J. Phys. A: Math. Gen. **33** (2000) 2971–2995. Printed in the UK PII: S0305-4470(00)07677-0

Right eigenvalue equation in quaternionic quantum mechanics

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Received 10 September 1999

Abstract. We study the right eigenvalue equation for quaternionic and complex linear matrix operators defined in n-dimensional quaternionic vector spaces. For quaternionic linear operators the eigenvalue spectrum consists of n complex values. For these operators we give a necessary and sufficient condition for the diagonalization of their quaternionic matrix representations. Our discussion is also extended to complex linear operators, whose spectrum is characterized by $2n$ complex eigenvalues. We show that a consistent analysis of the eigenvalue problem for complex linear operators requires the choice of a complex geometry in defining inner products. Finally, we introduce some examples of the left eigenvalue equations and highlight the main difficulties in their solution.

1. Introduction

Over the last decade, after the fundamental works of Finkelstein *et al* [1–3] on foundations of quaternionic quantum mechanics (qQM) and gauge theories, we have witnessed a renewed interest in algebrization and geometrization of physical theories by non-commutative fields [4, 5]. Among the numerous references on this subject, we recall the important paper of Horwitz and Biedenharn [6], where the authors showed that the assumption of a complex projection of the scalar product, also called *complex geometry* [7], permits the definition of a suitable tensor product [8] between single-particle quaternionic wavefunctions. We also mention quaternionic applications in special relativity [9], group representations [10–13], non-relativistic [14, 15] and relativistic dynamics [16, 17], field theory [18], Lagrangian formalism [19], electroweak model [20], grand unification theories [21] and the preonic model [22]. A clear and detailed discussion of qQM together possible topics for future developments in field theory and particle physics is found in the recent book by Adler [23].

In writing this paper, the main objective has been to address the lack of clarity among mathematical physicists on the proper choice of quaternionic eigenvalue equation within a qQM with *complex* or *quaternionic geometry*. In the past, interesting papers have addressed the mathematical discussion of the quaternionic eigenvalue equation, and related topics. For example, we find in the literature works on quaternionic eigenvalues and the characteristic equation [24, 25], diagonalization of matrices [26], the Jordan form and q-determinant [27, 28]. More recently, some of these problems have been also discussed for the octonionic field [29, 30].

Our approach aims to give a practical method to solve the quaternionic right eigenvalue equation in view of increasing interest in quaternionic [5, 23, 31] and octonionic [4, 32–36] applications in physics. Given quaternionic and complex linear operators on n-dimensional quaternionic vector spaces, we explicitly formulate practical rules to obtain the eigenvalues and the corresponding eigenvectors for their n-dimensional quaternionic matrix representations. In discussing the right complex eigenvalue problem in qQM, we find two obstacles. The first one is related to the difficulty in obtaining a suitable definition of the determinant for quaternionic matrices, the second one is represented by the loss, for non-commutative fields, of the fundamental theorem of the algebra. The lack of these tools, essentials in solving the eigenvalue problem in the complex world, makes the problem over the quaternionic field a complicated puzzle. We overcome the difficulties in approaching the eigenvalue problem in a quaternionic world by discussing the eigenvalue equation for $2n$-dimensional complex matrices obtained by translation from n-dimensional quaternionic matrix operators. We shall show that quaternionic linear operators, defined on quaternionic Hilbert space with quaternionic geometry, are diagonalizable if and only if the corresponding complex operators are diagonalizable. The spectral theorems, extended to quaternionic Hilbert spaces [1], are recovered in a more general context. We also study the linear independence on $\mathbb{H}$ of quaternionic eigenvectors by studying their (complex) eigenvalues and discuss the spectrum choice for quaternionic quantum systems. Finally, we construct the Hermitian operator associated with any anti-Hermitian matrix operator and show that a coherent discussion of the eigenvalue problem for complex linear operators requires a complex geometry. A brief discussion concerning the possibility of having left eigenvalue equations is also proposed and some examples presented. We point out, see also [29, 36], that left eigenvalues of Hermitian quaternionic matrices need not be real.

This paper is organized as follows. In section 2, we introduce the basic notation and mathematical tools. In particular, we discuss similarity transformations, symplectic decompositions and the left/right action of quaternionic imaginary units. In section 3, we give the basic framework of qQM and translation rules between complex and quaternionic matrices. In section 4, we approach the right eigenvalue problem by discussing the eigenvalue spectrum for $2n$-dimensional complex matrices obtained by translating n-dimensional quaternionic matrix representations. We give a practical method for diagonalizing n-dimensional quaternionic linear matrix operators and overcoming previous problems in the spectrum choice for quaternionic quantum systems. We also discuss the right eigenvalue problem for complex linear operators within a qQM with complex geometry. In section 5, we introduce the left eigenvalue equation and analyse the eigenvalue spectrum for Hermitian operators. We explicitly solve some examples of left/right eigenvalue equations for two-dimensional quaternionic and complex linear operators. Our conclusions and outlooks are drawn in the final section.

2. Basic notation and mathematical tools

A quaternion, $q \in \mathbb{H}$, is expressed by four real quantities [37, 38]

$$q = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d \qquad a, b, c, d \in \mathbb{R} \tag{1}$$

and three imaginary units

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1.$$

The quaternion skew-field $\mathbb{H}$ is an associative but non-commutative algebra of rank four over $\mathbb{R}$, endowed with an involutory antiautomorphism

$$q \to \overline{q} = a - \mathrm{i}b - \mathrm{j}c - \mathrm{k}d. \tag{2}$$

This conjugation implies a reversed-order product, namely

$$\overline{pq} = \overline{q}\,\overline{p} \qquad p, q \in \mathbb{H}.$$

Every non-zero quaternion is invertible, and the unique inverse is given by $1/q = \overline{q}/|q|^2$, where the quaternionic norm $|q|$ is defined by

$$|q|^2 = q\overline{q} = a^2 + b^2 + c^2 + d^2.$$

Similarity transformation

Two quaternions q and p belong to the same eigenclass when the following relation:

$$q = s^{-1}ps \qquad s \in \mathbb{H}$$

is satisfied. Quaternions of the same eigenclass have the same real part and the same norm,

$$\mathrm{Re}(q) = \mathrm{Re}\left(s^{-1}ps\right) = \mathrm{Re}(p) \qquad |q| = |s^{-1}ps| = |p|$$

consequently, they have the same absolute value of the imaginary part. The previous equations can be rewritten in terms of unitary quaternions as follows:

$$q = s^{-1}ps = \frac{\overline{s}}{|s|}p\frac{s}{|s|} = \overline{u}pu \qquad u \in \mathbb{H} \quad \overline{u}u = 1. \tag{3}$$

In equation (3), the unitary quaternion,

$$u = \cos\tfrac{1}{2}\theta + \vec{h}\cdot\vec{u}\sin\tfrac{1}{2}\theta \qquad \vec{h} \equiv (\mathrm{i}, \mathrm{j}, \mathrm{k}) \quad \vec{u} \in \mathbb{R}^3 \quad |\vec{u}| = 1$$

can be expressed in terms of the imaginary parts of q and p. In fact, given two quaternions belonging to the same eigenclass,

$$q = q_0 + \vec{h}\cdot\vec{q} \qquad p = p_0 + \vec{h}\cdot\vec{p} \qquad q_0 = p_0 \qquad |\vec{q}| = |\vec{p}|$$

we find [39], for $\vec{q} \neq \pm\vec{p}$,

$$\cos\theta = \frac{\vec{q}\cdot\vec{p}}{|\vec{q}||\vec{p}|} \qquad \text{and} \qquad \vec{u} = \frac{\vec{q}\times\vec{p}}{|\vec{q}||\vec{p}|\sin\theta}. \tag{4}$$

The remaining cases $\vec{q} = \vec{p}$ and $\vec{q} = -\vec{p}$ represent, respectively, the trivial similarity transformation, i.e. $uq\bar{u} = q$, and the similarity transformation between a quaternion q and its conjugate $\bar{q}$, i.e. $uq\bar{u} = \bar{q}$. In the first case the unitary quaternion is given by the identity quaternion. In the last case the similarity transformation is satisfied $\forall u = \vec{h}\cdot\vec{u}$ with $\vec{u}\cdot\vec{q} = 0$ and $|\vec{u}| = 1$.

Symplectic decomposition

Complex numbers can be constructed from real numbers by

$$z = \alpha + \mathrm{i}\beta \qquad \alpha, \beta \in \mathbb{R}.$$

In a similar way, we can construct quaternions from complex numbers by

$$q = z + \mathrm{j}w \qquad z, w \in \mathbb{C}$$

symplectic decomposition of quaternions.

Left/right action

Due to the non-commutative nature of quaternions we must distinguish between

$$q\vec{h} \qquad \text{and} \qquad \vec{h}q.$$

Thus, it is appropriate to consider left- and right-actions for our imaginary units i, j and k. Let us define the operators

$$\vec{L} = \left(L_{\mathrm{i}}, L_{\mathrm{j}}, L_{\mathrm{k}}\right) \tag{5}$$

and

$$\vec{R} = \left(R_{\mathrm{i}}, R_{\mathrm{j}}, R_{\mathrm{k}}\right) \tag{6}$$

which act on quaternionic states in the following way:

$$\vec{L} : \mathbb{H} \to \mathbb{H} \qquad \vec{L}q = \vec{h}q \in \mathbb{H} \tag{7}$$

and

$$\vec{R} : \mathbb{H} \to \mathbb{H} \qquad \vec{R}q = q\vec{h} \in \mathbb{H}. \tag{8}$$

The algebra of left/right generators can be expressed concisely by

$$L_{\mathrm{i}}^2 = L_{\mathrm{j}}^2 = L_{\mathrm{k}}^2 = L_{\mathrm{i}}L_{\mathrm{j}}L_{\mathrm{k}} = R_{\mathrm{i}}^2 = R_{\mathrm{j}}^2 = R_{\mathrm{k}}^2 = R_{\mathrm{k}}R_{\mathrm{j}}R_{\mathrm{i}} = -\mathbb{1}$$

and by the commutation relations

$$\left[L_{\mathrm{i,j,k}}, R_{\mathrm{i,j,k}}\right] = 0.$$

From these operators we can construct the following vector space:

$$\mathbb{H}_L \otimes \mathbb{H}_R$$

whose generic element will be characterized by left and right actions of quaternionic imaginary units i, j, k. In this paper we will work with two sub-spaces of $\mathbb{H}_L \otimes \mathbb{H}_R$, namely

$$\mathbb{H}^L \qquad \text{and} \qquad \mathbb{H}^L \otimes \mathbb{C}^R$$

whose elements are represented, respectively, by left actions of i, j, k

$$a + \vec{b} \cdot \vec{L} \in \mathbb{H}^L \qquad a, \vec{b} \in \mathbb{R} \tag{9}$$

and by left actions of i, j, k and right action of the only imaginary unit i

$$a + \vec{b} \cdot \vec{L} + cR_{\mathrm{i}} + \vec{d} \cdot \vec{L}R_{\mathrm{i}} \in \mathbb{H}^L \otimes \mathbb{C}^R \qquad a, \vec{b}, c, \vec{d} \in \mathbb{R}. \tag{10}$$

3. States and operators in qQM

The states of qQM will be described by vectors, $|\psi\rangle$, of a quaternionic Hilbert space, $V_{\mathbb{H}}$. First of all, due to the non-commutative nature of quaternionic multiplication, we must specify whether the quaternionic Hilbert space is to be formed by right or left multiplication of quaternionic vectors by scalars. The two different conventions give isomorphic versions of the theory [40]. We adopt the convention of right multiplication by scalars.

In quaternionic Hilbert spaces, we can define quaternionic and complex linear operators, which will be, respectively, denoted by $\mathcal{O}_{\mathbb{H}}$ and $\mathcal{O}_{\mathbb{C}}$. They will act on quaternionic vectors, $|\psi\rangle$, in the following way:

$$\mathcal{O}_{\mathbb{H}}(|\psi\rangle q) = (\mathcal{O}_{\mathbb{H}}|\psi\rangle)\, q \qquad q \in \mathbb{H}.$$

and

$$\mathcal{O}_{\mathbb{C}}(|\psi\rangle\lambda) = (\mathcal{O}_{\mathbb{C}}|\psi\rangle)\,\lambda \qquad \lambda \in \mathbb{C}.$$

Such operators are $\mathbb{R}$-linear from the left.

As a concrete illustration, let us consider the case of a finite n-dimensional quaternionic Hilbert space. The ket state $|\psi\rangle$ will be represented by a quaternionic n-dimensional column vector

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} = \begin{pmatrix} x_1 + \mathrm{j}y_1 \\ \vdots \\ x_n + \mathrm{j}y_n \end{pmatrix} \qquad x_1, y_1, \ldots, x_n, y_n \in \mathbb{C}. \tag{11}$$

Quaternionic linear operators, $\mathcal{O}_{\mathbb{H}}$, will be represented by $n \times n$ matrices with entries in $\mathbb{H}^L$, whereas complex linear operators, $\mathcal{O}_{\mathbb{C}}$, by $n \times n$ matrices with entries in $\mathbb{H}^L \otimes \mathbb{C}^R$.

By using the *symplectic* complex representation, the n-dimensional quaternionic vector

$$|\psi\rangle = |x\rangle + \mathrm{j}|y\rangle = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \mathrm{j} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

can be translated into the $2n$-dimensional complex column vector

$$|\psi\rangle \leftrightarrow \begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{pmatrix}. \tag{12}$$

The matrix representation of L_{i}, L_{j} and L_{k} consistent with the above identification is

$$\begin{aligned} L_{\mathrm{i}} &\leftrightarrow \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = \mathrm{i}\sigma_3 \\ L_{\mathrm{j}} &\leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathrm{i}\sigma_2 \\ L_{\mathrm{k}} &\leftrightarrow \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} = -\mathrm{i}\sigma_1. \end{aligned} \tag{13}$$

These translation rules allow us to represent quaternionic n-dimensional linear operators by $2n \times 2n$ complex matrices.

The right quaternionic imaginary unit

$$R_{\mathrm{i}} \leftrightarrow \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \tag{14}$$

adds four additional degrees of freedom,

$$R_{\mathrm{i}} \quad L_{\mathrm{i}}R_{\mathrm{i}} \quad L_{\mathrm{j}}R_{\mathrm{i}} \quad L_{\mathrm{k}}R_{\mathrm{i}}$$

and so, by observing that the Pauli matrices together with the identity matrix form a basis over $\mathbb{C}$, we have a set of rules which allows us to translate one-dimensional complex linear operators using 2×2 complex matrices [41]. Consequently, we can construct $2n \times 2n$ complex matrix representations for n-dimensional complex linear operators.

Let us note that the identification in equation (14) is consistent only for complex inner products (complex geometry) [41]. We observe that the right complex imaginary unit, R_{i}, does not have a well defined Hermiticity within a qQM with quaternionic inner product (quaternionic geometry),

$$\langle\varphi|\psi\rangle = |\varphi\rangle^{\dagger}|\psi\rangle = \sum_{l=1}^{n} \bar{\varphi}_l \psi_l. \tag{15}$$

In fact, anti-Hermitian operators must satisfy

$$\langle\varphi|\mathcal{A}\psi\rangle = -\langle\mathcal{A}\varphi|\psi\rangle.$$

For the right imaginary unit R_{i}, we have

$$|R_{\mathrm{i}}\psi\rangle \equiv R_{\mathrm{i}}|\psi\rangle = |\psi\rangle\mathrm{i} \qquad \langle R_{\mathrm{i}}\varphi| \equiv (R_{\mathrm{i}}|\varphi\rangle)^{\dagger} = -\mathrm{i}\langle\varphi|$$

and consequently

$$\langle\varphi|\psi\rangle\mathrm{i} = \langle\varphi|R_{\mathrm{i}}\psi\rangle \neq -\langle R_{\mathrm{i}}\varphi|\psi\rangle = \mathrm{i}\langle\varphi|\psi\rangle.$$

Nevertheless, by adopting a complex geometry, i.e. a complex projection of the quaternionic inner product,

$$\langle\varphi|\psi\rangle_{\mathbb{C}} = \tfrac{1}{2}(\langle\varphi|\psi\rangle - \mathrm{i}\langle\varphi|\psi\rangle\mathrm{i}) \tag{16}$$

we recover the anti-Hermiticity of the operator R_{i},

$$\langle\varphi|R_{\mathrm{i}}\psi\rangle_{\mathbb{C}} = -\langle R_{\mathrm{i}}\varphi|\psi\rangle_{\mathbb{C}}.$$

4. The right complex eigenvalue problem in qQM

The right eigenvalue equation for a generic quaternionic linear operator, $O_{\mathbb{H}}$, is written as

$$O_{\mathbb{H}}|\Psi\rangle = |\Psi\rangle q \tag{17}$$

where $|\Psi\rangle \in V_{\mathbb{H}}$ and $q \in \mathbb{H}$. By adopting quaternionic scalar products in our quaternionic Hilbert spaces, $V_{\mathbb{H}}$, we find states in one-to-one correspondence with unit rays of the form

$$|\boldsymbol{r}\rangle = \{|\Psi\rangle u\} \tag{18}$$

where $|\Psi\rangle$ is a normalized vector and u is a quaternionic phase of unity magnitude. The state vector, $|\Psi\rangle u$, corresponding to the same physical state $|\Psi\rangle$, is an $O_{\mathbb{H}}$-eigenvector with eigenvalue $\overline{u}qu$

$$O_{\mathbb{H}}|\Psi\rangle u = |\Psi\rangle u(\overline{u}qu).$$

For real values of q, we find only one eigenvalue, otherwise quaternionic linear operators will be characterized by an infinite eigenvalue spectrum

$$\{q, \overline{u}_1 q u_1, \ldots, \overline{u}_l q u_l, \ldots\}$$

with u_l unitary quaternions. The related set of eigenvectors

$$\big\{|\Psi\rangle, |\Psi\rangle u_1, \ldots, |\Psi\rangle u_l, \ldots\big\}$$

represents a ray. We can characterize our spectrum by choosing a representative ray

$$|\psi\rangle = |\Psi\rangle u_\lambda$$

so that the corresponding eigenvalue $\lambda = \bar{u}_\lambda q u_\lambda$ is complex. For this state the right eigenvalue equation becomes

$$O_{\mathbb{H}}|\psi\rangle = |\psi\rangle\lambda \tag{19}$$

with $|\psi\rangle \in V_{\mathbb{H}}$ and $\lambda \in \mathbb{C}$.

We now give a systematic method to determine the complex eigenvalues of quaternionic matrix representations for $O_{\mathbb{H}}$ operators.

4.1. Quaternionic linear operators and quaternionic geometry

In n-dimensional quaternionic vector spaces, $\mathbb{H}^n$, quaternionic linear operators, $O_{\mathbb{H}}$, are represented by $n \times n$ quaternionic matrices, $\mathcal{M}_n(\mathbb{H}^L)$, with elements in $\mathbb{H}^L$. Such quaternionic matrices admit $2n$-dimensional complex counterparts by the translation rules given in equation (13). Such complex matrices characterize a subset of the $2n$-dimensional complex matrices

$$\widetilde{M}_{2n}(\mathbb{C}) \subset M_{2n}(\mathbb{C}).$$

The eigenvalue equation for $O_{\mathbb{H}}$ reads

$$\mathcal{M}_{\mathbb{H}}|\psi\rangle = |\psi\rangle\lambda \tag{20}$$

where $\mathcal{M}_{\mathbb{H}} \in \mathcal{M}_n(\mathbb{H}^{\mathbb{L}})$, $|\psi\rangle \in \mathbb{H}^n$ and $\lambda \in \mathbb{C}$.

The one-dimensional eigenvalue problem

In order to introduce the reader to our general method of quaternionic matrix diagonalization, let us discuss one-dimensional right complex eigenvalue equations. In this case equation (20) becomes

$$Q_{\mathbb{H}}|\psi\rangle = |\psi\rangle\lambda \tag{21}$$

where $Q_{\mathbb{H}} = a + \vec{b} \cdot \vec{L} \in \mathbb{H}^L$, $|\psi\rangle = |x\rangle + \mathrm{j}|y\rangle \in \mathbb{H}$ and $\lambda \in \mathbb{C}$. By using the translation rules, given in section 3, we can generate the quaternionic algebra from the commutative complex algebra (Cayley–Dickson process). The complex counterpart of equation (21) reads

$$\begin{pmatrix} z & -w^* \\ w & z^* \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \qquad z = a + \mathrm{i}b_1 \qquad w = b_2 - \mathrm{i}b_3 \in \mathbb{C}. \tag{22}$$

Equation (22) is the eigenvalue equation for a complex matrix whose characteristic equation has real coefficients. For this reason, the translated complex operator admits λ and λ^* as eigenvalues.

Given the eigenvector corresponding to the eigenvalue λ, we can immediately obtain the eigenvector associated with the eigenvalue λ^* by taking the complex conjugate of equation (22) and then applying a similarity transformation by the matrix

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In this way, we find

$$\begin{pmatrix} z & -w^* \\ w & z^* \end{pmatrix} \begin{pmatrix} -y^* \\ x^* \end{pmatrix} = \lambda^* \begin{pmatrix} -y^* \\ x^* \end{pmatrix}. \tag{23}$$

So, for $\lambda \neq \lambda^* \in \mathbb{C}$, we obtain the eigenvalue spectrum $\{\lambda, \lambda^*\}$ with eigenvectors

$$\begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} -y^* \\ x^* \end{pmatrix}. \tag{24}$$

What happens when $\lambda \in \mathbb{R}$? In this case the eigenvalue spectrum will be determined by two equal eigenvalues λ. To show that, we remark that the eigenvectors (24) associated with the same eigenvalues λ, are linearly independent on $\mathbb{C}$. In fact,

$$\left\| \begin{matrix} x & -y^* \\ y & x^* \end{matrix} \right\| = |x|^2 + |y|^2 = 0 \qquad \text{if and only if} \quad x = y = 0.$$

So in the quaternionic world, by translation, we find two complex eigenvalues, respectively, λ and λ^*, associated with the following quaternionic eigenvectors:

$$|\psi\rangle \qquad \text{and} \qquad |\psi\rangle j \in |\boldsymbol{r}\rangle.$$

The infinite quaternionic eigenvalue spectrum can be characterized by the complex eigenvalue λ and the ray representative will be $|\psi\rangle$. In the next section, by using the same method, we will discuss eigenvalue equations in n-dimensional quaternionic vector spaces.

The n-dimensional eigenvalue problem

Let us formulate two theorems which generalize the previous results for quaternionic n-dimensional eigenvalue problems. The first theorem [T1] analyses the eigenvalue spectrum of the $2n$-dimensional complex matrix $\widetilde{M}$, the complex counterpart of the n-dimensional quaternionic matrix $\mathcal{M}_{\mathbb{H}}$. In this theorem we give the matrix S which allows us to construct the complex eigenvector $|\phi_{\lambda^*}\rangle$ from the eigenvector $|\phi_\lambda\rangle$. The explicit construction of $|\phi_{\lambda^*}\rangle$ enables us to show the linear independence on $\mathbb{C}$ of $|\phi_\lambda\rangle$ and $|\phi_{\lambda^*}\rangle$ when $\lambda \in \mathbb{R}$ and represents the main tool in constructing similarity transformations for diagonalizable quaternionic matrices. The second theorem [T2] discusses linear independence on $\mathbb{H}$ for $\mathcal{M}_{\mathbb{H}}$ eigenvectors.

Theorem T1. *Let $\widetilde{M}$ be the complex counterpart of a generic $n \times n$ quaternionic matrix $\mathcal{M}_{\mathbb{H}}$. Its eigenvalues appear in conjugate pairs.*

Proof. Let

$$\widetilde{M}|\phi_\lambda\rangle = \lambda|\phi_\lambda\rangle \tag{25}$$

be the eigenvalue equation for $\widetilde{M}$, where

$$\widetilde{M} \in M_{2n}(\mathbb{C}) \qquad |\phi_\lambda\rangle = \begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{pmatrix} \in \mathbb{C}^{2n} \qquad \lambda \in \mathbb{C}.$$

By taking the complex conjugate of equation (25),

$$\widetilde{M}^*|\phi_\lambda\rangle^* = \lambda^*|\phi_\lambda\rangle^*$$

and applying a similarity transformation by the matrix

$$S = \mathbb{1}_n \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we obtain

$$S\widetilde{M}^*S^{-1}S|\phi_\lambda\rangle^* = \lambda^*S|\phi_\lambda\rangle^*. \tag{26}$$

From the block structure of the complex matrix $\widetilde{M}$ it is easily checked that

$$S\widetilde{M}^*S^{-1} = \widetilde{M}$$

and consequently equation (26) reads

$$\widetilde{M}|\phi_{\lambda^*}\rangle = \lambda^*|\phi_{\lambda^*}\rangle \tag{27}$$

where

$$|\phi_{\lambda^*}\rangle = S|\phi_\lambda\rangle^* = \begin{pmatrix} -y_1^* \\ x_1^* \\ \vdots \\ -y_n^* \\ x_n^* \end{pmatrix}.$$

Let us show that the eigenvalues appear in conjugate pairs (this implies a double multiplicity for real eigenvalues). To do this, we need to prove that $|\phi_\lambda\rangle$ and $|\phi_{\lambda^*}\rangle$ are linearly independent on $\mathbb{C}$. In order to demonstrate the linear independence of such eigenvectors, trivial for $\lambda \neq \lambda^*$, we observe that the linear dependence, possible in the case $\lambda = \lambda^*$, should require

$$\left\| \begin{matrix} x_i & -y_i^* \\ y_i & x_i^* \end{matrix} \right\| = |x_i|^2 + |y_i|^2 = 0 \qquad i = 1, \ldots, n$$

verified only for null eigenvectors. The linear independence of $|\phi_\lambda\rangle$ and $|\phi_{\lambda^*}\rangle$ ensures an even multiplicity for real eigenvalues. □

We shall use the results of the first theorem to obtain information about the $\mathcal{M}_{\mathbb{H}}$ right complex eigenvalue spectrum. Due to the non-commutative nature of the quaternionic field we cannot give a suitable definition of the determinant for quaternionic matrices and consequently we cannot write a characteristic polynomial $P(\lambda)$ for $\mathcal{M}_{\mathbb{H}}$. Another difficulty is represented by the right position of the complex eigenvalue λ.

Theorem T2. *$\mathcal{M}_{\mathbb{H}}$ admits n linearly independent eigenvectors on $\mathbb{H}$ if and only if its complex counterpart $\widetilde{M}$ admits $2n$ linearly independent eigenvectors on $\mathbb{C}$.*

Proof. Let

$$\left\{|\phi_{\lambda_1}\rangle, |\phi_{\lambda_1^*}\rangle, \ldots, |\phi_{\lambda_n}\rangle, |\phi_{\lambda_n^*}\rangle\right\} \tag{28}$$

be a set of $2n$ $\widetilde{M}$-eigenvectors, linearly independent on $\mathbb{C}$, and α_l, β_l $(l = 1, \dots, n)$ be generic complex coefficients. By definition,

$$\sum_{l=1}^{n}\big(\alpha_l|\phi_{\lambda_l}\rangle + \beta_l|\phi_{\lambda_l^*}\rangle\big) = 0 \quad \Leftrightarrow \quad \alpha_l = \beta_l = 0. \tag{29}$$

By translating the complex eigenvector set (28) in the quaternionic formalism we find

$$\big\{|\psi_{\lambda_1}\rangle, |\psi_{\lambda_1^*}\rangle, \dots, |\psi_{\lambda_n}\rangle, |\psi_{\lambda_n^*}\rangle\big\}. \tag{30}$$

By eliminating the eigenvectors, $|\psi_{\lambda_l^*}\rangle = |\psi_{\lambda_l}\rangle\mathrm{j}$, corresponding for complex eigenvalues to ones with a negative imaginary part, linearly dependent with $|\psi_{\lambda_l}\rangle$ on $\mathbb{H}$, we obtain

$$\big\{|\psi_{\lambda_1}\rangle, \dots, |\psi_{\lambda_n}\rangle\big\}.$$

This set is formed by n linearly independent vectors on $\mathbb{H}$. In fact, by taking an arbitrary quaternionic linear combination of such vectors, we have

$$\sum_{l=1}^{n}\big[|\psi_{\lambda_l}\rangle(\alpha_l + \mathrm{j}\beta_l)\big] = \sum_{l=1}^{n}\big(|\psi_{\lambda_l}\rangle\alpha_l + |\psi_{\lambda_l^*}\rangle\beta_l\big) = 0 \quad \Leftrightarrow \quad \alpha_l = \beta_l = 0. \tag{31}$$

Note that equation (31) represents the quaternionic counterpart of equation (29). □

The $\mathcal{M}_{\mathbb{H}}$ complex eigenvalue spectrum is thus obtained by taking from the $2n$-dimensional $\widetilde{M}$-eigenvalues spectrum

$$\{\lambda_1, \lambda_1^*, \dots, \lambda_n, \lambda_n^*\}$$

the reduced n-dimensional spectrum

$$\{\lambda_1, \dots, \lambda_n\}.$$

We stress here the fact that, the choice of positive, rather than negative, imaginary part is a simple convention. In fact, from the quaternionic eigenvector set (30), we can extract different sets of quaternionic linearly independent eigenvectors

$$\big\{\big[|\psi_{\lambda_1}\rangle \text{ or } |\psi_{\lambda_1^*}\rangle\big], \dots, \big[|\psi_{\lambda_n}\rangle \text{ or } |\psi_{\lambda_n^*}\rangle\big]\big\}$$

and consequently we have a free choice in characterizing the n-dimensional $\mathcal{M}_{\mathbb{H}}$-eigenvalue spectrum. A direct consequence of the previous theorems, is the following corollary.

Corollary T2. *Two $\mathcal{M}_{\mathbb{H}}$ quaternionic eigenvectors with complex eigenvalues, λ_1 and λ_2, with $\lambda_2 \neq \lambda_1 \neq \lambda_2^*$, are linearly independent on $\mathbb{H}$.*

Proof. Let

$$|\psi_{\lambda_1}\rangle(\alpha_1 + \mathrm{j}\beta_1) + |\psi_{\lambda_2}\rangle(\alpha_2 + \mathrm{j}\beta_2) \tag{32}$$

be a quaternionic linear combination of such eigenvectors. By taking the complex translation of equation (32), we obtain

$$\alpha_1|\phi_{\lambda_1}\rangle + \beta_1|\phi_{\lambda_1^*}\rangle + \alpha_2|\phi_{\lambda_2}\rangle + \beta_2|\phi_{\lambda_2^*}\rangle. \tag{33}$$

The set of $\widetilde{M}$-eigenvectors

$$\big\{|\phi_{\lambda_1}\rangle, |\phi_{\lambda_1^*}\rangle, |\phi_{\lambda_2}\rangle, |\phi_{\lambda_2^*}\rangle\big\}$$

is linearly independent on $\mathbb{C}$. In fact, theorem T1 ensures linear independence between eigenvectors associated with conjugate pairs of eigenvalues, and the condition $\lambda_2 \neq \lambda_1 \neq \lambda_2^*$ completes the proof by ensuring the linear independence between

$$\left\{|\phi_{\lambda_1}\rangle, |\phi_{\lambda_1^*}\rangle\right\} \qquad \text{and} \qquad \left\{|\phi_{\lambda_2}\rangle, |\phi_{\lambda_2^*}\rangle\right\}.$$

Thus the linear combination in equation (33), the complex counterpart of equation (32), is null if and only if $\alpha_{1,2} = \beta_{1,2} = 0$, and consequently the quaternionic linear eigenvectors $|\psi_{\lambda_1}\rangle$ and $|\psi_{\lambda_2}\rangle$ are linear independent on $\mathbb{H}$. $\square$

A brief discussion about the choice of spectrum

What happens to the eigenvalue spectrum when we have two simultaneous diagonalizable quaternionic linear operators? We show that for complex operators the choice of a common quaternionic eigenvector set reproduces in qQM the standard results of complex quantum mechanics (cQM). Let

$$\mathcal{A}_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} E \qquad \text{and} \qquad \mathcal{A}_2 = \tfrac{1}{2}\hbar \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \tag{34}$$

be anti-Hermitian complex operators associated, respectively, with energy and spin. In cQM, the corresponding eigenvalue spectrum is

$$\{\mathrm{i}E, \mathrm{i}E\}_{\mathcal{A}_1} \qquad \text{and} \qquad \left\{\mathrm{i}\tfrac{1}{2}\hbar, -\mathrm{i}\tfrac{1}{2}\hbar\right\}_{\mathcal{A}_2} \tag{35}$$

and physically we can describe a particle with positive energy E and spin $\frac{1}{2}$. What happens in qQM with quaternionic geometry? The complex operators in equation (34) also represent two-dimensional quaternionic linear operators and so we can translate them in the complex world and then extract the eigenvalue spectrum. By following the method given in this section, we find the following eigenvalues:

$$\{\mathrm{i}E, -\mathrm{i}E, \mathrm{i}E, -\mathrm{i}E\}_{\mathcal{A}_1} \qquad \text{and} \qquad \left\{\mathrm{i}\tfrac{1}{2}\hbar, -\mathrm{i}\tfrac{1}{2}\hbar, \mathrm{i}\tfrac{1}{2}\hbar, -\mathrm{i}\tfrac{1}{2}\hbar\right\}_{\mathcal{A}_2}$$

and adopting the positive imaginary part convention we extract

$$\{\mathrm{i}E, \mathrm{i}E\}_{\mathcal{A}_1} \qquad \text{and} \qquad \left\{\mathrm{i}\tfrac{1}{2}\hbar, \mathrm{i}\tfrac{1}{2}\hbar\right\}_{\mathcal{A}_2}.$$

It seems that we lose the physical meaning of a spin-$\frac{1}{2}$ particle with positive energy. How can we recover the different sign in the spin eigenvalues? The solution to this apparent puzzle is represented by the choice of a *common* quaternionic eigenvector set. In fact, we observe that the previous eigenvalue spectra are related to the following eigenvector sets:

$$\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}_{\mathcal{A}_1} \qquad \text{and} \qquad \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ j \end{pmatrix}\right\}_{\mathcal{A}_2}.$$

By fixing a common set of eigenvectors

$$\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}_{\mathcal{A}_{1,2}} \tag{36}$$

we recover the standard results of equation (35). Obviously,

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}_{\mathcal{A}_{1,2}} \quad \to \quad \left\{+\mathrm{i}E, +\mathrm{i}E\right\}_{\mathcal{A}_1} \qquad \left\{+\mathrm{i}\tfrac{1}{2}\hbar, -\mathrm{i}\tfrac{1}{2}\hbar\right\}_{\mathcal{A}_2}$$

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ j \end{pmatrix} \right\}_{\mathcal{A}_{1,2}} \quad \to \quad \left\{+\mathrm{i}E, -\mathrm{i}E\right\}_{\mathcal{A}_1} \qquad \left\{+\mathrm{i}\tfrac{1}{2}\hbar, +\mathrm{i}\tfrac{1}{2}\hbar\right\}_{\mathcal{A}_2}$$

$$\left\{ \begin{pmatrix} j \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}_{\mathcal{A}_{1,2}} \quad \to \quad \left\{-\mathrm{i}E, +\mathrm{i}E\right\}_{\mathcal{A}_1} \qquad \left\{-\mathrm{i}\tfrac{1}{2}\hbar, -\mathrm{i}\tfrac{1}{2}\hbar\right\}_{\mathcal{A}_2}$$

$$\left\{ \begin{pmatrix} j \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ j \end{pmatrix} \right\}_{\mathcal{A}_{1,2}} \quad \to \quad \left\{-\mathrm{i}E, -\mathrm{i}E\right\}_{\mathcal{A}_1} \qquad \left\{-\mathrm{i}\tfrac{1}{2}\hbar, +\mathrm{i}\tfrac{1}{2}\hbar\right\}_{\mathcal{A}_2}$$

represent equivalent choices. Thus, in this particular case, the different possibilities in choosing our quaternionic eigenvector set will give the following outputs.

$$\begin{array}{llcll}
\text{Energy:} & +E, +E & \text{and} & \tfrac{1}{2}\text{-spin:} & \uparrow, \downarrow \\
 & +E, -E & \text{and} & & \uparrow, \uparrow \\
 & -E, +E & \text{and} & & \downarrow, \downarrow \\
 & -E, -E & \text{and} & & \downarrow, \uparrow.
\end{array}$$

Thus, we can also describe a $\frac{1}{2}$-spin particle with positive energy by re-interpreting spin-up/down negative energy as spin down/up positive energy solutions

$$-E, \uparrow (\downarrow) \quad \to \quad E, \downarrow (\uparrow).$$

From an anti-Hermitian to a Hermitian matrix operator

Let us remark on an important difference between the structure of an anti-Hermitian operator in complex and in quaternionic quantum mechanics. In cQM, we can always trivially relate an anti-Hermitian operator, $\mathcal{A}$ to a Hermitian operator, $\mathcal{H}$, by removing a factor i

$$\mathcal{A} = \mathrm{i}\mathcal{H}.$$

In qQM, we must take care. For example,

$$\mathcal{A} = \begin{pmatrix} -\mathrm{i} & 3\mathrm{j} \\ 3\mathrm{j} & \mathrm{i} \end{pmatrix} \tag{37}$$

is an anti-Hermitian operator, nevertheless, i$\mathcal{A}$ does not represent a Hermitian operator. The reason is simple: given any independent (over $\mathbb{H}$) set of normalized eigenvectors $|v_l\rangle$ of $\mathcal{A}$ with complex imaginary eigenvalues λ_l,

$$\mathcal{A} = \sum_l |v_l\rangle |\lambda_l| \mathrm{i} \langle v_l|$$

the corresponding Hermitian operator $\mathcal{H}$ is soon obtained by

$$\mathcal{H} = \sum_l |v_l\rangle |\lambda_l| \langle v_l|$$

since both the factors are independent of the particular representative $|v_l\rangle$ chosen. Due to the non-commutative nature of $|v_l\rangle$, we cannot extract the complex imaginary unit i. Our approach to quaternionic eigenvalue equations contains a practical method to find eigenvectors $|v_l\rangle$ and eigenvalues λ_l and consequently solves the problem to determine, given a quaternionic anti-Hermitian operator, the corresponding Hermitian operator. An easy computation shows that

$$\{\mathrm{i}|\lambda_1|, \mathrm{i}|\lambda_2|\} = \{2\mathrm{i}, 4\mathrm{i}\} \qquad \text{and} \qquad \{|v_1\rangle, |v_2\rangle\} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{i} \\ \mathrm{j} \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{k} \\ 1 \end{pmatrix} \right\}.$$

So, the Hermitian operator corresponding to the anti-Hermitian operator of equation (37) is

$$\mathcal{H} = \begin{pmatrix} 3 & \mathrm{k} \\ -\mathrm{k} & 3 \end{pmatrix}. \tag{38}$$

A practical rule for diagonalization

We know that $2n$-dimensional complex operators, are diagonalizable if and only if they admit $2n$ linear independent eigenvectors. It is easy to demonstrate that the diagonalization matrix for $\widetilde{M}$

$$\widetilde{S}\widetilde{M}\widetilde{S}^{-1} = \widetilde{M}^{\text{diag}}$$

is given by

$$\widetilde{S} = \text{Inverse}\left[\begin{pmatrix} x_1^{(\lambda_1)} & x_1^{(\lambda_1^*)} & \dots & x_1^{(\lambda_n)} & x_1^{(\lambda_n^*)} \\ y_1^{(\lambda_1)} & y_1^{(\lambda_1^*)} & \dots & y_1^{(\lambda_n)} & y_1^{(\lambda_n^*)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^{(\lambda_1)} & x_n^{(\lambda_1^*)} & \dots & x_n^{(\lambda_n)} & x_n^{(\lambda_n^*)} \\ y_n^{(\lambda_1)} & y_n^{(\lambda_1^*)} & \dots & y_n^{(\lambda_n)} & y_n^{(\lambda_n^*)} \end{pmatrix} \right]. \tag{39}$$

Such a matrix is in the same subset of $\widetilde{M}$, i.e. $\widetilde{S} \in \widetilde{M}_{2n}(\mathbb{C})$. In fact, by recalling the relationship between $|\phi_\lambda\rangle$ and $|\phi_{\lambda^*}\rangle$, we can rewrite the previous diagonalization matrix as

$$\widetilde{S} = \text{Inverse}\left[\begin{pmatrix} x_1^{(\lambda_1)} & -y_1^{*\,(\lambda_1)} & \dots & x_1^{(\lambda_n)} & -y_1^{*\,(\lambda_n)} \\ y_1^{(\lambda_1)} & x_1^{*\,(\lambda_1)} & \dots & y_1^{(\lambda_n)} & x_1^{*\,(\lambda_n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^{(\lambda_1)} & -y_n^{*\,(\lambda_1)} & \dots & x_n^{(\lambda_n)} & -y_n^{*\,(\lambda_n)} \\ y_n^{(\lambda_1)} & x_n^{*\,(\lambda_1)} & \dots & y_n^{(\lambda_n)} & x_n^{*\,(\lambda_n)} \end{pmatrix} \right]. \tag{40}$$

The linear independence of the $2n$ complex eigenvectors of $\widetilde{M}$ guarantees the existence of $\widetilde{S}^{-1}$ and the isomorphism between the group of $n \times n$ invertible quaternionic matrices $\mathsf{GL}(n, \mathbb{H})$ and the complex counterpart group $\widetilde{\mathsf{GL}}(2n, \mathbb{C})$ ensures $\widetilde{S}^{-1} \in \widetilde{M}_{2n}(\mathbb{C})$. So, the quaternionic n-dimensional matrix which diagonalizes $\mathcal{M}_{\mathbb{H}}$

$$\mathcal{S}_{\mathbb{H}}\mathcal{M}_{\mathbb{H}}\mathcal{S}_{\mathbb{H}}^{-1} = \mathcal{M}_{\mathbb{H}}^{\text{diag}}$$

can be obtained directly by translating equation (40) in

$$\mathcal{S}_{\mathbb{H}} = \text{Inverse}\left[\begin{pmatrix} x_1^{(\lambda_1)} + \mathrm{j}y_1^{(\lambda_1)} & \dots & x_1^{(\lambda_n)} + \mathrm{j}y_1^{(\lambda_n)} \\ \vdots & \ddots & \vdots \\ x_n^{(\lambda_1)} + \mathrm{j}y_n^{(\lambda_1)} & \dots & x_n^{(\lambda_n)} + \mathrm{j}y_n^{(\lambda_n)} \end{pmatrix} \right]. \tag{41}$$

In translating complex matrices in quaternionic language, we remember that an appropriate mathematical notation should require the use of the left/right quaternionic operators $L_{\mathrm{i,j,k}}$ and R_{i}. In this case, due to the particular form of our complex matrices,

$$\widetilde{M}, \widetilde{S}, \widetilde{S}^{-1} \in \widetilde{M}_{2n}(\mathbb{C})$$

their quaternionic translation is performed by left operators and so we use the simplified notation i, j, k instead of L_{i}, L_{j}, L_{k}.

This diagonalization quaternionic matrix is strictly related to the choice of a particular set of quaternionic linear independent eigenvectors

$$\{|\psi_{\lambda_1}\rangle, \ldots, |\psi_{\lambda_n}\rangle\}.$$

So, the diagonalized quaternionic matrix reads

$$\mathcal{M}_{\mathbb{H}}^{\mathrm{diag}} = \mathrm{diag}\,\{\lambda_1, \ldots, \lambda_n\}.$$

The choice of a different quaternionic eigenvector set

$$\{[|\psi_{\lambda_1}\rangle \text{ or } |\psi_{\lambda_1^*}\rangle], \ldots, [|\psi_{\lambda_n}\rangle \text{ or } |\psi_{\lambda_n^*}\rangle]\}$$

will give, for not real eigenvalues, a different diagonalization matrix and consequently a different diagonalized quaternionic matrix

$$\mathcal{M}_{\mathbb{H}}^{\mathrm{diag}} = \mathrm{diag}\,\{[\lambda_1 \text{ or } \lambda_1^*], \ldots, [\lambda_n \text{ or } \lambda_n^*]\}.$$

In conclusion,

$$\mathcal{M}_{\mathbb{H}} \text{ diagonalizable} \quad \Leftrightarrow \quad \widetilde{M} \text{ diagonalizable}$$

and the diagonalization quaternionic matrix can be easily obtained from the quaternionic eigenvector set.

4.2. Complex linear operators and complex geometry

In this section, we discuss the right eigenvalue equation for complex linear operators. In n-dimensional quaternionic vector spaces, $\mathbb{H}^n$, the complex linear operator, $O_{\mathbb{C}}$, is represented by $n \times n$ quaternionic matrices, $\mathcal{M}_n(\mathbb{H}^L \otimes \mathbb{C}^R)$, with elements in $\mathbb{H}^L \otimes \mathbb{C}^R$. Such quaternionic matrices admit $2n$-dimensional complex counterparts which recover the *full* set of $2n$-dimensional complex matrices, $M_{2n}(\mathbb{C})$. It is immediate to check that quaternionic matrices $\mathcal{M}_{\mathbb{H}} \in \mathcal{M}_n(\mathbb{H}^L)$ are characterized by $4n^2$ real parameters and so a natural translation gives the complex matrix $\widetilde{M}_{2n}(\mathbb{C}) \subset M_{2n}(\mathbb{C})$, whereas a generic $2n$-dimensional complex matrix $M \in M_{2n}(\mathbb{C})$, characterized by $8n^2$ real parameters needs to double the $4n^2$ real parameters of $\mathcal{M}_{\mathbb{H}}$. By allowing right action for the imaginary units i we recover the missing real parameters. So, the $2n$-dimensional complex eigenvalue equation

$$M|\phi\rangle = \lambda|\phi\rangle \qquad M \in M_{2n}(\mathbb{C}) \quad |\phi\rangle \in \mathbb{C}^{2n} \quad \lambda \in \mathbb{C} \tag{42}$$

becomes, in the quaternionic formalism,

$$\mathcal{M}_{\mathbb{C}}|\psi\rangle = |\psi\rangle\lambda \qquad \mathcal{M}_{\mathbb{C}} \in \mathcal{M}_n(\mathbb{H}^{\mathbb{L}} \otimes \mathbb{C}^R) \quad |\psi\rangle \in \mathbb{H}^n \quad \lambda \in \mathbb{C}. \tag{43}$$

The right position of the complex eigenvalue λ is justified by the translation rule

$$\mathrm{i}\,\mathbb{1}_{2n} \quad \leftrightarrow \quad R_{\mathrm{i}}\,\mathbb{1}_n.$$

By solving the complex eigenvalue problem of equation (42), we find $2n$ eigenvalues and we have no possibilities to classify or characterize such a complex eigenvalue spectrum. Is it possible to extract a suitable quaternionic eigenvectors set? What happens when the complex spectrum is characterized by $2n$ *different* complex eigenvalues? To give satisfactory answers to these questions we must adopt a complex geometry [6, 7]. In this case

$$|\psi\rangle \qquad \text{and} \qquad |\psi\rangle \mathrm{j}$$

represent orthogonal vectors and so we cannot kill the eigenvectors $|\psi\rangle\mathrm{j}$. So, for n-dimensional quaternionic matrices $\mathcal{M}_{\mathbb{C}}$ we must consider the *full* eigenvalue spectrum

$$\{\lambda_1, \ldots, \lambda_{2n}\}. \tag{44}$$

The corresponding quaternionic eigenvector set is then given by

$$\{|\psi_{\lambda_1}\rangle, \ldots, |\psi_{\lambda_{2n}}\rangle\} \tag{45}$$

which represents the quaternionic translation of the M-eigenvector set

$$\{|\phi_{\lambda_1}\rangle, \ldots, |\phi_{\lambda_{2n}}\rangle\}. \tag{46}$$

In conclusion, within a qQM with complex geometry [31] we find for quaternionic linear operators, $M_{\mathbb{H}}$, and complex linear operators, $M_{\mathbb{C}}$, a $2n$-dimensional complex eigenvalue spectrum and consequently $2n$ quaternionic eigenvectors. Let us now give a practical method to diagonalize complex linear operators. Complex $2n$-dimensional matrices, M, are diagonalizable if and only if admit $2n$ linear independent eigenvectors. The diagonalizable matrix can be written in terms of M-eigenvectors as follows:

$$S = \text{Inverse}\left[\begin{pmatrix} x_1^{(\lambda_1)} & x_1^{(\lambda_2)} & \ldots & x_1^{(\lambda_{2n-1})} & x_1^{(\lambda_{2n})} \\ y_1^{(\lambda_1)} & y_1^{(\lambda_2)} & \ldots & y_1^{(\lambda_{2n-1})} & y_1^{(\lambda_{2n})} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^{(\lambda_1)} & x_n^{(\lambda_2)} & \ldots & x_n^{(\lambda_{2n-1})} & x_n^{(\lambda_{2n})} \\ y_n^{(\lambda_1)} & y_n^{(\lambda_2)} & \ldots & y_n^{(\lambda_{2n-1})} & y_n^{(\lambda_{2n})} \end{pmatrix}\right]. \tag{47}$$

This matrix admits a quaternionic counterpart [41] by complex linear operators

$$\mathcal{S}_{\mathbb{C}} = \text{Inverse}\left[\begin{pmatrix} q_1^{[1,2]} + p_1^{[1,2]} R_{\mathrm{i}} & \ldots & q_1^{[2n-1,2n]} + p_1^{[2n-1,2n]} R_{\mathrm{i}} \\ \vdots & \ddots & \vdots \\ q_n^{[1,2]} + p_n^{[1,2]} R_{\mathrm{i}} & \ldots & q_n^{[2n-1,2n]} + p_n^{[2n-1,2n]} R_{\mathrm{i}} \end{pmatrix}\right] \tag{48}$$

where

$$q_l^{[m,n]} = \frac{x_l^{(\lambda_m)} + y_l^{(\lambda_n)*}}{2} + \mathrm{j}\,\frac{y_l^{(\lambda_m)} - x_l^{(\lambda_n)*}}{2}$$

and

$$p_l^{[m,n]} = \frac{x_l^{(\lambda_m)} - y_l^{(\lambda_n)*}}{2\mathrm{i}} + \mathrm{j}\,\frac{y_l^{(\lambda_m)} + x_l^{(\lambda_n)*}}{2\mathrm{i}}.$$

To simplify the notation we use i, j, k instead of L_{i}, L_{j}, L_{k}. The right operator R_{i} indicates the right action of the imaginary unit i. The diagonalized quaternionic matrix reproduces the quaternionic translation of the complex matrix

$$M^{\mathrm{diag}} = \mathrm{diag}\,\{\lambda_1, \ldots, \lambda_{2n}\}$$

into

$$\mathcal{M}_{\mathbb{C}}^{\mathrm{diag}} = \mathrm{diag}\left\{\frac{\lambda_1 + \lambda_2^*}{2} + \frac{\lambda_1 - \lambda_2^*}{2\mathrm{i}} R_{\mathrm{i}}, \ldots, \frac{\lambda_{2n-1} + \lambda_{2n}^*}{2} + \frac{\lambda_{2n-1} - \lambda_{2n}^*}{2\mathrm{i}} R_{\mathrm{i}}\right\}. \tag{49}$$

5. Quaternionic eigenvalue equation

By working with quaternions we have different possibilities to write eigenvalue equations. In fact, in solving such equations, we could consider quaternionic or complex, left or right eigenvalues. In this section, we briefly introduce the problem inherent to quaternionic eigenvalue equations and emphasize the main difficulties present in such an approach.

5.1. Right quaternionic eigenvalue equation for complex linear operators

As seen in the previous sections, the right eigenvalue equation for quaternionic linear operators, $\mathcal{O}_{\mathbb{H}}$, reads

$$\mathcal{M}_{\mathbb{H}}|\tilde{\psi}\rangle = |\tilde{\psi}\rangle q \qquad q \in \mathbb{H}.$$

Such an equation can be converted into a right complex eigenvalue equation by rephasing the quaternionic eigenvalues, q,

$$\mathcal{M}_{\mathbb{H}}|\tilde{\psi}\rangle u = |\tilde{\psi}\rangle u\bar{u}qu = |\tilde{\psi}\rangle\lambda \qquad \lambda \in \mathbb{C}.$$

This trick fails for complex linear operators. In fact, by discussing right quaternionic eigenvalue equations for complex linear operators,

$$\mathcal{M}_{\mathbb{C}}|\psi\rangle = |\psi\rangle q \tag{50}$$

due to the presence of the right imaginary unit i in $\mathcal{M}_{\mathbb{C}}$, we cannot apply quaternionic similarity transformations,

$$\big(\mathcal{M}_{\mathbb{C}}|\tilde{\psi}\rangle\big)u \neq \mathcal{M}_{\mathbb{C}}\big(|\tilde{\psi}\rangle u\big) \qquad u \in \mathbb{H}.$$

Within a qQM with complex geometry [20, 31, 41], a generic anti-Hermitian operator must satisfy

$$\langle\phi|\mathcal{A}_{\mathbb{C}}\psi\rangle_{\mathbb{C}} = -\langle\mathcal{A}_{\mathbb{C}}\phi|\psi\rangle_{\mathbb{C}}. \tag{51}$$

We can immediately find a constraint on our $\mathcal{A}_{\mathbb{C}}$-eigenvalues by putting in the previous equation $|\phi\rangle = |\psi\rangle$,

$$\begin{aligned}&\langle\psi|\psi q_{\psi}\rangle_{\mathbb{C}} = -\langle\psi q_{\psi}|\psi\rangle_{\mathbb{C}} \quad \Rightarrow \quad q_{\psi} = \mathrm{i}\alpha_{\psi} + \mathrm{j}w_{\psi}\\ &\alpha_{\psi} \in \mathbb{R} \qquad w_{\psi} \in \mathbb{C}.\end{aligned} \tag{52}$$

Thus, complex linear anti-Hermitian operators, $\mathcal{A}_{\mathbb{C}}$, will be characterized by purely imaginary quaternions. An important property must be satisfied for complex linear anti-Hermitian operators, namely eigenvectors $|\phi\rangle$ and $|\psi\rangle$ associated with different eigenvalues, $q_{\phi} \neq q_{\psi}$, have to be orthogonal in $\mathbb{C}$. By combining equations (51) and (52), we find

$$\langle\phi|\psi q_{\psi}\rangle_{\mathbb{C}} = \langle q_{\phi}\phi|\psi\rangle_{\mathbb{C}}.$$

To guarantee the complex orthogonality of the eigenvectors $|\phi\rangle$ and $|\psi\rangle$, namely $\langle\phi|\psi\rangle_{\mathbb{C}} = 0$, we must require a complex projection for the eigenvalues, $(q)_{\mathbb{C}}$,

$$q_{\psi,\phi} \longrightarrow \lambda_{\psi,\phi} \in \mathbb{C}.$$

In conclusion, a consistent discussion of right eigenvalue equations within a qQM with complex geometry requires complex eigenvalues.

5.2. Left quaternionic eigenvalue equation

What happens for left quaternionic eigenvalue equations? In solving such equations for quaternionic and complex linear operators,

$$\mathcal{M}_{\mathbb{H}}|\tilde{\psi}\rangle = \tilde{q}|\tilde{\psi}\rangle \qquad \mathcal{M}_{\mathbb{C}}|\tilde{\psi}\rangle = \tilde{q}|\tilde{\psi}\rangle \qquad \tilde{q} \in \mathbb{H}$$

we do not have a systematic way to approach the problem. In this case, due to the presence of left quaternionic eigenvalues (translated in complex formalism by two-dimensional matrices), the translation trick does not apply and so we must solve the problem directly in the quaternionic world.

In discussing left quaternionic eigenvalue equations, we underline the difficulty hidden in diagonalizing such operators. Let us suppose that the matrix representations of our operators are diagonalized by a matrix $\mathcal{S}_{\mathbb{H}/\mathbb{C}}$

$$\mathcal{S}_{\mathbb{H}}\mathcal{M}_{\mathbb{H}}\mathcal{S}_{\mathbb{H}}^{-1} = \mathcal{M}_{\mathbb{H}}^{\text{diag}} \qquad \text{and} \qquad \mathcal{S}_{\mathbb{C}}\mathcal{M}_{\mathbb{C}}\mathcal{S}_{\mathbb{C}}^{-1} = \mathcal{M}_{\mathbb{C}}^{\text{diag}}.$$

The eigenvalue equation will be modified in

$$\mathcal{M}_{\mathbb{H}}^{\text{diag}}\mathcal{S}_{\mathbb{H}}|\tilde{\psi}\rangle = \mathcal{S}_{\mathbb{H}}\tilde{q}\mathcal{S}_{\mathbb{H}}^{-1}\mathcal{S}_{\mathbb{H}}|\tilde{\psi}\rangle$$
$$\mathcal{M}_{\mathbb{C}}^{\text{diag}}\mathcal{S}_{\mathbb{C}}|\tilde{\psi}\rangle = \mathcal{S}_{\mathbb{C}}\tilde{q}\mathcal{S}_{\mathbb{C}}^{-1}\mathcal{S}_{\mathbb{C}}|\tilde{\psi}\rangle$$

and now, due to the non-commutative nature of $\widetilde{q}$,

$$\mathcal{S}_{\mathbb{H}/\mathbb{C}}\widetilde{q}\mathcal{S}_{\mathbb{H}/\mathbb{C}}^{-1} \neq \widetilde{q}.$$

So, we can have operators with the same left quaternionic eigenvalues spectrum but no similarity transformation relating them. This is shown explicitly in appendix B, where we discuss examples of two-dimensional quaternionic linear operators. Let us now analyse other difficulties in solving the left quaternionic eigenvalue equation. The Hermitian quaternionic linear operators satisfy

$$\langle\widetilde{\phi}|\mathcal{H}_{\mathbb{H}}\widetilde{\psi}\rangle = \langle\mathcal{H}_{\mathbb{H}}\widetilde{\phi}|\widetilde{\psi}\rangle.$$

By setting $|\widetilde{\phi}\rangle = |\widetilde{\psi}\rangle$ in the previous equation we find constraints on the quaternionic eigenvalues $\widetilde{q}$

$$\langle\widetilde{\psi}|\widetilde{q}\widetilde{\psi}\rangle = \langle\widetilde{q}\widetilde{\psi}|\widetilde{\psi}\rangle.$$

From this equation we cannot extract the conclusion that $\tilde{q}$ must be real, $\widetilde{q} = \widetilde{q}^{\dagger}$. In fact,

$$\langle\widetilde{\psi}|(\widetilde{q} - \widetilde{q}^{\dagger})|\widetilde{\psi}\rangle = 0$$

could admit quaternionic solutions for $\tilde{q}$ (see the example in appendix B). So, the first complication is represented by the possibility of finding Hermitian operators with quaternionic eigenvalues. Within a qQM, we can overcome this problem by choosing anti-Hermitian operators to represent observable quantities. In fact,

$$\langle\widetilde{\phi}|\mathcal{A}_{\mathbb{H}}\widetilde{\psi}\rangle = -\langle\mathcal{A}_{\mathbb{H}}\widetilde{\phi}|\widetilde{\psi}\rangle$$

will imply, for $|\widetilde{\phi}\rangle = |\widetilde{\psi}\rangle$,

$$\langle\widetilde{\psi}|(\widetilde{q} + \widetilde{q}^{\dagger})|\widetilde{\psi}\rangle = 0. \tag{53}$$

In this case, the real quantity, $\widetilde{q} + \widetilde{q}^{\dagger}$, commutes with $|\widetilde{\psi}\rangle$, and so equation (53) gives the constraint

$$\widetilde{q} = \mathrm{i}\alpha + \mathrm{j}w.$$

We could work with anti-Hermitian operators and choose $|\widetilde{q}|$ as the observable output. Examples of the left/right eigenvalue equation for two-dimensional anti-Hermitian operators will be discussed in appendix B. In this appendix, we explicitly show an important difference between the left and right eigenvalue equation for anti-Hermitian operators: left and right eigenvalues can have different absolute values and so cannot represent the same physical quantity.

6. Conclusions

The study undertaken in this paper demonstrates the possibility of constructing a practical method to diagonalize quaternionic and complex linear operators on quaternionic vector spaces. Quaternionic eigenvalue equations have to be right eigenvalue equations. As shown in our paper, the choice of a right position for quaternionic eigenvalues is fundamental in searching for a diagonalization method. A left position of quaternionic eigenvalues gives unwanted surprises. For example, we find operators with the same eigenvalues which are not related by a similarity transformation, Hermitian operators with quaternionic eigenvalues, etc.

Quaternionic linear operators in n-dimensional vector spaces take infinite spectra of quaternionic eigenvalues. Nevertheless, the *complex translation trick* ensures that such spectra are related by similarity transformations and this gives the possibility of choosing n representative complex eigenvalues to perform calculations. The complete set of quaternionic eigenvalues spectra can be generated from the complex eigenvalue spectrum,

$$\{\lambda_1, \ldots, \lambda_n\}$$

by quaternionic similarity transformations,

$$\{\overline{u}_1\lambda_1 u_1, \ldots, \overline{u}_n\lambda_n u_n\}.$$

Such a symmetry is broken when we consider a set of diagonalizable operators. In this case the freedom in constructing the eigenvalue spectrum for the first operator, and consequently the free choice in determining an eigenvectors basis, will fix the eigenvalue spectrum for the other operators.

The power of the complex translation trick gives the possibility of studying general properties for quaternionic and complex linear operators. Complex linear operators play an important role within a qQM with complex geometry by reproducing the standard complex results in reduced quaternionic vector spaces [31]. The method of diagonalization becomes very useful in the resolution of quaternionic differential equations [42]. Consequently, an immediate application is found in solving the Schrödinger equation with quaternionic potentials [43].

Mathematical topics to be developed are represented by the discussion of the eigenvalue equation for real linear operators, $\mathcal{O}_{\mathbb{R}}$, and by a detailed study of the left eigenvalue equation. Real linear operators are characterized by left and right actions of the quaternionic imaginary units i, j, k. The translation trick now needs to be applied in the real world and so, for a coherent discussion, it will require the adoption of a real geometry.

Acknowledgments

The authors wish to express their gratitude to Nir Cohen for several helpful comments concerning quaternionic matrix theory and Jordan form. They are also indebted to Gisele Ducati for suggestions on possible applications of the diagonalization method to quaternionic differential equations and for many stimulating conversations. GS gratefully acknowledges the Department of Applied Mathematics, IMECC-Unicamp, for the invitation and hospitality. This work was partially supported by a fellowship of the Department of Physics, Lecce University, (GS) and by a research grant of the Fapesp, São Paulo State (SdL). Finally, the authors wish to thank the referees for drawing attention to interesting references and for their remarks which helped to clarify the notation and improve the discussion presented in this paper.

Appendix A. Two-dimensional right complex eigenvalue equations

In this appendix we explicitly solve the right eigenvalues equations for quaternionic, $\mathcal{O}_{\mathbb{H}}$, and complex $\mathcal{O}_{\mathbb{C}}$, linear operators, in two-dimensional quaternionic vector spaces.

Quaternionic linear operators

Let

$$\mathcal{M}_{\mathbb{H}} = \begin{pmatrix} \mathrm{i} & \mathrm{j} \\ \mathrm{k} & \mathrm{i} \end{pmatrix} \tag{A1}$$

be the quaternionic matrix representation associated with a quaternionic linear operator in a two-dimensional quaternionic vector space. Its complex counterpart reads

$$\widetilde{M} = \begin{pmatrix} \mathrm{i} & 0 & 0 & -1 \\ 0 & -\mathrm{i} & 1 & 0 \\ 0 & -\mathrm{i} & \mathrm{i} & 0 \\ -\mathrm{i} & 0 & 0 & -\mathrm{i} \end{pmatrix}.$$

In order to solve the right eigenvalue problem

$$\mathcal{M}_{\mathbb{H}}|\psi\rangle = |\psi\rangle\lambda \qquad \lambda \in \mathbb{C}$$

let us determine the $\widetilde{M}$-eigenvalue spectrum. From the constraint

$$\det\left[\widetilde{M} - \lambda\,\mathbb{1}_4\right] = 0$$

we find for the $\widetilde{M}$-eigenvalues the following solutions:

$$\left\{\lambda_1, \lambda_1^*, \lambda_2, \lambda_2^*\right\}_{\widetilde{M}} = \left\{2^{1/4}\mathrm{e}^{3\mathrm{i}\pi/8}, 2^{1/4}\mathrm{e}^{-3\mathrm{i}\pi/8}, -2^{1/4}\mathrm{e}^{-3\mathrm{i}\pi/8}, -2^{1/4}\mathrm{e}^{3\mathrm{i}\pi/8}\right\}_{\widetilde{M}}.$$

The $\widetilde{M}$-eigenvector set is given by

$$\left\{\begin{pmatrix} -1+\mathrm{i}\lambda_1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1-\mathrm{i}\lambda_1^* \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1-\mathrm{i}\lambda_1^* \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1-\mathrm{i}\lambda_1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}_{\widetilde{M}}.$$

The $\mathcal{M}_{\mathbb{H}}$-eigenvalue spectrum is soon obtained from that one of $\widetilde{M}$. For example, by adopting the positive imaginary part convention we find

$$\{\lambda_1, \lambda_2\}_{\mathcal{M}_{\mathbb{H}}} = \left\{2^{1/4}\mathrm{e}^{3\mathrm{i}\pi/8}, -2^{1/4}\mathrm{e}^{-3\mathrm{i}\pi/8}\right\}_{\mathcal{M}_{\mathbb{H}}} \tag{A2}$$

and the corresponding quaternionic eigenvector set, defined up to a right complex phase, reads

$$\left\{ \begin{pmatrix} -1 + \mathrm{i}\lambda_1 \\ \mathrm{j} \end{pmatrix}, \begin{pmatrix} \mathrm{j}(1 - \mathrm{i}\lambda_1^*) \\ 1 \end{pmatrix} \right\}_{\mathcal{M}_{\mathbb{H}}}. \tag{A3}$$

The quaternionic matrix which diagonalizes $\mathcal{M}_{\mathbb{H}}$ is

$$\mathcal{S}_{\mathbb{H}} = \text{Inverse}\left[\begin{pmatrix} -1 + \mathrm{i}\lambda_1 & \mathrm{j}(1 - \mathrm{i}\lambda_1^*) \\ \mathrm{j} & 1 \end{pmatrix} \right] = -\frac{1}{2|\lambda_1|^2} \begin{pmatrix} \mathrm{i}\lambda_1^* & \mathrm{j}\left[\mathrm{i}\lambda_1 + |\lambda_1|^2\right] \\ \mathrm{k}\lambda_1^* & \mathrm{i}\lambda_1 - |\lambda_1|^2 \end{pmatrix}. \tag{A4}$$

As remarked in this paper, we have infinite possibilities for diagonalization

$$\{\overline{u}_1\lambda_1 u_1, \overline{u}_2\lambda_2 u_2\}.$$

Equivalent diagonalized matrices can be obtained from

$$\mathcal{M}_{\mathbb{H}}^{\text{diag}} = \text{diag}\{\lambda_1, \lambda_2\}$$

by performing a similarity transformation

$$\mathcal{U}^{-1}\mathcal{M}_{\mathbb{H}}^{\text{diag}}\mathcal{U} = \mathcal{U}^{\dagger}\mathcal{M}_{\mathbb{H}}^{\text{diag}}\mathcal{U}$$

and

$$\mathcal{U} = \text{diag}\{u_1, u_2\}.$$

The diagonalization matrix given in equation (A4) becomes

$$\mathcal{S}_{\mathbb{H}} \to \mathcal{U}^{\dagger}\mathcal{S}_{\mathbb{H}}.$$

Complex linear operators

Let

$$\mathcal{M}_{\mathbb{C}} = \begin{pmatrix} -\mathrm{i}R_{\mathrm{i}} + \mathrm{j} & -\mathrm{k}R_{\mathrm{i}} + 1 \\ -\mathrm{k}R_{\mathrm{i}} - 1 & \mathrm{i}R_{\mathrm{i}} + \mathrm{j} \end{pmatrix} \tag{A5}$$

be the quaternionic matrix representation associated with a complex linear operator in a two-dimensional quaternionic vector space. Its complex counterpart is

$$M = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}.$$

The right complex eigenvalue problem

$$\mathcal{M}_{\mathbb{C}}|\psi\rangle = |\psi\rangle\lambda \qquad \lambda \in \mathbb{C}$$

can be solved by determining the M-eigenvalue spectrum

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}_{M/\mathcal{M}_{\mathbb{C}}} = \{2, -2, 2\mathrm{i}, -2\mathrm{i}\}_{M/\mathcal{M}_{\mathbb{C}}}. \tag{A6}$$

Such eigenvalues also determine the $\mathcal{M}_{\mathbb{C}}$-eigenvalues spectrum. The $\mathcal{M}_{\mathbb{C}}$-eigenvector set is obtained by translating the complex M-eigenvector set

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -\mathrm{i} \\ \mathrm{i} \\ 1 \end{pmatrix}, \begin{pmatrix} -\mathrm{i} \\ 1 \\ -1 \\ -\mathrm{i} \end{pmatrix} \right\}_M$$

in the quaternionic formalism

$$\left\{ \begin{pmatrix} 1 \\ -\mathrm{j} \end{pmatrix}, \begin{pmatrix} \mathrm{j} \\ 1 \end{pmatrix}, \begin{pmatrix} 1+\mathrm{k} \\ \mathrm{i}+\mathrm{j} \end{pmatrix}, \begin{pmatrix} \mathrm{j}-\mathrm{i} \\ \mathrm{k}-1 \end{pmatrix} \right\}_{\mathcal{M}_{\mathbb{C}}}. \tag{A7}$$

The quaternionic matrix which diagonalizes $\mathcal{M}_{\mathbb{C}}$ is

$$\mathcal{S}_{\mathbb{C}} = \text{Inverse}\left[\begin{pmatrix} 1 & 1+\mathrm{k} \\ -\mathrm{j} & \mathrm{i}+\mathrm{j} \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} 1 & \mathrm{j} \\ \frac{1}{2}(1-\mathrm{k}) & -\frac{1}{2}(\mathrm{i}+\mathrm{j}) \end{pmatrix} \tag{A8}$$

and the diagonalized matrix is given by

$$\mathcal{M}_{\mathbb{C}}^{\mathrm{diag}} = 2 \begin{pmatrix} -\mathrm{i}R_{\mathrm{i}} & 0 \\ 0 & \mathrm{i} \end{pmatrix}. \tag{A9}$$

This matrix can be obtained directly from the $M/\mathcal{M}_{\mathbb{C}}$ eigenvalue spectrum by translating, in the quaternionic formalism, the matrix

$$M^{\mathrm{diag}} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2\mathrm{i} & 0 \\ 0 & 0 & 0 & -2\mathrm{i} \end{pmatrix}.$$

It is interesting to note that equivalent diagonalized matrices can be obtained from $\mathcal{M}_{\mathbb{C}}^{\mathrm{diag}}$ in equation (A9) by the similarity transformation

$$\mathcal{U}^{\dagger} \mathcal{M}_{\mathbb{C}}^{\mathrm{diag}} \mathcal{U}.$$

For example, by choosing

$$\mathcal{U} = \begin{pmatrix} -\mathrm{j} & 0 \\ 0 & \dfrac{1+\mathrm{k}}{\sqrt{2}} \end{pmatrix}$$

one finds

$$\mathcal{M}_{\mathbb{C}}^{\mathrm{diag}} \to 2 \begin{pmatrix} \mathrm{i}R_{\mathrm{i}} & 0 \\ 0 & \mathrm{j} \end{pmatrix}. \tag{A10}$$

Appendix B. Two-dimensional left quaternionic eigenvalue equations

Let us now examine left quaternionic eigenvalue equations for quaternionic linear operators.

Hermitian operators

Let

$$\mathcal{H}_{\mathbb{H}} = \begin{pmatrix} 0 & \mathrm{k} \\ -\mathrm{k} & 0 \end{pmatrix}$$

be the quaternionic matrix representation associated with a Hermitian quaternionic linear operator. We consider its left quaternionic eigenvalue equation

$$\mathcal{H}_{\mathbb{H}}|\widetilde{\psi}\rangle = \widetilde{q}|\widetilde{\psi}\rangle \tag{B1}$$

where

$$|\widetilde{\psi}\rangle = \begin{pmatrix} \widetilde{\psi}_1 \\ \widetilde{\psi}_2 \end{pmatrix} \in \mathbb{H}^2 \qquad \tilde{q} \in \mathbb{H}.$$

Equation (B1) can be rewritten by the following quaternionic system:

$$\mathrm{k}\widetilde{\psi}_2 = \tilde{q}\widetilde{\psi}_1 \qquad -\mathrm{k}\widetilde{\psi}_1 = \tilde{q}\widetilde{\psi}_2.$$

The solution is

$$\{\tilde{q}\}_{\mathcal{H}_{\mathbb{H}}} = \{z + \mathrm{j}\beta\}_{\mathcal{H}_{\mathbb{H}}}$$

where

$$z \in \mathbb{C} \qquad \beta \in \mathbb{R} \qquad |z|^2 + \beta^2 = 1.$$

The $\mathcal{H}_{\mathbb{H}}$-eigenvector set is given by

$$\left\{ \begin{pmatrix} \widetilde{\psi}_1 \\ -\mathrm{k}(z + \mathrm{j}\beta)\widetilde{\psi}_1 \end{pmatrix} \right\}_{\mathcal{H}_{\mathbb{H}}}.$$

It is easy to verify that in this case

$$\langle\widetilde{\psi}|(\tilde{q} - \tilde{q}^{\dagger})|\widetilde{\psi}\rangle = 0$$

is verified for quaternionic eigenvalues $\tilde{q} \neq \tilde{q}^{\dagger}$.

Anti-hermitian operators

Let

$$\mathcal{A}_{\mathbb{H}} = \begin{pmatrix} \mathrm{j} & \mathrm{i} \\ \mathrm{i} & \mathrm{k} \end{pmatrix}$$

be the quaternionic matrix representation associated with an anti-Hermitian quaternionic linear operator. Its right complex spectrum is given by

$$\{\lambda_1, \lambda_2\}_{\mathcal{H}_{\mathbb{H}}} = \left\{\mathrm{i}\sqrt{2-\sqrt{2}}, \mathrm{i}\sqrt{2+\sqrt{2}}\right\}_{\mathcal{H}_{\mathbb{H}}}.$$

We now consider the left quaternionic eigenvalue equation

$$\mathcal{A}_{\mathbb{H}}|\widetilde{\psi}\rangle = \widetilde{q}|\widetilde{\psi}\rangle \tag{B2}$$

where

$$|\widetilde{\psi}\rangle = \begin{pmatrix} \widetilde{\psi}_1 \\ \widetilde{\psi}_2 \end{pmatrix} \in \mathbb{H}^2 \qquad \tilde{q} \in \mathbb{H}.$$

By solving the following quaternionic system:

$$\mathrm{j}\widetilde{\psi}_1 + \mathrm{i}\widetilde{\psi}_2 = \tilde{q}\widetilde{\psi}_1 \qquad \mathrm{i}\widetilde{\psi}_1 + \mathrm{k}\widetilde{\psi}_2 = \tilde{q}\widetilde{\psi}_2$$

we find

$$\{\tilde{q}_1, \tilde{q}_2\}_{\mathcal{A}_{\mathbb{H}}} = \left\{ \frac{\mathrm{i}}{\sqrt{2}} + \frac{\mathrm{j}+\mathrm{k}}{2}, \frac{-\mathrm{i}}{\sqrt{2}} + \frac{\mathrm{j}+\mathrm{k}}{2} \right\}_{\mathcal{A}_{\mathbb{H}}}$$

and

$$\left\{ \begin{pmatrix} \widetilde{\psi}_1 \\ \left(\frac{1}{\sqrt{2}} + \frac{\mathrm{j}+\mathrm{k}}{2}\right)\widetilde{\psi}_1 \end{pmatrix}, \begin{pmatrix} \widetilde{\psi}_1 \\ \left(\frac{-1}{\sqrt{2}} + \frac{\mathrm{j}+\mathrm{k}}{2}\right)\widetilde{\psi}_1 \end{pmatrix} \right\}_{\mathcal{H}_{\mathbb{H}}}.$$

We observe that

$$\left\{|\overline{u}_1\lambda_1 u_1| = \sqrt{2-\sqrt{2}}, |\overline{u}_2\lambda_2 u_2 = \sqrt{2+\sqrt{2}}\right\}$$

and

$$\{|\tilde{q}_1| = 1, |\tilde{q}_2| = 1\}.$$

Thus, left and right eigenvalues cannot be associated with the same physical quantity.

A new possibility

In order to complete our discussion let us discuss for the quaternionic linear operator given in equation (A1) its left quaternionic eigenvalue equation

$$\mathcal{M}_{\mathbb{H}}|\widetilde{\psi}\rangle = \tilde{q}|\widetilde{\psi}\rangle \tag{B3}$$

where

$$|\widetilde{\psi}\rangle = \begin{pmatrix} \widetilde{\psi}_1 \\ \widetilde{\psi}_2 \end{pmatrix} \in \mathbb{H}^2 \qquad \tilde{q} \in \mathbb{H}.$$

Equation (B3) can be rewritten using the following quaternionic system:

$$\mathrm{i}\widetilde{\psi}_1 + \mathrm{j}\widetilde{\psi}_2 = \tilde{q}\widetilde{\psi}_1$$
$$\mathrm{k}\widetilde{\psi}_1 + \mathrm{i}\widetilde{\psi}_2 = \tilde{q}\widetilde{\psi}_2.$$

The solution gives for the quaternionic eigenvalue spectrum

$$\{\tilde{q}_1, \tilde{q}_2\}_{\mathcal{M}_{\mathbb{H}}} = \left\{ \mathrm{i} + \frac{\mathrm{j}+\mathrm{k}}{\sqrt{2}}, \mathrm{i} - \frac{\mathrm{j}+\mathrm{k}}{\sqrt{2}} \right\}_{\mathcal{M}_{\mathbb{H}}} \tag{B4}$$

and for the eigenvector set

$$\left\{ \begin{pmatrix} 1 \\ \frac{1-\mathrm{i}}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{\mathrm{i}-1}{\sqrt{2}} \end{pmatrix} \right\}_{\mathcal{M}_{\mathbb{H}}}. \tag{B5}$$

Let us now consider the following quaternionic linear operator:

$$\mathcal{N}_{\mathbb{H}} = \begin{pmatrix} \mathrm{i} + \dfrac{\mathrm{j}+\mathrm{k}}{\sqrt{2}} & 0 \\ 0 & \mathrm{i} - \dfrac{\mathrm{j}+\mathrm{k}}{\sqrt{2}} \end{pmatrix}. \tag{B6}$$

This operator represents a diagonal operator and has the same left quaternionic eigenvalue spectrum of $\mathcal{M}_{\mathbb{H}}$, notwithstanding such an operator is not equivalent to $\mathcal{M}_{\mathbb{H}}^{\mathrm{diag}}$. In fact, the $\mathcal{N}_{\mathbb{H}}$-complex counterpart is characterized by the following eigenvalue spectrum:

$$\left\{\mathrm{i}\sqrt{2}, -\mathrm{i}\sqrt{2}, \mathrm{i}\sqrt{2}, -\mathrm{i}\sqrt{2}\right\}_N$$

different from the eigenvalue spectrum of the M-complex counterpart of $\mathcal{M}_{\mathbb{H}}$. Thus, there is no similarity transformation which relates these two operators in the complex world and consequently by translation there is no a quaternionic matrix which relates $\mathcal{N}_{\mathbb{H}}$ to $\mathcal{M}_{\mathbb{H}}$. So, in the quaternionic world, we can have quaternionic linear operators which have the same left quaternionic eigenvalue spectrum but are not related by a similarity transformation.

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