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Dirac–Hestenes Lagrangian

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We formulate the variational principle of the Dirac equation within the noncommutative even space-time subalgebra, the Clifford $\mathbb{R}$-algebra $Cl^{+}_{1,3}$. A fundamental ingredient in our multivectorial algebraic formulation is a $\mathbb{D}$-complex geometry, $\mathbb{D} \equiv \text{span}_{\mathbb{D}}\{1, \gamma_{21}\}$, $\gamma_{21} \in Cl^{+}_{1,3}$. We derive the Lagrangian for the Dirac–Hestenes equation and show that it must be mapped on $\mathbb{D} \otimes \mathcal{F}$, where $\mathcal{F}$ denotes an $\mathbb{R}$-algebra of functions.

1. INTRODUCTION

This introduction contains a brief summary of the translation between the Dirac [1, 2] and Dirac–Hestenes [3] equations. Throughout the paper we use the following notation: $\mathbb{R}$ is the real field; $\mathbb{C}$ is the complex field $\mathbb{C} \equiv \text{span}_{\mathbb{R}}\{1, i\}$, $i \in \mathbb{C}$, $i^2 = -1$; $\mathbb{D}$ is the field $\mathbb{D} \equiv \text{span}_{\mathbb{R}}\{1, \gamma_{21}\}$, $\gamma_{21} \in Cl^{+}_{1,3}$, $\gamma^3_{21} = -1$; $\mathcal{F}$ is an $\mathbb{R}$-algebra of functions from $\mathbb{R}^4$ to $\mathbb{R}$; and $\otimes \equiv \otimes_{\mathbb{R}}$, $x_\mu \in \mathcal{F}$, $\partial^\mu \in \text{der}\mathcal{F}$, $\partial^\mu x_\nu = \delta^\mu_\nu \in \mathcal{F}$ ($\mu, \nu = 0, 1, 2, 3$).

Let $g_{\mu\nu} = \text{diag}\,(+, -, -, -)$ be the Minkowski metric, $\mathcal{I} \lhd (\mathbb{C} \otimes Cl_{1,3})$ be the one-sided ideal, $\mathcal{I} \approx \mathbb{C}^4$, and $\Psi_D \in \mathcal{I} \otimes \mathcal{F}$,

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$$\Psi_D \equiv \begin{pmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ a_3 + ib_3 \\ a_4 + ib_4 \end{pmatrix} \equiv \begin{pmatrix} \psi_{D,1} \\ \psi_{D,2} \\ \psi_{D,3} \\ \psi_{D,4} \end{pmatrix}, \tag{1}$$

$$a_m, b_m \in \mathcal{F}, \quad \psi_{D,m} \in \mathbb{C} \otimes \mathcal{F}, \quad m = 1, 2, 3, 4$$

Let $\gamma_\mu \in \mathrm{End}(\mathcal{S}) \approx \mathrm{Mat}_4(\mathbb{C})$ be 4×4 complex matrices which satisfy the Dirac algebra:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} \mathbb{1}_4 \quad \in \mathrm{Mat}_4(\mathbb{C}), \qquad \mu, \nu = 0, 1, 2, 3$$

If $m \in \mathbb{R}^+$ is the mass of the particle, then the Dirac equation for a free particle Ψ_D reads

$$i\gamma_\mu \partial^\mu \Psi_D = m\Psi_D \quad \in \mathcal{S} \otimes \mathcal{F} \tag{2}$$

For the Dirac matrices a possible choice useful for the discussion presented in Section 2 is

$$\gamma^\mu \equiv \{\gamma_0, \gamma_k\}, \qquad k = 1, 2, 3$$

$$\gamma_0 \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma_1 \equiv \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2 \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_3 \equiv \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \tag{3}$$

A renewed interest exists in the formulation of the Dirac theory in terms of the Clifford algebra [4–9]. An interesting result is the possibility to write the Dirac equation in the even space-time subalgebra, $Cl^+_{1,3}$. This result is achieved by working only with the general properties of the Clifford $\mathbb{R}$-algebra, in particular the concept of an even subalgebra [10].

An alternative possibility in rewriting the Dirac equation in $Cl^+_{1,3}$ is represented by the direct *translation* of the elements which characterize the standard "complex" formulation. In the present section, we summarize this translation and introduce the notion of $\mathbb{D}$-complex geometry [11–14].

Consider a subalgebra isomorphic to the complex field $\mathbb{C}$,

$$Cl^+_{1,3} > \mathbb{D} \equiv \mathrm{span}_{\mathbb{R}}\{1, \gamma_{21}\} \approx \mathbb{C} = \mathrm{span}_{\mathbb{R}}\{1, i\}, \qquad \gamma_{21} \leftrightarrow i$$

By $\mathbb{D}$-complex geometry we mean an $\mathbb{R}$-linear mapping

$$\chi: \quad Cl^+_{1,3} \otimes \mathcal{F} \to \mathbb{D} \otimes \mathcal{F}, \quad \mathbb{D} < Cl^+_{1,3}$$
$$\chi \in \text{lin}_{\mathbb{R}}(Cl^+_{1,3} \otimes \mathcal{F}, \mathbb{D} \otimes \mathcal{F}), \qquad \Psi \in Cl^+_{1,3} \otimes \mathcal{F}, \tag{4}$$
$$\chi(\Psi) \equiv (\Psi)_{\mathcal{F}} - \gamma_{21} (\gamma_{21} \Psi)_{\mathcal{F}}$$

where the subscript $\mathcal{F}$ denotes the mapping on $\mathbb{R} \otimes \mathcal{F} \approx \mathcal{F}$ of the quantity within the brackets.

The Dirac spinor fields are elements of the $\mathbb{C}$-space $\mathbb{C}^4 \otimes \mathcal{F}$, thus are characterized by eight real functions of four real variables (e.g., ref. 15)

$$\dim_{\mathbb{R}} \mathcal{I} = \dim_{\mathbb{R}} Cl^+_{1,3} = 8$$

A possible basis of the Clifford $\mathbb{R}$-algebra $Cl^+_{1,3}$ is

$$1,$$
$$\gamma_{01}, \gamma_{02}, \gamma_{03},$$
$$\gamma_{21}, \gamma_{31}, \gamma_{23},$$
$$\gamma_5 \equiv \gamma_{0123} \quad \in Cl^+_{1,3}$$

An arbitrary element in $Cl^+_{1,3} \otimes \mathcal{F}$ can be written as

$$\alpha_0 + \gamma_{01}\alpha_1 + \gamma_{02}\alpha_2 + \gamma_{03}\alpha_3 + \gamma_{21}\alpha_4 + \gamma_{31}\alpha_5 + \gamma_{23}\alpha_6 + \gamma_5\alpha_7, \qquad \alpha_m \in \mathcal{F}, \quad m = 0, \ldots, 7 \tag{5}$$

The Hestenes spinor, solution of the Dirac–Hestenes equation,

$$\psi_{H,m} = a_m + \gamma_{21} b_m \in \mathbb{D} \otimes \mathcal{F}, \qquad m = 1, 2, 3, 4$$
$$\Psi_H \equiv \psi_{H,1} + \gamma_{31}\psi_{H,2} + \gamma_5(\psi_{H,3} + \gamma_{31}\psi_{H,4}) \quad \in Cl^+_{1,3} \otimes \mathcal{F} \tag{6}$$

will represent the counterpart in the even space-time subalgebra of the complex Dirac spinor Ψ_D. The *isomorphism*

$$\rho: \quad \mathcal{I} \otimes \mathcal{F} \to Cl^+_{1,3} \otimes \mathcal{F} \tag{7}$$

requires the identification

$$\rho \in \text{lin}_{\mathbb{R}} (\mathcal{I} \otimes \mathcal{F}, Cl^+_{1,3} \otimes \mathcal{F}), \qquad i \in \mathcal{I}, \quad \gamma_{21} \in Cl^+_{1,3}, \quad \rho(i) = \gamma_{21}$$

Let β be the main antiautomorphism of the Clifford $\mathbb{C}$-algebra $\mathbb{C} \otimes Cl_{1,3}$; then there exists a Hermitian sesquilinar form [16] in the space of the Dirac spinors

$$h: \quad \mathcal{I}^+ \otimes \mathcal{I} \to \mathbb{C}, \qquad h \in \mathcal{I}^* \otimes (\mathcal{I}^+)^*, \qquad h(\mathcal{I}^+ \otimes \mathcal{I}) \equiv (\beta\mathcal{I})\mathcal{I} \in \mathbb{C} \tag{8}$$

There exists a basis in $\mathcal{I}$ such that

$$\Phi_D^\dagger h \Psi_D \equiv (\varphi_{D,1}^*, \varphi_{D,2}^*, \varphi_{D,3}^*, \varphi_{D,4}^*) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_{D,1} \\ \psi_{D,2} \\ \psi_{D,3} \\ \psi_{D,4} \end{pmatrix}$$
$$\equiv \varphi_{D,1}^*\psi_{D,1} + \varphi_{D,2}^*\psi_{D,2} - \varphi_{D,3}^*\psi_{D,3} - \varphi_{D,4}^*\psi_{D,4} \tag{9}$$
$$\varphi_{D,m}, \psi_{D,m} \in \mathbb{C} \otimes \mathcal{F}, \qquad m = 1, 2, 3, 4$$

We recall that $\gamma_\mu \in \mathrm{End}(\mathcal{I}) \approx \mathrm{Mat}_4(\mathbb{C})$,

$$\gamma_\mu: \quad \mathcal{I} \to \mathcal{I}, \qquad \gamma_\mu \in \mathcal{I} \otimes \mathcal{I}^*$$

Under change of basis

$$\gamma_\mu \to \mathcal{S}\gamma_\mu\mathcal{S}^{-1}, \qquad h \to \mathcal{S}^\dagger h \mathcal{S}^*$$

Therefore, h and γ_μ are different tensors and the identification $h = \gamma_0$ is not correct.

To translate the Hermitian product (8), (9) into the $Cl_{1,3}^+$ formalism, we need to single out the conjugation which characterizes the standard Hermitian conjugate and impose an appropriate geometry. In order to translate $\Phi_D^\dagger$ into $Cl_{1,3}^+ \otimes \mathcal{F}$, we must determine the possible automorphisms (α) and antiautomorphisms (β) of $Cl_{1,3}^+$:

$$\alpha \in \mathrm{aut}(Cl_{1,3}^+ \otimes \mathcal{F}), \qquad \beta \in \mathrm{antiaut}(Cl_{1,3}^+ \otimes \mathcal{F})$$
$$\alpha(\Psi_H\Phi_H) = \alpha\,(\Psi_H)\alpha(\Phi_H), \qquad \beta(\Psi_H\Phi_H) = \beta(\Phi_H)\,\beta(\Psi_H)$$

We find the following:

α, grade involution:

$$\gamma_{0i} \to -\gamma_{0i}, \qquad \gamma_{ij} \to +\gamma_{ij}, \qquad \gamma_5 \to -\gamma_5$$
$$\alpha(\Phi_H) \equiv \hat{\Phi}_H = \varphi_{H,1} + \gamma_{31}\varphi_{H,2} - \gamma_5(\varphi_{H,3} + \gamma_{31}\varphi_{H,4})$$

β, reversion:

$$\gamma_{0i} \to +\gamma_{0i}, \qquad \gamma_{ij} \to -\gamma_{ij}, \qquad \gamma_5 \to -\gamma_5$$
$$\beta(\Phi_H) \equiv \tilde{\Phi}_H = \varphi_{H,1}^* - \varphi_{H,2}^*\gamma_{31} - \gamma_5(\varphi_{H,3}^* - \varphi_{H,4}^*\gamma_{31})$$

$\alpha \circ \beta$, Clifford conjugation:

$$\gamma_{0i} \to -\gamma_{0i}, \qquad \gamma_{ij} \to -\gamma_{ij}, \qquad \gamma_5 \to +\gamma_5$$

$$\alpha \circ \beta(\Phi_{\mathrm{H}}) \equiv \bar{\Phi}_{\mathrm{H}} = \varphi^*_{\mathrm{H},1} - \varphi^*_{\mathrm{H},2}\gamma_{31} + \gamma_5(\varphi^*_{\mathrm{H},3} - \varphi^*_{\mathrm{H},4}\gamma_{31})$$

$$i, j = 1, 2, 3, \quad i \neq j, \qquad \varphi_{\mathrm{H},m} \in \mathbb{D} \otimes \mathcal{F}, \quad m = 1, 2, 3, 4$$

The Hermitian sesquilinear form $\Phi^\dagger_{\mathrm{D}} h \Psi_{\mathrm{D}} \in \mathbb{C} \otimes \mathcal{F}$ can be translated by using the reversion and grade involution and adopting a $\mathbb{D}$-complex mapping (4),

$$\mathbb{C} \otimes \mathcal{F} \ni \Phi^\dagger_{\mathrm{D}} h \Psi_{\mathrm{D}} \leftrightarrow \chi(\tilde{\Phi}_{\mathrm{H}}\hat{\Psi}_{\mathrm{H}}) \equiv [\tilde{\Phi}_{\mathrm{H}}\hat{\Psi}_{\mathrm{H}}]_{\mathcal{F}} - \gamma_{21}\,[\gamma_{21}\tilde{\Phi}_{\mathrm{H}}\hat{\Psi}_{\mathrm{H}}]_{\mathcal{F}} \in \mathbb{D} \otimes \mathcal{F} \tag{10}$$

The $\mathbb{D}$-complex geometry, mapping on $\mathbb{D} \otimes \mathcal{F}$, is also justified by the following argument: We can define an anti-self-adjoint operator $\bar{\partial}$ with all the properties of a translation operator, but, by imposing noncomplex geometries, there is no corresponding self-adjoint operator with all the properties expected for a momentum operator [17]. The identification of i with the bivector γ_{21} gives us two possibilities for defining the momentum operator, respectively left and right action of the bivector γ_{21},

$$\rho(i\Psi_{\mathrm{D}}) = \gamma_{21}\Psi_{\mathrm{H}} \qquad \text{or} \qquad \rho(i\Psi_{\mathrm{D}}) = \Psi_{\mathrm{H}}\gamma_{21}, \qquad [\gamma_{21}, \Psi_{\mathrm{H}}] \neq 0$$

and thus we can define the following momentum operators:

$$-\gamma_{21}\bar{\bar{\partial}}\Psi_{\mathrm{H}} \quad \text{or} \quad -\bar{\bar{\partial}}\Psi_{\mathrm{H}}\gamma_{21}$$

By introducing the concept of left/right operators

$$\mathbb{O}^{l,r} \in \mathrm{End}\,(Cl^+_{1,3} \otimes \mathcal{F}), \qquad \mathbb{O}^l\Psi_{\mathrm{H}} \equiv \mathbb{O}\Psi_{\mathrm{H}},$$

$$\mathbb{O}^r\Psi_{\mathrm{H}} \equiv \Psi_{\mathrm{H}}\mathbb{O}, \qquad \Psi_{\mathrm{H}} \in Cl^+_{1,3} \otimes \mathcal{F}$$

necessary within noncommutative algebraic structures where we must distinguish between the left and right multiplication, we can express the momentum operator in $Cl^+_{1,3} \otimes \mathcal{F}$ as

$$-\gamma^l_{21}\,\bar{\bar{\partial}} \equiv -\gamma^l_{21} \otimes \bar{\bar{\partial}} \qquad \text{or} \qquad -\gamma^r_{21}\,\bar{\bar{\partial}} \equiv -\gamma^r_{21} \otimes \bar{\bar{\partial}}$$

In translating the Dirac equation, the first choice (γ_{21}-left action) must be rejected because such an operator, due to the term

$$\gamma^l_{01}\,\partial_{x_1} + \gamma^l_{02}\,\partial_{x_2} + \gamma^l_{03}\,\partial_{x_3} \quad \in \mathrm{End}\,(Cl^+_{1,3} \otimes \mathcal{F})$$

does not commute with the Dirac–Hestenes Hamiltonian $\mathcal{H}_{\mathrm{H}}$.

In the second case (γ_{21}-right action), the operator $\gamma^r_{21}\,\bar{\bar{\partial}}$ is real on the left, and thus commutes with $\mathcal{H}_{\mathrm{H}}$,

$$[\gamma_{21}^r \bar{\partial}, \mathcal{H}_H]\Psi_H \equiv \bar{\partial}(\mathcal{H}_H\Psi_H)\gamma_{21} - \mathcal{H}_H\bar{\partial}\Psi_H\gamma_{31} = 0$$

It remains to prove the hermiticity of the momentum operator, $-\gamma_{21}^r\, \bar{\partial}$. To do this we need to define an appropriate mapping for scalar products. If probability amplitudes are assumed to be element of nondivision algebras (in this case $Cl_{1,3}^+$), we cannot give a satisfactory probability interpretation [17].

It is seen that a $\mathbb{D}$-complex mapping (4),

$$\chi(\langle\Phi_H|\Psi_H\rangle) \equiv \left(\int d^3x\ \tilde{\Phi}_H\Psi_H\right)_{\mathbb{D}\otimes\mathcal{F}}$$

overcomes the previous problem and gives the required Hermiticity properties for the momentum operator

$$\chi(\langle\Phi_H|\bar{\partial}\Psi_H\gamma_{21}\rangle) = \chi(\langle\bar{\partial}\Phi_H\gamma_{21}|\Psi_H\rangle) \tag{11}$$

Equation (11) implies

$$\left(\int d^3x\ \tilde{\Phi}_H\bar{\partial}\Psi_H\right)_{\mathbb{D}\otimes\mathcal{F}}\gamma_{21} = -\gamma_{21}\left(\int d^3x\ \bar{\partial}\tilde{\Phi}_H\Psi_H\right)_{\mathbb{D}\otimes\mathcal{F}}$$

Now, to prove the Hermiticity of our momentum operator it is sufficient to perform integration by parts and use the $\mathbb{D}$-complex mapping.

We conclude this section by observing that there is a *difference* in translating complex operators and states. For example, the complex imaginary unit i can be interpreted as operator

$$i\mathbb{1}_4 \in \text{End}\ (\mathbb{C}^4)$$

or state

$$\begin{pmatrix} i\\0\\0\\0\end{pmatrix}, \begin{pmatrix} 0\\i\\0\\0\end{pmatrix}, \begin{pmatrix} 0\\0\\i\\0\end{pmatrix}, \begin{pmatrix} 0\\0\\0\\i\end{pmatrix} \in \mathcal{I}$$

The translation will be, respectively,

$$\gamma_{21}^r \in \text{End}\ (Cl_{1,3}^+)$$

or

$$\gamma_{21}, \qquad \gamma_{31}\gamma_{21} = \gamma_{32}, \qquad \gamma_5\gamma_{21} = \gamma_{03}, \qquad \gamma_5\gamma_{31}\gamma_{21} = \gamma_{01} \quad \in Cl_{1,3}^+$$

Let ρ^{End} be the endomorphism linear mapping

$$\rho^{\text{End}}: \quad \text{End}(\mathcal{I}) \rightarrow \text{End}(Cl_{1,3}^+) \tag{12}$$

We require

$$\rho(i\Psi_D) = \rho(\Psi_D i), \qquad [i, \Psi_D] = 0$$

The previous relation is satisfied because

$$\rho(i\Psi_D) = \rho^{\text{End}}(i\mathbb{1}_4)\rho(\Psi_D) = \gamma_{21}^r \, \Psi_H = \Psi_H\gamma_{21}$$

$$\rho(\Psi_D i) = \rho(\Psi'_D) = \Psi'_H = \Psi_H \, \gamma_{21}$$

2. DIRAC EQUATION

Once we have obtained the translation from the Dirac spinor field $\Psi_D \in \mathcal{I} \otimes \mathcal{F}$ to the Hestenes spinor field $\Psi_H \in Cl^+_{1,3} \otimes \mathcal{F}$, we can translate the standard complex Dirac equation into the even space-time subalgebra.

For convenience, we multiply the left- and right-hand sides of Eq. (2) by γ_0,

$$i(\partial_t + \gamma_0\gamma_k\partial_k)\,\Psi_D = m\gamma_0\Phi_D, \qquad k = 1, 2, 3 \tag{13}$$

We shall prove that this equation can be translated in the $Cl^+_{1,3}$ formalism; we take the ρ given by (7) as

$$\rho[i(\partial_t + \gamma_0\gamma_k\partial_k)\Psi_D] = m\rho(\gamma_0\Psi_D), \qquad k = 1, 2, 3 \tag{14}$$

In the previous section, we established the maps

$$\rho(\Psi_D) = \Psi_H \qquad \text{and} \qquad \rho(i\Psi_D) = \gamma_{21}^r\Psi_H \equiv \Psi_H\gamma_{21} \tag{15}$$

Thus to complete the translation of Eq. (13) in the $Cl^+_{1,3}$ formalism, it remains to calculate

$$\rho(\gamma_0\Psi_D) \qquad \text{and} \qquad \rho(\gamma_0\gamma_k\Psi_D), \quad k = 1, 2, 3$$

By using the explicit form of the Dirac matrices given in Eq. (3), we find

$$\gamma_0\gamma_1\Psi_D \equiv i\begin{pmatrix}\psi_{D,4}\\ \psi_{D,3}\\ -\psi_{D,2}\\ -\psi_{D,1}\end{pmatrix}, \qquad \gamma_0\gamma_2\Psi_D \equiv \begin{pmatrix}-\psi_{D,4}\\ \psi_{D,3}\\ \psi_{D,2}\\ -\psi_{D,1}\end{pmatrix}$$

$$\gamma_0\gamma_3\Psi_D \equiv i\begin{pmatrix}-\psi_{D,3}\\ \psi_{D,4}\\ \psi_{D,1}\\ -\psi_{D,2}\end{pmatrix}, \qquad \gamma_0\Psi_D \equiv \begin{pmatrix}\psi_{D,1}\\ \psi_{D,2}\\ -\psi_{D,3}\\ -\psi_{D,4}\end{pmatrix}$$

The task is to obtain their counterpart in $Cl^+_{1,3} \otimes \mathcal{F}$. The solution is

$$\gamma_{01}\Psi_H = \gamma_{21}\gamma_5\gamma_{31}\Psi_H \equiv [\psi_{H,4} + \gamma_{31}\psi_{H,3} - \gamma_5(\psi_{H,2} + \gamma_{31}\psi_{H,1})]\gamma_{21}$$
$$\gamma_{02}\Psi_H = -\gamma_5\gamma_{31}\Psi_H \equiv -\psi_{H,4} + \gamma_{31}\psi_{H,3} + \gamma_5(\psi_{H,2} - \gamma_{31}\psi_{H,1})$$
$$\gamma_{03}\Psi_H = \gamma_{21}\gamma_5\Psi_H \equiv [-\psi_{H,3} + \gamma_{31}\psi_{H,4} + \gamma_5(\psi_{H,1} - \gamma_{31}\psi_{H,1})]\gamma_{21}$$
$$\alpha(\Psi_H) = \hat{\Psi}_H \equiv \psi_{H,1} + \gamma_{31}\psi_{H,2} - \gamma_5(\psi_{H,3} + \gamma_{31}\psi_{H,4})$$

We now have all the tools needed to complete the translation of the Dirac equation in the $Cl^{+}_{1,3}$ formalism. The isomorphisms

$$\rho(\gamma_0\Psi_D) = \alpha(\Psi_H) = \hat{\Psi}_H \qquad \text{and} \qquad \rho\,(\gamma_0\gamma_k\Psi_D) = \gamma_{0k}\Psi_H, \quad k = 1, 2, 3$$

together with Eq. (15) allow us to write the $Cl^{+}_{1,3}$ counterpart of Eq. (13). Finally, the translated Dirac–Hestenes equation reads

$$(\partial_t + \gamma_{0k}\partial_k)\,\Psi_H\gamma_{21} = m\hat{\Psi}_H \quad \in Cl^{+}_{1,3} \otimes \mathcal{F}, \qquad k = 1, 2, 3 \tag{16}$$

The choice of the Dirac matrices (3) was *ad hoc* to obtain a simple translation for the complex Dirac matrices $\gamma_0\gamma_k$,

$$\rho(\gamma_0\gamma_k\Psi_D) = \gamma_{0k}\Psi_H, \qquad k = 1, 2, 3$$

What happens if we change our basis, $\gamma^{\text{new}}_\mu = \mathcal{S}\gamma_\mu\mathcal{S}^{-1}$? We shall show that it is possible to construct a set of translation rules which enables us to obtain for a generic 4×4 complex matrix its counterpart in the even space-time subalgebra. Thus, the problem concerning the translation of γ^{new}_μ is overcome.

A generic 4×4 complex matrix is characterized by 32 real elements, whereas $\dim_{\mathbb{R}} Cl^{+}_{1,3} = 8$, so it seems that we do not have the needed real freedom degrees to perform our translation. Nevertheless, we must observe that the space $Cl^{+}_{1,3} \otimes \mathcal{F}$ is a $Cl^{+}_{1,3}$-bimodule of the Hestenes spinors $\Psi_H \in Cl^{+}_{1,3} \otimes \mathcal{F}$. This implies, due to the noncommutativity of the Clifford algebra $Cl^{+}_{1,3}$, a left/right action on Ψ_H. So we must consider along with the standard eight left generators

$$1^l,\ \gamma^l_{01},\ \gamma^l_{02},\ \gamma^l_{03},\ \gamma^l_{21},\ \gamma^l_{31},\ \gamma^l_{23},\ \gamma^l_5 \in \text{End}\,(Cl^{+}_{1,3}) \tag{17}$$

the right generators

$$\gamma^r_{21},\ \gamma^r_{31},\ \gamma^r_{23} \in \text{End}\,(Cl^{+}_{1,3}) \tag{18}$$

It is not necessary to consider $\gamma^r_{01}, \gamma^r_{02}, \gamma^r_{03}$, because these operators can be obtained from the previous ones (18) by γ_5 multiplication, $[\gamma_5, Cl^{+}_{1,3}] = 0$. By using left, (17), and right, (18), generators we can write the operators

$$o^l_1 + o^l_2\gamma^r_{21} + o^l_3\gamma^r_{31} + o^l_4\gamma^r_{23}, \qquad o^l_m \in \text{End}\,(Cl^{+}_{1,3}), \quad m = 1, 2, 3, 4$$

characterized by 32 real parameters. *This does not imply necessarily the possibility of a translation*. In the standard Dirac theory, the operators are given in terms of 4×4 complex matrices and so represent i-complex linear operators

$$\mathbb{O}_{\mathrm{D}}\,[\Psi_{\mathrm{D}}(a + ib)] = (\mathbb{O}_{\mathrm{D}}\Psi_{\mathrm{D}})(a + ib), \qquad a, b \in \mathbb{R}$$

To perform our translation we must require a $\mathbb{D}$-complex linearity for our operators

$$\mathbb{O}_{\mathrm{H}}[\Psi_{\mathrm{H}}(a + ib)] = (\mathbb{O}_{\mathrm{H}}\,\Psi_{\mathrm{H}})(a + \gamma_{21}b)$$

This implies that the only acceptable right generator is γ^r_{21}. The problem is now the lack of 16 real degrees of freedom

$$\dim_{\mathbb{R}}(o^l_1 + o^l_2\gamma^r_{21}) = 16$$

The solution is achieved by recalling that in the Clifford algebra $Cl^+_{1,3}$, the grade involution $\alpha \in$ aut $(Cl^+_{1,3} \otimes \mathcal{F})$ represents a $\mathbb{D}$-complex linear operator

$$\begin{aligned}\alpha[\Psi_{\mathrm{H}}(a + \gamma_{21})] &= \alpha(\Psi_{\mathrm{H}})\alpha(a + \gamma_{21}b), \qquad \alpha(\gamma_{21}) = \gamma_{21}, \quad a, b \in \mathbb{R}\\ &= \alpha(\Psi_{\mathrm{H}})(a + \gamma_{21}b)\end{aligned}$$

To obtain the set of translation rules it is sufficient to give explicitly the matrix counterpart of the operators

$$1^l,\ \gamma^l_{21},\ \gamma^l_{31},\ \gamma^l_5,\ \gamma^r_{21},\ \alpha \in \mathrm{End}_{\mathbb{D}}(Cl^+_{1,3}) \tag{19}$$

The other operators can be found by suitable multiplications of the previous ones. It is evident that

$$1^l \leftrightarrow \mathbb{1}_4 \qquad \text{and} \qquad \gamma^r_{21} \leftrightarrow i\mathbb{1}_4 \tag{20}$$

A computation shows that

$$\begin{aligned}\gamma^l_{21} &\leftrightarrow \begin{pmatrix} i & 0 & 0 & 0\\ 0 & -i & 0 & 0\\ 0 & 0 & i & 0\\ 0 & 0 & 0 & -i \end{pmatrix}, &\quad \gamma^l_{31} &\leftrightarrow \begin{pmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0 \end{pmatrix}\\ \gamma^l_5 &\leftrightarrow \begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix}, &\quad \alpha &\leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}\end{aligned} \tag{21}$$

By using Eqs. (20)–(21) we can write the matrix counterpart for a generic left/right generator. For example,

$$\gamma^l_{01} = \gamma^l_{21}\gamma^l_{31}\gamma^l_5 \leftrightarrow \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^l_{02} = -\gamma^l_{31}\gamma^l_5 \leftrightarrow -\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^l_{03} = \gamma^l_{21}\gamma^l_5 \leftrightarrow \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

The complete set of translation rules is given in Appendix A.

3. THE DIRAC–HESTENES LAGRANGIAN

Our main objective in this work is to derive the Lagrangian $\mathscr{L}_{\mathrm{H}}$ that yields the Dirac–Hestenes equation

$$\mathscr{D}_+\Psi_{\mathrm{H}}\gamma_{21} = m\hat{\Psi}_{\mathrm{H}} \quad \in CL^+_{1,3} \otimes \mathscr{F}$$

$$\mathscr{D}_\pm \equiv \partial_t \pm \gamma^l_{0k}\partial_k, \qquad k = 1, 2, 3$$

We shall obtain the Dirac–Hestenes Lagrangian $\mathcal{L}_H$ by translation. To do that, let us start by considering the traditional form for the complex Dirac Lagrangian,

$$\mathcal{L}_D \equiv \Psi_D^{\dagger} h\, \Phi_D \in \mathbb{C} \otimes \mathcal{F}, \qquad \Phi_D \equiv (i\gamma^{\mu}\,\partial_{\mu} - m)\Psi_D \in \mathcal{I} \otimes \mathcal{F} \tag{23}$$

We showed in (10) that

$$\mathbb{C} \otimes \mathcal{F} \ni \Psi_D^{\dagger} h\, \Phi_D \leftrightarrow \chi(\tilde{\Psi}_H \hat{\Phi}_H) \in \mathbb{D} \otimes \mathcal{F}$$

Thus, to obtain the desired translation we need to calculate

$$\hat{\Phi}_H = \alpha(\Phi_H) = \rho(\gamma_0 \Phi_D) \quad \in Cl_{1,3}^{+} \otimes \mathcal{F}$$

By using the results presented in the previous section, we find

$$\rho(\gamma_0 \Phi_D) = \rho[(i\gamma_0\gamma^{\mu}\,\partial_{\mu} - m\gamma_0)\Psi_D] = \mathcal{D}_{+}\Psi_H - m\hat{\Psi}_H$$

and consequently

$$\mathbb{C} \otimes \mathcal{F} \ni \mathcal{L}_D \leftrightarrow \mathcal{L}_H \equiv \chi(\tilde{\Psi}_H \mathcal{D}_{+} \Psi_H \gamma_{21} - m\tilde{\Psi}_H \hat{\Psi}_H) \in \mathbb{D} \otimes \mathcal{F} \tag{24}$$

Let us now discuss the hermiticity of the Dirac–Hestenes Lagrangian $\mathcal{L}_H$. By applying the reversion involution to $\mathcal{L}_H$ we get

$$\tilde{\mathcal{L}}_H = \chi(-\gamma_{21}\tilde{\Psi}_H \overleftarrow{\mathcal{D}}_{+}\, \Psi_H - m\overline{\Psi}_H \Psi_H)$$

where $\overleftarrow{\mathcal{D}}_{+}$ indicates the left-action on $\tilde{\Psi}_H$ of the derivation which appears in the operator $\mathcal{D}_{+}$. By observing that

$$\chi(\overline{\Psi}_H \Psi_H) = \chi(\tilde{\Psi}_H \hat{\Psi}_H)$$

and performing integration by parts, we obtain

$$\tilde{\mathcal{L}}_H = \chi(\gamma_{21}\tilde{\Psi}_H \mathcal{D}_{+} \Psi_H - m\tilde{\Psi}_H \hat{\Psi}_H) \tag{25}$$

Due to the $\mathbb{D}$-complex geometry, the bivector γ_{21} can be removed from the extreme left to right, $\tilde{\Psi}_H \mathcal{D}_{+} \Psi_H$, in Eq. (25), and so the hermiticity of the Dirac–Hestenes Lagrangian is proved,

$$\mathcal{L}_H = \tilde{\mathcal{L}}_H$$

In order to formulate the variational principle within the algebraic formalism, let us rewrite Eq. (24) by using the projection operator

$$\mathrm{End}_{\mathbb{D}}(Cl_{1,3}^{+}) \ni \mathcal{P} \equiv \tfrac{1}{2}(1 - \gamma_{21}^{l}\gamma_{21}^{r})$$

and the grade involution α. The new expression for the Dirac–Hestenes Lagrangian reads

$$\mathscr{L}_{\mathrm{H}} = \tfrac{1}{2}\{\mathscr{P}(\tilde{\Psi}_{\mathrm{H}}\mathscr{D}_{+}\Psi_{\mathrm{H}}\gamma_{21} - m\tilde{\Psi}_{\mathrm{H}}\hat{\Psi}_{\mathrm{H}}) + \alpha[\mathscr{P}(\tilde{\Psi}_{\mathrm{H}}\mathscr{D}_{+}\Psi_{\mathrm{H}}\gamma_{21} - m\tilde{\Psi}_{\mathrm{H}}\hat{\Psi}_{\mathrm{H}})]\} \quad \in \mathbb{D} \otimes \mathscr{F}$$

or by making explicit the action of the $\mathscr{P}$-operator and α-involution,

$$\begin{aligned}\mathscr{L}_{\mathrm{H}} = \tfrac{1}{4}(&\tilde{\Psi}_{\mathrm{H}}\,\mathscr{D}_{+}\Psi_{\mathrm{H}}\gamma_{21} - m\tilde{\Psi}_{\mathrm{H}}\hat{\Psi}_{\mathrm{H}} \\ &+ \gamma_{21}\tilde{\Psi}_{\mathrm{H}}\mathscr{D}_{+}\Psi_{\mathrm{H}} + m\gamma_{21}\tilde{\Psi}_{\mathrm{H}}\hat{\Psi}_{\mathrm{H}}\gamma_{21} \\ &+ \overline{\Psi}_{\mathrm{H}}\mathscr{D}_{-}\hat{\Psi}_{\mathrm{H}}\gamma_{21} - m\overline{\Psi}_{\mathrm{H}}\Psi_{\mathrm{H}} \\ &+ \gamma_{21}\overline{\Psi}_{\mathrm{H}}\mathscr{D}_{-}\hat{\Psi}_{\mathrm{H}} + m\gamma_{21}\overline{\Psi}_{\mathrm{H}}\Psi_{\mathrm{H}}\gamma_{21})\end{aligned} \tag{26}$$

It is here that appeal to the variational principle must be made. A variation $\delta\Psi_{\mathrm{H}}$ in Ψ_{H} from Eq. (26) cannot be brought to the extreme right because of the bivector γ_{21} in the first term of the previous expression. The only consistent procedure is to generalize the variational rule that says that Ψ_{H} and $\overline{\Psi}_{\mathrm{H}}$ must be varied *independently* [18]. We thus apply independent variations to

$$\Psi_{\mathrm{H}}, \quad \Psi_{\mathrm{H}}\gamma_{21}, \quad \hat{\Psi}_{\mathrm{H}}, \quad \hat{\Psi}_{\mathrm{H}}\gamma_{21} \tag{27}$$

and

$$\bar{\Psi}_{\mathrm{H}}, \quad \gamma_{21}\overline{\Psi}_{\mathrm{H}}, \quad \tilde{\Psi}_{\mathrm{H}}, \quad \gamma_{21}\tilde{\Psi}_{\mathrm{H}} \tag{28}$$

This generalization of the variational principle is discussed in Appendix B. The variations applied to fields (27) field the adjoint Dirac–Hestenes equation

$$-\gamma_{21}\tilde{\Psi}_{\mathrm{H}}\overleftarrow{\mathscr{D}}_{+} = m\overline{\Psi}_{\mathrm{H}} \tag{29}$$

whereas applying thems to fields (28) yields the Dirac–Hestenes equation

$$\mathscr{D}_{+}\Psi_{\mathrm{H}}\gamma_{21} = m\tilde{\Psi}_{\mathrm{H}} \tag{30}$$

Let us discuss an interesting point. The Dirac–Hestenes Lagrangian (24) is $\mathbb{D}$-complex and Hermitian. The situation is more subtle with a classical field, for now $\mathscr{L}_{\mathrm{H}}^{\mathrm{new}}$, defined by

$$\mathscr{L}_{\mathrm{H}}^{\mathrm{new}} = \tfrac{1}{2}(\tilde{\Psi}_{\mathrm{H}}\mathscr{D}_{+}\Psi_{\mathrm{H}}\gamma_{21} + \gamma_{21}\tilde{\Psi}_{\mathrm{H}}\mathscr{D}_{+}\Psi_{\mathrm{H}} - m\tilde{\Psi}_{\mathrm{H}}\hat{\Psi}_{\mathrm{H}} - m\bar{\Psi}_{\mathrm{H}}\Psi_{\mathrm{H}}) \quad \in \mathscr{F}$$

is both Hermitian and *real*. Thus it may be objected that the complex projection in the previous classical Lagrangian is superfluous. For $\mathscr{L}_{\mathrm{H}}^{\mathrm{new}}$ itself this true, but for multivectorial algebraic variations in the fields, $\delta\Psi_{\mathrm{H}}$, etc., a difference exists. The variation $\delta\mathscr{L}_{\mathrm{H}}^{\mathrm{new}} \in Cl_{1,3}^{+} \otimes \mathscr{F}$, while $\delta\mathscr{L}_{\mathrm{H}}$ from (24) is always $\mathbb{D}$-complex. Furthermore, $\mathscr{L}_{\mathrm{H}}^{\mathrm{new}}$ does not yield the correct field equation through the variational principle unless we limit $\delta\Psi_H$, etc., to $\mathbb{D}$-complex variations notwithstanding $\Psi_H \in Cl_{1,3}^+ \otimes \mathcal{F}$. We consider this latter option unjustified and thus select for the formal structure of the classical Lagrangian that of Eq. (24). Let us summarize the situation concerning fields and variations:

$$\Psi_H,\ \hat{\Psi}_H,\ \check{\Psi}_H,\ \overline{\Psi}_H,\ \tilde{\Psi}_H \in Cl_{1,3}^+ \otimes \mathcal{F}$$

$$\delta(\Psi_H),\ \delta(\Psi_H\gamma_{21}),\ \delta(\hat{\Psi}_H),\ \delta(\hat{\Psi}_H\ \gamma_{21}) \in Cl_{1,3}^+ \otimes \mathcal{F}$$

$$\delta(\overline{\Psi}_H),\ \delta(\gamma_{21}\overline{\Psi}_H),\ \delta(\tilde{\Psi}_H),\ \delta(\gamma_{21}\tilde{\Psi}_H) \in Cl_{1,3}^+ \otimes \mathcal{F}$$

$$\mathcal{L}_H \in \mathbb{D} \otimes \mathcal{F}$$

$$\delta\mathcal{L}_H \in \mathbb{D} \otimes \mathcal{F}$$

We conclude this section by discussing an alternative way to obtain the field equations from the Dirac–Hestenes Lagrangian. Let us rewrite the α-involution by using the operator $\gamma_0^l\gamma_0^r$,

$$\alpha(\Psi_H) \equiv \hat{\Psi}_H = \gamma_0\Psi_H\gamma_0 \quad \in Cl_{1,3}^+ \otimes \mathcal{F}$$

By adopting this notation we can express the Dirac–Hestenes Lagrangian as

$$\mathcal{L}_H = \mathcal{P}\mathcal{P}_\alpha[\tilde{\Psi}_H\mathcal{D}_+\Psi_H\gamma_{21} - m\tilde{\Psi}_H\gamma_0\Psi_H\gamma_0] \quad \in \mathbb{D} \otimes \mathcal{F} \tag{31}$$

where

$$\mathcal{P}_\alpha \equiv \tfrac{1}{2}(1 + \gamma_0^l\gamma_0^r), \qquad [\mathcal{P}, \mathcal{P}_\alpha] = 0$$

By making the variation

$$\Psi_H \to \Psi_H + \delta\Psi_H \tag{32}$$

we can put $\delta\Psi_H$ on the extreme right because, due to our mapping on $\mathbb{D} \otimes \mathcal{F}$, we can bring γ_{21} and γ_0 from the extreme right to left in Eq. (31). In fact,

$$\mathcal{P}(\mathcal{A}\gamma_{21}) = \mathcal{P}(\gamma_{21}\mathcal{A})$$

$$\mathcal{P}_\alpha(\mathcal{A}\gamma_0) = \mathcal{P}_\alpha(\gamma_0\mathcal{A})$$

with

$$\mathcal{A} \in Cl_{1,3} \otimes \mathcal{F}$$

The variation (32) implies

$$\delta\mathcal{L}_H = \mathcal{P}\mathcal{P}_\alpha[\tilde{\Psi}_H\mathcal{D}_+\delta\Psi_H\gamma_{21} - m\tilde{\Psi}_H\gamma_0\delta\Psi_H\gamma_0]$$

which, after integration by parts and by moving γ_{21} and γ_0 from the extreme right to left, becomes

$$\delta\mathcal{L}_{\rm H} = \mathcal{P}\mathcal{P}_\alpha\,[-\gamma_{21}\tilde{\Psi}_{\rm H}\overleftarrow{\mathcal{D}}_+\delta\Psi_{\rm H} - m\gamma_0\tilde{\Psi}_{\rm H}\gamma_0\delta\Psi_{\rm H}]$$

Finally, $\delta\mathcal{L}_{\rm H} = 0$ implies

$$-\gamma_{21}\tilde{\Psi}_{\rm H}\overleftarrow{\mathcal{D}}_+ = m\gamma_0\tilde{\Psi}_{\rm H}\gamma_0$$

and so we obtain the adjoint Dirac–Hestenes equation (29), as required.

4. THE INVARIANCE GROUP OF $\mathcal{L}_{\rm H}$

Having obtained the Dirac–Hestenes Lagrangian in the previous section, we may ask which global group leaves this Lagrangian invariant. Remembering that $\Psi_{\rm H} \in Cl_{1,3}^+ \otimes \mathcal{F}$, we have that the most general $\mathbb{D}$-complex linear transformation on $\Psi_{\rm H}$ is given by

$$\Psi_{\rm H} \to (A^l + B^l\gamma_{21}^r + C^l\gamma_0^l\gamma_0^r + D^l\gamma_{21}^r\gamma_0^l\gamma_0^r)\Psi_{\rm H} \quad \in Cl_{1,3}^+ \otimes \mathcal{F} \tag{33}$$

with

$$A^l,\ B^l,\ C^l,\ D^l \in Cl_{1,3}^{+(l)}$$

Now, the algebraic structure of the Dirac operator $\mathcal{D}_+$ strongly limits the left action on $\Psi_{\rm H}$; this leads to the conclusion that

$$A^l = a\cdot 1^l, \qquad B^l = b\cdot 1^l, \qquad C^l = 0, \qquad D^l = 0, \qquad a, b \in \mathbb{R}$$

So Eq. (33) will be modified as

$$\Psi_{\rm H} \to (a\cdot 1^l + b\cdot\gamma_{21}^r)\Psi_{\rm H} \equiv \Psi_{\rm H}(a + \gamma_{21}b) \tag{34}$$

and consequently

$$\hat{\Psi}_{\rm H} \to \hat{\Psi}_{\rm H}(a + \gamma_{21}b)$$
$$\tilde{\Psi}_{\rm H} \to (a - \gamma_{21}b)\tilde{\Psi}_{\rm H} \tag{35}$$

Applying the global transformations (34)–(35), we find that the Dirac–Hestenes Lagrangian becomes

$$\mathcal{L}'_{\rm H} \equiv \chi[z^*(\tilde{\Psi}_{\rm H}\mathcal{D}_+\Psi_{\rm H}\gamma_{21} - m\tilde{\Psi}_{\rm H}\hat{\Psi}_{\rm H})z]$$
$$\equiv z^*z\chi(\tilde{\Psi}_{\rm H}\mathcal{D}_+\Psi_{\rm H}\gamma_{21} - m\tilde{\Psi}_{\rm H}\hat{\Psi}_{\rm H}), \qquad z, z^* \in \mathbb{D}$$

Thus by requiring $z^*z = 1$, we find that the only invariance group is defined by

$$U(1,\ \gamma_{21}^r)$$

where the previous notation means the right action of the $\mathbb{D}$-complex unitary group on the algebraic spinor $\Psi_{\rm H}$,

$$\Psi_{\mathrm{H}} \to e^{\gamma_{21}^{r}\delta}\,\Psi_{\mathrm{H}} \equiv \Psi_{\mathrm{H}} e^{\gamma_{21}\delta}, \qquad \delta \in \mathbb{R} \tag{36}$$

Remembering that the Glashow group [19] for the Salam–Weinberg theory [20, 21] is $SU(2) \otimes U(1)$, we observe that this $U(1)$ group may be identified with our $U(1, \gamma_{21}^{r})$, and our field $\Psi_{\mathrm{H}} \in Cl_{1,3}^{+} \otimes \mathcal{F}$ must necessarily be a singlet (scalar) under $SU(2)$.

The interesting feature is what happens if we select a field in the full space–time algebra $Cl_{1,3} \otimes \mathcal{F}$. Now the number of fermionic particles is two,

$$\Psi_{\mathrm{H}}^{(1)} + \Psi_{\mathrm{H}}^{(2)}\gamma_0, \qquad \Psi_{\mathrm{H}}^{(1,2)} \in Cl_{1,3}^{+} \otimes \mathcal{F}$$

For example the leptons of the first family (electronic neutrino ν_e, electron e) can be concisely rewritten in $Cl_{1,3} \otimes \mathcal{F}$ as

$$\Psi_{\mathrm{Lep}}^{(\mathrm{1st\ fam})} = \Psi_{\mathrm{H}}^{(\nu_e)} + \Psi_{\mathrm{H}}^{(e)}\,\gamma_0, \qquad \Psi_{\mathrm{H}}^{(\nu_e, e)} \in Cl_{1,3}^{+} \otimes \mathcal{F} \tag{37}$$

The orthogonality of the fields $\Psi_{\mathrm{H}}^{(\nu_e)}$, $\Psi_{\mathrm{H}}^{(e)}\gamma_0$ is guaranteed by our $\mathbb{D}$-complex mapping,

$$\chi(\tilde{\Phi}\hat{\Psi}\gamma_0) = 0, \qquad \Phi, \Psi \in Cl_{1,3}^{+} \otimes \mathcal{F}$$

Now it is still not obvious, due to the presence of the Dirac operator $\mathcal{D}_{+}$, that an invariance group isomorphic to $SU(2)$ exists. We remark that to obtain a global invariance isomorphic to $SU(2)$ we must choose suitable combinations of

$$\gamma_5^{l,r}, \quad \gamma_0^{r}, \quad \gamma_{21}^{r}$$

These operators satisfy

$$[\gamma_5^{l}, \mathcal{D}_{+}] = 0$$

and

$$\chi([\mathcal{A}, \gamma_5]) = \chi([\mathcal{A}, \gamma_0]) = \chi([\mathcal{A}, \gamma_{21}]) = 0, \qquad \mathcal{A} \in Cl_{1,3} \otimes \mathcal{F}$$

Consequently, the infinitesimal transformation

$$\begin{aligned} \Psi_{\mathrm{Lep}} \to (1 &+ \alpha_1\gamma_0^{r}\gamma_{21}^{r} + \alpha_2\gamma_5^{l}\gamma_5^{r}\gamma_0^{r} + \alpha_3\gamma_5^{l}\gamma_5^{r}\gamma_{21}^{r} \\ &+ \beta\gamma_{21}^{r})\,\Psi_{\mathrm{Lep}}, \qquad 1 \gg \alpha_{1,2,3}, \quad \beta \in \mathbb{R} \end{aligned}$$

leaves invariant the zero-mass Lagrangian

$$\mathcal{L}_{\mathrm{Lep}} \equiv \chi(\tilde{\Psi}_{\mathrm{Lep}}\mathcal{D}_{+}\Psi_{\mathrm{Lep}}\gamma_{21}) \quad \in \mathbb{D} \otimes \mathcal{F} \tag{38}$$

The zero-mass fields will gain mass by spontaneous symmetry breaking [22, 23].

The anti-Hermitian generators

$$\gamma_0^r \gamma_{21}^r, \quad \gamma_5^l \gamma_5^r \gamma_0^r, \quad \gamma_5^l \gamma_5^r \gamma_{21}^r, \quad \gamma_{21}^r$$

represent the multivectorial $Cl_{1,3}$ counterpart of the generators of the standard (complex) Glashow group

$$SU(2) \otimes U(1)$$

5. CONCLUSIONS

We begin our discussion with the last results of the last section. We have shown that by working within a multivectorial formalism it is possible to impose a Glashow group invariance and that this occurs by merely adopting $Cl_{1,3}$ fields. Our viewpoint is that the $SU(2) \otimes U(1)$ invariance in particle physics could be better understood by working in the $Cl_{1,3}$ formalism, where each element is suitable to *geometric interpretation*. For example, a better understanding of the geometric meaning of the generators of the invariance Glashow group could be very important in reaching grand unification groups. The adoption of a $\mathbb{D}$-complex geometry represents a fundamental ingredient of the multivectorial algebraic approach to quantum mechanics. Such a mapping gives the desired *electromagnetic invariance* $U(1, \gamma_{21}^r)$ and suggests an invariance group isomorphic to the Glashow group. By passing from $Cl_{1,3}^+$ to $Cl_{1,3}$ fields, the $\mathbb{D}$-complex geometry guarantees the right orthogonality between electron and neutrino field and gives the possibility to find four $Cl_{1,3}^{(l/r)}$ elements which are isomorphic to the generators of the electroweak group $SU(2) \otimes U(1)$. A complete discussion on the Salam–Weinberg model in the multivectorial formalism will be presented in a forthcoming paper [24].

Let us recall the other result of this paper. We discussed and generalized the application of the variational principle to Lagrangians with $Cl_{1,3}^+$ fields. In order to obtain the Dirac–Hestenes equation we proved the need to adopt a $\mathbb{D}$-complex mapping for our Lagrangians or apply, due to the noncommutative nature of the Clifford algebras, different variations for the fields Ψ_H, $\Psi_H \gamma_{21}$, etc.

We also recall the possibility to perform a *translation* between 4×4 complex matrices and left/right elements of the even space-time subalgebra. This allows an immediate translation of the Dirac equation in the multivectorial formalism. Obviously this approach can be used to reproduce other standard results of quantum mechanics. We conclude by emphasizing that this translation represents only a *partial translation*, for example, it does not apply to odd-dimensional complex matrices. Different outputs can be obtained by working with Clifford algebras. *New geometric interpretations* naturally appear in the space-time algebraic approach and this could be very useful in reaching fundamental symmetries in unification Lagrangians.

APPENDIX A. TRANSLATION RULES

Let us define the projectors

$$\alpha_\pm \equiv \tfrac{1}{2}(\mathrm{id} \pm \alpha) \in \mathrm{End}_{\mathbb{D}}(Cl^+_{1,3})$$

The 16 linear independent 4 × 4 matrices have the following counterparts in the even space-time subalgebra:

$$\alpha_+ \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^l_{21}\gamma^r_{21}\alpha_+ \leftrightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^l_{23}\gamma^r_{21}\alpha_+ \leftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^l_{31}\alpha_+ \leftrightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\alpha_- \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma^l_{21}\gamma^r_{21}\alpha_- \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\gamma^l_{23}\gamma^r_{21}\alpha_- \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \gamma^l_{31}\alpha_- \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$-\gamma^l_5\alpha_+ \leftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^l_{03}\gamma^r_{21}\alpha_+ \leftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$-\gamma^l_{01}\gamma^r_{21}\alpha_+ \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^l_{02}\alpha_+ \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^l_5\alpha_- \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma^l_{03}\gamma^r_{21}\alpha_- \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\gamma^l_{01}\gamma^r_{21}\alpha_- \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^l_{02}\alpha_- \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

The remaining 16 "complex" matrices are obtained by $\gamma^r_{21} \leftrightarrow i\mathbb{1}_4$ multiplication. The 16 operators

$$(1^l,\ \gamma^l_{21}\gamma^r_{21},\ \gamma^l_{23}\gamma^r_{21},\ \gamma^l_{31},\ \gamma^l_5,\ \gamma^l_{03}\gamma^r_{21},\ \gamma^l_{02},\ \gamma^l_{01}\gamma^r_{21})\alpha_\pm \quad \in \mathrm{End}_{\mathbb{D}}(Cl^+_{1,3})$$

are $\mathbb{D}$-complex linearly independent,

$$\dim_{\mathbb{D}} Cl^{+(l/r)}_{1,3} = \dim_{\mathbb{C}} \mathrm{Mat}_4(\mathbb{C}) = 16$$

The proof is based on the *i*-complex linear independence of the listed 4×4 real matrices.

APPENDIX B. VARIATIONAL PRINCIPLE

Consider one of the simplest of all particle Lagrangian densities, that for two classical scalar fields, $\varphi_{1,2} \in \mathcal{F}$, without interactions

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 - \frac{m^2}{2}\varphi_1^2 + \tfrac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 - \frac{m^2}{2}\varphi_2^2 \tag{B1}$$

$$\equiv \partial_\mu\phi^\dagger\,\partial^\mu\phi - m^2\phi^\dagger\phi$$

where

$$\phi \equiv \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) \in \mathbb{C} \otimes \mathcal{F}, \qquad \phi^\dagger \equiv \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2) \in \mathbb{C} \otimes \mathcal{F}$$

The corresponding Euler–Lagrange equations are

$$(\partial_\mu\partial^\mu + m^2)\,\varphi_{1,2} = 0 \tag{B2}$$

or, equivalently,

$$(\partial_\mu\partial^\mu + m^2)\phi = 0 \tag{B3}$$

Now to obtain "directly" the last equation one performs very particular variations of ϕ and $\phi^\dagger$,

$$\phi \to \phi, \qquad \phi^\dagger \to \phi^\dagger + \delta\phi^\dagger \tag{B4}$$

i.e., in order to obtain the corresponding Euler–Lagrangian equation one treats ϕ and $\phi^\dagger$ as *independent* fields. In second quantization these fields indeed contain independent creation and annihilation operators corresponding to positive and negative charged particles. To satisfy Eq. (B4) we must necessarily have

$$\delta\varphi_1 + i\delta\varphi_2 = 0 \tag{B5}$$

and this means that the *variations* in the originally real $\varphi_{1,2}$ fields are complex (if $\delta\varphi_1$ is real, then $\delta\varphi_2$ is pure imaginary, etc.).

In this Appendix we generalize the variational rule given for "complex" fields. Let $\Psi \in Cl^+_{1,3} \otimes \mathcal{F}$ be expressed by

$$\Psi = \psi_0 + \gamma_{01}\psi_1 + \gamma_{02}\psi_2 + \gamma_{03}\psi_3 + \gamma_{21}\psi_4 + \gamma_{31}\psi_5 + \gamma_{23}\psi_6 + \gamma_5\psi_7, \qquad \psi_{0,\ldots,7} \in \mathcal{F}$$

As shown in the Introduction, we can define the involutions

$$\hat{\Psi} = \psi_0 - \gamma_{01}\psi_1 - \gamma_{02}\psi_2 - \gamma_{03}\psi_3 + \gamma_{21}\psi_4 + \gamma_{31}\psi_5 + \gamma_{23}\psi_6 - \gamma_5\psi_7$$

$$\tilde{\Psi} = \psi_0 + \gamma_{01}\psi_1 + \gamma_{02}\psi_2 + \gamma_{03}\psi_3 - \gamma_{21}\psi_4 - \gamma_{31}\psi_5 - \gamma_{23}\psi_6 - \gamma_5\psi_7$$

$$\overline{\Psi} = \psi_0 - \gamma_{01}\psi_1 - \gamma_{02}\psi_2 - \gamma_{03}\psi_3 - \gamma_{21}\psi_4 - \gamma_{31}\psi_5 - \gamma_{23}\psi_6 + \gamma_5\psi_7$$

The complex variational principle which treats Φ and $\Phi^\dagger$ as independent fields is now generalized by applying different variations to Ψ, $\hat{\Psi}$, $\tilde{\Psi}$, $\overline{\Psi}$. Nevertheless, by working within the noncommutative algebra $Cl^+_{1,3}$ we must also analyze the following fields:

$$-\gamma_{21}\Psi\gamma_{21}, \quad -\gamma_{31}\Psi\gamma_{31}, \quad -\gamma_{23}\Psi\gamma_{23}$$

$$-\gamma_{21}\hat{\Psi}\gamma_{21}, \quad -\gamma_{31}\hat{\Psi}\gamma_{31}, \quad -\gamma_{23}\hat{\Psi}\gamma_{23}$$

$$-\gamma_{21}\tilde{\Psi}\gamma_{21}, \quad -\gamma_{31}\tilde{\Psi}\gamma_{31}, \quad -\gamma_{23}\tilde{\Psi}\gamma_{23}$$

In fact, we can treat Ψ and $\Phi = -\gamma_{21}\Psi\gamma_{21}$ as independent fields

$$\Psi \to \Psi, \qquad \Phi \to \Phi + \delta\Phi$$

The previous equation is satisfied by requiring

$$\delta\psi_0 + \gamma_{01}\delta\psi_1 + \gamma_{02}\delta\psi_2 + \gamma_{03}\delta\psi_3 + \gamma_{21}\delta\psi_4 + \gamma_{31}\delta\psi_5 + \gamma_{23}\delta\psi_6 + \gamma_5\delta\psi_7 = 0$$

and this means that the variations in the originally real fields $\psi_{0,\ldots,7}$ are in $Cl^+_{1,3}$. In conclusion, we must apply different variations to the fields

$$\Psi, \quad \hat{\Psi}, \quad \tilde{\Psi}, \quad \bar{\Psi}, \quad \Phi, \quad \hat{\Phi}, \quad \tilde{\Phi}, \quad \overline{\Phi}$$

which appear in the Dirac–Hestenes Lagrangian (26),

$$\mathcal{L}_{\mathrm{H}} = \tfrac{1}{4}(\tilde{\Psi}_{\mathrm{H}}\mathcal{D}_+\gamma_{21}\Phi_{\mathrm{H}} - m\tilde{\Psi}_{\mathrm{H}}\hat{\Psi}_{\mathrm{H}} + \tilde{\Phi}_{\mathrm{H}}\gamma_{21}\mathcal{D}_+\Psi_{\mathrm{H}} - m\tilde{\Phi}_{\mathrm{H}}\hat{\Phi}_{\mathrm{H}}$$

$$+ \bar{\Psi}_H \mathcal{D}_- \gamma_{21} \hat{\Phi}_H - m\overline{\Psi}_H \Psi_H$$
$$+ \bar{\Phi}_H \gamma_{21} \mathcal{D}_- \hat{\Psi}_H - m\overline{\Phi}_H \Phi_H) \qquad \text{(B6)}$$

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