leftrightarrow \tilde{M}_{\mathbb{C}} \tag{20}$$

is an isomorphism of the ring of the n -dimensional \mathbb{C} -linear matrices into the ring of $2n$ -dimensional complex matrices.

Indeed, if

$$A_{\mathbb{C}} = A_{0,\mathbb{H}} + A_{1,\mathbb{H}}R_i \quad \text{and} \quad B_{\mathbb{C}} = B_{0,\mathbb{H}} + B_{1,\mathbb{H}}R_i$$

are two \mathbb{C} -linear matrices, their corresponding complex counterparts are given by

$$g(A_{\mathbb{C}}) = f(A_{0,\mathbb{H}}) + if(A_{1,\mathbb{H}}), \quad g(B_{\mathbb{C}}) = f(B_{0,\mathbb{H}}) + if(B_{1,\mathbb{H}}).$$

Then,

$$\begin{aligned} A_{\mathbb{C}}B_{\mathbb{C}} &= C_{\mathbb{C}} = A_{0,\mathbb{H}}B_{0,\mathbb{H}} + A_{1,\mathbb{H}}R_iB_{0,\mathbb{H}} + A_{0,\mathbb{H}}B_{1,\mathbb{H}}R_i + A_{1,\mathbb{H}}R_iB_{1,\mathbb{H}}R_i \\ &= A_{0,\mathbb{H}}B_{0,\mathbb{H}} - A_{1,\mathbb{H}}B_{1,\mathbb{H}} + (A_{0,\mathbb{H}}B_{1,\mathbb{H}} + A_{1,\mathbb{H}}B_{0,\mathbb{H}})R_i \end{aligned}$$

and

$$\begin{aligned}
 g(A_C)g(B_C) &= f(A_{0,H})f(B_{0,H}) - f(A_{1,H})f(B_{1,H}) + if(A_{1,H})f(B_{0,H}) + if(A_{0,H})f(B_{1,H}) \\
 &= f(A_{0,H}B_{0,H} - A_{1,H}B_{1,H}) + if(A_{0,H}B_{1,H} + A_{1,H}B_{0,H}) = g(C_C).
 \end{aligned}$$

□

By using this isomorphism, the right eigenvalue spectrum of C-linear quaternionic matrices can easily be determined.¹ The following result,

“A C-linear matrix is diagonalizable if the corresponding complex counterpart is diagonalizable,”

was proven in Ref. 1 (where a preliminary discussion of the eigenvalue problem for C-linear quaternionic matrix operators was given). It is worth pointing out that the converse of the previous statement is, in general, not true. For instance, let us consider the complex matrix

$$\tilde{G}_C = \begin{pmatrix} z_1 & 0 & 1 & 0 \\ 0 & z_2 & 0 & 1 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & z_2 \end{pmatrix}. \tag{21}$$

This matrix admits a corresponding diagonalizable C-linear quaternionic matrix given by

$$G_C = D_C + N_C = \begin{pmatrix} \text{Re}(z_1) + \text{Im}(z_1)R_i & 0 \\ 0 & \text{Re}(z_2) + \text{Im}(z_2)R_i \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -j + kR_i & 0 \\ 0 & -j + kR_i \end{pmatrix}, \tag{22}$$

where N_C is nilpotent, diagonal, and commutes with D_C .

The normal form of a C-linear quaternionic matrix can easily be calculated. Indeed, given any C-linear transformation M_C and its corresponding complex counterpart \tilde{M}_C [see Eq. (19)], from the known properties of the Jordan form of complex matrices, we can immediately obtain

$$\tilde{S}_C^{-1} \tilde{M}_C \tilde{S}_C = \tilde{J}_C = \tilde{D}_C + \tilde{N}_C,$$

where \tilde{D}_C is diagonal, \tilde{N}_C is nilpotent, and $[\tilde{D}_C, \tilde{N}_C] = 0$. Then, the quaternionic C-linear matrices M_C, S_C, D_C , and N_C are uniquely determined by the isomorphism stated in Proposition 1.

C. R-linear matrices

In the n -dimensional quaternionic vector space $V[n, H]$, the R-linear transformations are represented by

$$M_R = \sum_{\mu=0}^3 M_{\mu,H} R_\mu, \tag{23}$$

where $M_{\mu,H}$ represent H-linear quaternionic matrices and R_μ are the right acting operators defined

in the second section. Any M_R is then characterized by $16n^2$ real parameters. We can translate \mathbb{R} -linear $n \times n$ matrices into equivalent $4n \times 4n$ real matrices, and vice versa, by the following translation rules:

$$\begin{aligned}
 R_i \leftrightarrow \mathbf{I} &= \begin{pmatrix} 0 & -\mathbf{1}_n & 0 & 0 \\ \mathbf{1}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_n \\ 0 & 0 & -\mathbf{1}_n & 0 \end{pmatrix}, \\
 R_j \leftrightarrow \mathbf{J} &= \begin{pmatrix} 0 & 0 & -\mathbf{1}_n & 0 \\ 0 & 0 & 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 & 0 & 0 \\ 0 & \mathbf{1}_n & 0 & 0 \end{pmatrix}, \\
 R_k \leftrightarrow \mathbf{K} &= \begin{pmatrix} 0 & 0 & 0 & -\mathbf{1}_n \\ 0 & 0 & \mathbf{1}_n & 0 \\ 0 & -\mathbf{1}_n & 0 & 0 \\ \mathbf{1}_n & 0 & 0 & 0 \end{pmatrix},
 \end{aligned} \tag{24}$$

and

$$M_{\mu, \mathbb{H}} = M_0 + iM_1 + jM_2 + kM_3 \leftrightarrow \hat{M}_{\mu, \mathbb{H}} = \begin{pmatrix} M_0 & -M_1 & -M_2 & -M_3 \\ M_1 & M_0 & -M_3 & M_2 \\ M_2 & M_3 & M_0 & -M_1 \\ M_3 & -M_2 & M_1 & M_0 \end{pmatrix}, \tag{25}$$

where $M_{\mu, \mathbb{H}} \in M[n, \mathbb{H}]$, $M_0, \dots, M_3 \in M[n, \mathbb{R}]$, and $\hat{M}_{\mu, \mathbb{H}} \in M[4n, \mathbb{R}]$. It is easy to verify that $\mathbf{I}, \mathbf{J}, \mathbf{K}$ commute with $\hat{M}_{\mu, \mathbb{H}}$:

$$[\mathbf{I}, \hat{M}_{\mu, \mathbb{H}}] = [\mathbf{J}, \hat{M}_{\mu, \mathbb{H}}] = [\mathbf{K}, \hat{M}_{\mu, \mathbb{H}}] = 0$$

and

$$\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = \mathbf{KJI} = -\mathbf{1}.$$

The following proposition holds.

Proposition 2: Let M_R be a \mathbb{R} -linear matrix and \hat{M}_R its real counterpart. Then the mapping

$$h: M_R = M_{0, \mathbb{H}} + M_{1, \mathbb{H}}R_i + M_{2, \mathbb{H}}R_j + M_{3, \mathbb{H}}R_k \mapsto \hat{M}_R = \hat{M}_{0, \mathbb{H}} + \hat{M}_{1, \mathbb{H}}\mathbf{I} + \hat{M}_{2, \mathbb{H}}\mathbf{J} + \hat{M}_{3, \mathbb{H}}\mathbf{K}$$

is an isomorphism of the ring of the n -dimensional \mathbb{R} -linear matrices into the ring of $4n$ -dimensional real matrices \hat{M}_R .

Observe that $\widehat{M_{\mathbb{H}}M'_{\mathbb{H}}} = \hat{M}_{\mathbb{H}}\hat{M}'_{\mathbb{H}}$. Let

$$A_R = A_{0, \mathbb{H}} + A_{1, \mathbb{H}}R_i + A_{2, \mathbb{H}}R_j + A_{3, \mathbb{H}}R_k \quad \text{and} \quad B_R = B_{0, \mathbb{H}} + B_{1, \mathbb{H}}R_i + B_{2, \mathbb{H}}R_j + B_{3, \mathbb{H}}R_k$$

be two \mathbb{R} -linear quaternionic matrices. Their corresponding real counterparts are given by

$$h(A_R) = \hat{A}_{0, \mathbb{H}} + \mathbf{I}\hat{A}_{1, \mathbb{H}} + \mathbf{J}\hat{A}_{2, \mathbb{H}} + \mathbf{K}\hat{A}_{3, \mathbb{H}} \quad \text{and} \quad h(B_R) = \hat{B}_{0, \mathbb{H}} + \mathbf{I}\hat{B}_{1, \mathbb{H}} + \mathbf{J}\hat{B}_{2, \mathbb{H}} + \mathbf{K}\hat{B}_{3, \mathbb{H}}.$$

In particular, the diagonal elements of $D_{\mathbb{R}}$ corresponding to the quaternionic translation of the real blocks

$$\begin{aligned} \hat{D}_{1,\mathbb{R}} &= \begin{pmatrix} \lambda_m & -\mu_m & 0 & 0 \\ \mu_m & \lambda_m & 0 & 0 \\ 0 & 0 & \lambda_{m+1} & -\mu_{m+1} \\ 0 & 0 & \mu_{m+1} & \lambda_{m+1} \end{pmatrix}, \\ \hat{D}_{2,\mathbb{R}} &= \begin{pmatrix} \lambda_m & -\mu_m & 0 & 0 \\ \mu_m & \lambda_m & 0 & 0 \\ 0 & 0 & \lambda_{m+1} & 0 \\ 0 & 0 & 0 & \lambda_{m+2} \end{pmatrix}, \\ \hat{D}_{3,\mathbb{R}} &= \begin{pmatrix} \lambda_m & 0 & 0 & 0 \\ 0 & \lambda_{m+1} & 0 & 0 \\ 0 & 0 & \lambda_{m+2} & 0 \\ 0 & 0 & 0 & \lambda_{m+3} \end{pmatrix}, \end{aligned} \tag{28}$$

are respectively given by

$$\begin{aligned} D_{1,\mathbb{R}} &= \frac{1}{2}[\lambda_m(1-L_iR_i) + \mu_m(L_i+R_i) + \lambda_{m+1}(1+L_iR_i) + \mu_{m+1}(L_i-R_i)], \\ D_{2,\mathbb{R}} &= \frac{1}{2}[\lambda_m(1-L_iR_i) + \mu_m(L_i+R_i) + \frac{1}{2}\lambda_{m+1}(1+L_iR_i-L_jR_j+L_kR_k) \\ &\quad + \frac{1}{2}\lambda_{m+2}(1+L_iR_i+L_jR_j-L_kR_k)], \\ D_{3,\mathbb{R}} &= \frac{1}{4}[\lambda_m(1-L_iR_i-L_jR_j-L_kR_k) + \lambda_{m+1}(1-L_iR_i+L_jR_j+L_kR_k) \\ &\quad + \lambda_{m+2}(1+L_iR_i-L_jR_j+L_kR_k) + \lambda_{m+3}(1+L_iR_i+L_jR_j-L_kR_k)]. \end{aligned} \tag{29}$$

As happens for \mathbb{C} -linear quaternionic matrices, an \mathbb{R} -linear quaternionic matrix is diagonalizable if the corresponding real counterpart is diagonalizable. The converse is not necessarily true.

V. THE EIGENVALUE PROBLEM FOR \mathbb{R} -LINEAR MATRICES

Let us now consider the eigenvalue problem for \mathbb{R} -linear quaternionic matrices. Equation (13) is obviously too restrictive. In fact, such an equation sets the real eigenvalue spectrum of \mathbb{R} -linear quaternionic operators. No information is given about the remaining eigenvalues. In particular, if the real counterpart $\hat{M}_{\mathbb{R}}$ of the \mathbb{R} -linear quaternionic matrix $M_{\mathbb{R}}$ does not have real eigenvalues, Eq. (13) does not admit solution. This is very embarrassing if we consider, for example, \mathbb{R} -linear anti-Hermitian quaternionic operators. Thus, we need to modify Eq. (13). The discussion regarding the ‘‘pseudo-triangular’’ form of the matrices $\hat{A}_{\mathbb{R}}$ [see Eq. (26)] suggests as a \mathbb{R} -linear eigenvalue problem the following system of *coupled* equations:

$$\begin{aligned} M_{\mathbb{R}}\psi &= a\psi + b\varphi, \\ M_{\mathbb{R}}\varphi &= c\varphi + d\psi, \end{aligned} \tag{30}$$

where

$$\begin{aligned} a, b, c, d \in \mathbb{R}, \quad \psi &= \psi_0 + i\psi_1 + j\psi_2 + k\psi_3, \quad \varphi = \varphi_0 + i\varphi_1 + j\varphi_2 + k\varphi_3, \\ \psi_0, \dots, \psi_3, \varphi_0, \dots, \varphi_3 &\in V[n, \mathbb{R}]. \end{aligned}$$

It can be shown that the real coefficients a, b, c, d are related to the real and imaginary parts of the \hat{M}_R eigenvalues. In fact, by translating the system (30) into its real matrix counterpart, we find

$$\begin{pmatrix} \hat{M}_R & 0 \\ 0 & \hat{M}_R \end{pmatrix} \begin{pmatrix} \hat{\psi} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} a\mathbf{1}_{4n} & b\mathbf{1}_{4n} \\ c\mathbf{1}_{4n} & d\mathbf{1}_{4n} \end{pmatrix} \begin{pmatrix} \hat{\psi} \\ \hat{\phi} \end{pmatrix}, \tag{31}$$

where

$$\hat{\psi} = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad \hat{\phi} = \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \in V[4n, \mathbb{R}].$$

The matrix equation (31) admits nontrivial solutions if and only if

$$\det \left[\begin{pmatrix} \hat{M}_R - a\mathbf{1}_{4n} & -b\mathbf{1}_{4n} \\ -c\mathbf{1}_{4n} & \hat{M}_R - d\mathbf{1}_{4n} \end{pmatrix} \right] = 0. \tag{32}$$

By rewriting the matrix \hat{M}_R in terms of the similarity matrix \hat{S}_R and of its Jordan form \hat{J}_R , i.e.,

$$\hat{M}_R = \hat{S}_R \hat{J}_R \hat{S}_R^{-1},$$

and by using the cyclic property of the determinant, we reduce Eq. (32) to

$$\det \left[\begin{pmatrix} \hat{J}_R - a\mathbf{1}_{4n} & -b\mathbf{1}_{4n} \\ -c\mathbf{1}_{4n} & \hat{J}_R - d\mathbf{1}_{4n} \end{pmatrix} \right] = 0. \tag{33}$$

By simple algebraic manipulations,²⁰ we obtain

$$\begin{aligned} \det \left[\begin{pmatrix} \hat{J}_R - a\mathbf{1}_{4n} & -b\mathbf{1}_{4n} \\ -c\mathbf{1}_{4n} & \hat{J}_R - d\mathbf{1}_{4n} \end{pmatrix} \right] &= \det[(\hat{J}_R - a\mathbf{1}_{4n})(\hat{J}_R - d\mathbf{1}_{4n}) - bc\mathbf{1}_{4n}] \\ &= \det[\hat{J}_R^2 - (a+d)\hat{J}_R + (ad-bc)\mathbf{1}_{4n}] \\ &= \prod_i [z_i^2 - (a+d)z_i + (ad-bc)] = 0, \end{aligned}$$

where z_i represent the eigenvalues of the real matrix \hat{M}_R . The previous equation explicitly shows the relation between the real coefficients a, b, c, d (which appear in the \mathbb{R} -linear eigenvalue problem) and the eigenvalues z of the real counterpart of the quaternionic matrix M_R . In the case of complex eigenvalues z , we find

$$\Delta = (a+d) - 4(ad-bc) = (a-d)^2 + bc < 0. \tag{34}$$

This condition guarantees that the eigenvalues of the real matrix

$$Z = \begin{pmatrix} a\mathbf{1}_{4n} & b\mathbf{1}_{4n} \\ c\mathbf{1}_{4n} & d\mathbf{1}_{4n} \end{pmatrix}$$

appear in conjugate pairs. Consequently, we can find a real similarity transformation T such that

$$TZZT^{-1} = \begin{pmatrix} \lambda \mathbf{1}_{4n} & -\mu \mathbf{1}_{4n} \\ \mu \mathbf{1}_{4n} & \lambda \mathbf{1}_{4n} \end{pmatrix}.$$

Finally, without loss of generality, we can consider the following eigenvalue problem for \mathbb{R} -linear transformations

$$\begin{aligned} M_{\mathbb{R}}\psi &= \lambda\psi - \mu\varphi, \\ M_{\mathbb{R}}\varphi &= \lambda\varphi + \mu\psi. \end{aligned} \quad (35)$$

VI. FINAL REMARKS

These final remarks aim to give a concluding discussion on the “coupled” eigenvalue problem and a brief summary of mathematical and physical applications motivating our interest in this research. In particular, we are interested to bring together two areas: quaternionic differential operators and quantum mechanics.

A. Coupled eigenvalue equations

In the previous section, we have introduced, for \mathbb{R} -linear transformations, the eigenvalue problem (35) which represents the *natural* generalization of (13). In particular, as we observed above, the study of system (35) instead of Eq. (13) allows us to take into account the existence of complex eigenvalues and, consequently, complete the eigenvalue spectrum of \mathbb{R} -linear quaternionic operators. Actually, the eigenvalue problem (35) also applies to \mathbb{H} - and \mathbb{C} -linear transformations. It can be considered as an equivalent formulation of Eqs. (9) and (12). To show that, let us consider the equation

$$M_{\mathbb{C}}\psi = \psi z = \psi\lambda + \psi i \mu. \quad (36)$$

We limit ourselves to discussing \mathbb{C} -linear transformations. Obviously, if a preferred complex direction is chosen for the eigenvalues of \mathbb{H} -linear quaternionic operators, all the arguments in what follows also hold for \mathbb{H} -linear transformations. By using the \mathbb{C} -linearity, we find

$$(M_{\mathbb{C}}\psi)i = M_{\mathbb{C}}(\psi i) = \psi i \lambda - \psi \mu.$$

The pair of eigenvector $(\psi, \varphi = \psi i)$, where ψ is solution of Eq. (36), satisfies the coupled equations

$$\begin{aligned} M_{\mathbb{C}}\psi &= \lambda\psi - \mu\varphi, \\ M_{\mathbb{C}}\varphi &= \mu\psi + \lambda\varphi. \end{aligned} \quad (37)$$

Vice versa, let $M_{\mathbb{C}}$ and (37) be respectively a \mathbb{C} -linear transformation and the corresponding eigenvalue problem. We denote by (ψ, φ) a solution of the system (37). If ψ satisfies Eq. (36), too, then, comparing Eq. (36) and the first equation in (37), one immediately obtains $\varphi = \psi i$. If, on the contrary, ψ is not a solution of Eq. (36), by using the \mathbb{C} -linearity, we obtain

$$M_{\mathbb{C}}(\varphi i) = (M_{\mathbb{C}}\varphi)i = \mu\psi i + \varphi i.$$

Thus,

$$M_{\mathbb{C}}(\psi + \varphi i) = (\psi + \varphi i)z. \quad (38)$$

Hence, it is possible to associate to any solution (ψ, φ) of the system (37) a corresponding eigenvector of $M_{\mathbb{C}}$.

It is worth pointing out that the coupled system (35) can be obtained by solving the eigenvalue problem (13) for *complexified* quaternionic eigenvectors and *complexified* real eigenvalues. In fact, by imposing that

$$\psi \rightarrow \Psi = \psi + I\varphi \in \mathbb{H}(1, i, j, k) \otimes \mathbb{C}(1, I),$$

and

$$r \rightarrow Z = \lambda + I\mu \in \mathbb{C}(1, I),$$

from the *complexified* eigenvalue problem (13),

$$M_{\mathbb{R}}\Psi = \Psi Z,$$

we immediately get the coupled system (35).

B. Applications

Many physical problems dealing with differential operators are greatly simplified by using the matrix formalism and solving the corresponding eigenvalue problem.

Let us first consider a very simple case, that is, the \mathbb{H} -linear second order homogeneous ordinary differential equation

$$\ddot{\psi}(x) - \alpha\dot{\psi}(x) - \beta\psi(x) = 0, \quad \alpha, \beta \in \mathbb{H}, \quad x \in \mathbb{R}. \tag{39}$$

In looking for quaternionic exponential solution $\psi(x) = \exp[qx]$ and observing that the derivative of $\exp[qx]$ with respect to the real variable x is $q \exp[qx]$, we reduce the previous problem to find the solutions of the following quadratic equation

$$q^2 = \alpha q + \beta. \tag{40}$$

This equation can be rewritten in matrix form as follows:

$$M_{\mathbb{H}} \begin{pmatrix} q \\ 1 \end{pmatrix} = \begin{pmatrix} q^2 \\ q \end{pmatrix}, \quad M_{\mathbb{H}} = \begin{pmatrix} \alpha & \beta \\ 1 & 0 \end{pmatrix}. \tag{41}$$

As seen in this article, the \mathbb{H} -linear quaternionic matrix $M_{\mathbb{H}}$ satisfies a right (complex) eigenvalue equation

$$M_{\mathbb{H}} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v \\ w \end{pmatrix} z, \quad z \in \mathbb{C}, \quad v, w \in \mathbb{H}, \quad w\bar{w} = 1. \tag{42}$$

Due to the particular form of $M_{\mathbb{H}}$, the components of the $M_{\mathbb{H}}$ -eigenvectors satisfy the following condition

$$v = wz. \tag{43}$$

Multiplying (from the right) Eq. (42) by \bar{w} and using the constraint (43), we obtain

$$M_{\mathbb{H}} \begin{pmatrix} wz\bar{w} \\ 1 \end{pmatrix} = \begin{pmatrix} wz^2\bar{w} \\ wz\bar{w} \end{pmatrix}. \tag{44}$$

Comparing Eq. (41) with Eq. (44), we immediately get

$$q = wz\bar{w}. \tag{45}$$

The problem of finding exponential solutions for a \mathbb{H} -linear differential with constant coefficients and, consequently, zeros of \mathbb{H} -linear polynomial equations,²¹ is thus equivalent to solving the right (complex) eigenvalue problem for the associated matrix. Obviously, the previous considerations also hold for the n -dimensional case.

The solutions of \mathbb{X} -linear quaternionic differential equations with constant coefficients

$$\psi^{(n)}(x) - A_{n-1,\mathbb{X}}\psi^{(n-1)}(x) - A_{n-2,\mathbb{X}}\psi^{(n-2)}(x) - \dots - A_{0,\mathbb{X}}\psi(x) = 0, \quad \mathbb{X} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \quad (46)$$

can be given in terms of the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} A_{n-1,\mathbb{X}} & A_{n-2,\mathbb{X}} & \dots & A_{0,\mathbb{X}} \\ 1 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Interesting \mathbb{C} -linear differential equations appear in quaternionic quantum mechanics.³ For example, by studying quaternionic tunneling effects as a candidate to possible phenomenological deviations from the standard (complex) theory, we have to solve the following \mathbb{H} -linear Schrödinger equation,

$$\partial_t \Psi(x,t) = \left[\frac{i}{\hbar} \left(\frac{\hbar^2}{2m} \partial_{xx} - iV \right) + \frac{j}{\hbar} W \right] \Psi(x,t), \quad (47)$$

where $(j/\hbar)W$ represents the *new* quaternionic perturbation. The quaternionic stationary state wave function

$$\Psi(x,t) = \psi(x) \exp \left[-\frac{i}{\hbar} Et \right] \quad (48)$$

is a solution of Eq. (47) on the condition that $\psi(x)$ be a solution of the following time-independent \mathbb{C} -linear (ordinary) differential equation:

$$i \frac{\hbar^2}{2m} \ddot{\psi}(x) - iV\psi(x) + jW\psi(x) + \psi(x)iE = 0. \quad (49)$$

Observe that the choice of the imaginary unit i in the Laplacian operator ∂_{xx} , Eq. (47), and in the time exponential, Eq. (48), is fundamental to recover the standard results in the complex limit. In this formalism, quaternionic potentials are treated as perturbation effects on standard quantum mechanics. We also point out that the right position of the time exponential is fundamental to perform the separation of variables.

The solution of Eq. (49) can be given in terms of the eigenvalues and eigenvectors of the \mathbb{C} -linear matrix

$$\begin{pmatrix} 0 & A_{0,\mathbb{C}} \\ 1 & 0 \end{pmatrix},$$

where

$$A_{0,\mathbb{C}} = \frac{2m}{\hbar^2} (V + L_k W + L_i R_i E).$$

A detailed phenomenological discussion of the quaternionic tunneling effect is found in the paper of Ref. 4.

C. Outlooks

As seen in this article, the choice of right (complex) eigenvalues for \mathbb{H} - and \mathbb{C} -linear operators plays a fundamental role in discussing canonical forms and in finding solutions of polynomial and differential equations. It was shown that the right (complex) eigenvalue problem is equivalent to a “coupled” system and this was extremely important to study the eigenvalue problem for \mathbb{R} -linear quaternionic matrices, where a pair of real eigenvalues must be introduced. This work was intended as an attempt at motivating the study of \mathbb{R} - and \mathbb{C} -linear quaternionic operators in view of possible applications in quantum mechanics and gauge theory. It would be desirable to give a complete theory of \mathbb{X} -linear quaternionic matrices and differential operators. More realistically, this article touches only a few aspects of the theory and shows how the choice of the right eigenvalue equation seems to be the best to investigate quaternionic formulations of physical theory. It was not our purpose to study here differential operators. The results in this field are far from being conclusive and some questions represent at present intriguing challenges: variations of parameters; order reduction; not invertible higher derivative \mathbb{R} - and \mathbb{C} -linear constant coefficients; variable coefficients; and integral transforms. Finally, it would be desirable to extend the discussion on the eigenvalue problem by matrix translation to the nonassociative case.^{22–24}

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