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Toward an Octonionic World

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In order to obtain a consistent formulation of octonionic quantum mechanics (OQM), we introduce left/right-barred operators. Such operators enable us to find the translation rules between octonionic numbers and 8×8 real matrices (a translation is also given for 4×4 complex matrices). The use of a complex geometry allows us to overcome the hermiticity problem and define an appropriate momentum operator within OQM. As an application of our results, we develop an octonionic relativistic free wave equation, linear in the derivatives. Even if the wave functions are only one-component, we show that four independent solutions, corresponding to those of the Dirac equation, exist.

1. INTRODUCTION

In the early 1930s, in order to explain the novel phenomena of that time, namely β-decay and the strong interactions, Jordan[(1)] introduced a nonassociative but commutative algebra as a basic building block for a new quantum theory. With the discovery that 3×3 hermitian octonionic matrices realize the Jordan postulate[(2,3)] octonions appeared in quantum mechanics for the first time. The hope of applying nonassociative algebras to physics was soon dashed with the Fermi theory of the β-decay and with the Yukawa model of nuclear forces. Octonions disappeared from physics soon after being introduced. Banished from physics, octonions continued their career in mathematics.[(4–7)] Semisimple Lie groups, classified into four categories—orthogonal groups, unitary groups, symplectic groups, and exceptional groups—were respectively associated with real, complex, quaternionic, and octonionic algebras. Thus, such algebras became the core of the classification of possible symmetries in physics.

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From the 1960s onward, there has been renewed and intense interest in the use of octonions in physics.[8] The octonionic algebra has been in fact linked with a number of interesting subjects: structure of interactions,[9] $SU(3)$ color symmetry and quark confinement,[10,11] standard model gauge group,[12] exceptional GUT groups,[13] Dirac–Clifford algebra,[14] nonassociative Yang–Mills theories,[15,16] space-time symmetries in ten dimensions,[17] and supersymmetry and supergravity theories.[18,19] Moreover, the recent successful application of quaternionic numbers in quantum mechanics,[20–24] in particular in formulating a quaternionic Dirac equation,[25–28] suggests going one step further and using octonions as underlying numerical field. Nonassociative numbers are difficult to manipulate and so the use of the octonionic field within octonionic quantum mechanics (OQM) and in particular in formulating the Dirac equation[29],[3] is nontrivial. Obviously, if we are not able to construct a suitable OQM, octonions will remain beautiful ghosts in search of a physical incarnation.

In this work, we overcome the problems due to the nonassociativity of the octonionic algebra by introducing left-right barred operators (which will be sometimes called generalized octonions). Such operators complete the mathematical material introduced in recent papers (on octonionic representations and nonassociative gauge theories) of Joshi *et al.*[15,16] Then we investigate their relations to $GL(8, \mathcal{R})$ and $GL(4, \mathcal{C})$. Establishing this relation, we find interesting translation rules, which gives us the opportunity to formulate a consistent OQM. Both the postulates of quantum mechanics and the octonion nonassociativity property will be respected.

The philosophy behind the translation can be concisely expressed by the following sentence: There exists at least one version of octonionic quantum mechanics where the standard quantum mechanics is reproduced. The use of a complex scalar product[30] (or complex geometry as called by Rembieliński[31]) will be the main tool to obtain such an OQM.

Is there any other acceptable octonionic quantum theory? Do octonionic quantum theories necessitate complex geometry? At this stage these questions lack answers and the aim of our work is to clarify these points.

We wish to stress that translation rules do not imply that our octonionic quantum world (with complex geometry) is equivalent to the standard quantum world. When translation fails, the two worlds are not equivalent. An interesting case is supersymmetry. Since the number of spinor components will be reduced from four to one, the number of degrees of freedom between

[3] In the literature we find a Dirac equation formulation by *complexified* ocotonions with an embarrassing doubling of solutions: "the wave function ψ is not a column matrix, but must be taken as an octonion. ψ therefore consists of eight wave functions, rather than the four wave functions of the Dirac equation."[29]

bosonic and fermionic fields matches. So we need just one fermion and one boson without any auxiliary field.

Similar translation rules, between quaternionic quantum mechanics (QQM) with complex geometry and standard quantum mechanics, have been recently found.[23] As an application, such rules can be exploited in reformulating in a natural way the electroweak sector of the standard model.[24]

This paper is organized as follows: In Section 2 we give a brief introduction to the octonionic division algebra. In Section 3 we discuss generalized numbers and introduce barred operators. Working with nonassociative numbers, we need to distinguish between left-barred and right-barred operators. In Section 4 we investigate the relation between generalized octonions and 8×8 real matrices. In this section, we also give the translation rules between octonionic barred operators and $GL(4, \mathcal{C})$, which will be very useful in formulating our OQM. After these mathematical sections, in Section 5 we show how the complex geometry allows us to overcome the hermiticity problem. In this section we also introduce the appropriate definition for the momentum operator (which satisfies the required commutation rules with our octonionic Hamiltonian) and the new completeness relations. As application of our results, in Section 6 we explicitly develop an octonionic Dirac equation and suggest possible differences between complex and octonionic quantum theories. Our conclusions are drawn in the final section.

2. OCTONIONIC ALGEBRA

A remarkable theorem of Albert[32] shows that the only algebras $\mathcal{A}$ over the reals with unit element and admitting a real modulus function $N(a)$ ($a \in \mathcal{A}$) with the properties

$$N(0) = 0 \tag{1a}$$

$$N(a) > 0 \qquad \text{if} \quad a \neq 0 \tag{1b}$$

$$N(ra) = |r| N(a) \qquad (r \in \mathcal{R}) \tag{1c}$$

$$N(a_1 + a_2) \leq N(a_1) + N(a_2) \tag{1d}$$

are the reals $\mathcal{R}$, the complex $\mathcal{C}$, the quaternions $\mathcal{H}$ (in honor of Hamilton[33]), and the octonions $\mathbb{O}$ (or Graves–Cayley numbers[34,35]). Albert's theorem generalizes famous nineteenth-century results of Frobenius[36] and Hurwitz,[37] who first reached the same conclusion, but with the additional assumption that $N(a)^2$ is a quadratic form.

In addition to Albert's theorem on algebras admitting a modulus function $N(a)$, we can characterize the algebras $\mathcal{R}$, $\mathcal{C}$, $\mathcal{H}$, and $\mathbb{O}$ by the concept of *division algebra* (in which one has no nonzero divisors of zero). A classical theorem[38,39] states that the only division algebra over the reals are algebras of dimensions 1, 2, 4, and 8, the only associative division algebras over the reals are $\mathcal{R}$, $\mathcal{C}$, and $\mathcal{H}$, whereas the *nonassociative* algebras include the octonions $\mathbb{O}$ (an interesting discussion concerning nonassociative algebras is presented in ref. 40). For a very nice review of aspects of the quaternionic and octonionic algebras see ref. 8 and the recent book of Adler.[20] In this paper we will deal with octonions and their generalizations.

We now summarize our notation for the octonionic algebra and introduce useful elementary properties to manipulate the nonassociative numbers. There are a number of equivalent ways to represent the octonion multiplication table. Fortunately, it is always possible to choose an orthonormal basis ($e_0, \ldots, e_7$) such that

$$o = r_0 + \sum_{m=1}^{7} r_m e_m \qquad (r_{0,\ldots,7} \text{ reals}) \tag{2}$$

where e_m are elements obeying the noncommutative and nonassociative algebra

$$e_m e_n = -\delta_{mn} + \varepsilon_{mnp} e_p \qquad (m, n, p = 1, \ldots, 7) \tag{3}$$

with ε_{mnp} totally antisymmetric and equal to unity for the seven combinations

123, 145, 176, 246, 257, 347, and 365

(each cycle represents a quaternionic subalgebra). The norm $N(o)$ for the octonions is defined by

$$N(o) = (o^\dagger o)^{1/2} = (o o^\dagger)^{1/2} = (r_0^2 + \ldots + r_7^2)^{1/2} \tag{4}$$

with the octonionic conjugate $o^\dagger$ given by

$$o^\dagger = r_0 - \sum_{m=1}^{7} r_m e_m \tag{5}$$

The inverse is then

$$o^{-1} = o^\dagger / N(o) \qquad (o \neq 0) \tag{6}$$

We can define an *associator* (analogous to the usual algebraic commutator) as follows:

$$\{x, y, z\} \equiv (xy)z - x(yz) \tag{7}$$

where, in each term on the right-hand side, we must first perform the multiplication in brackets. Note that for real, complex, and quaternionic numbers the associator is trivially null. For octonionic imaginary units we have

$$\{e_m, e_n, e_p\} \equiv (e_m e_n)e_p - e_m(e_n e_p) = 2\varepsilon_{mnps} e_s \tag{8}$$

with ε_{mnps} totally antisymmetric and equal to unity for the seven combinations

1247, 1265, 2345, 2376, 3146, 3157, and 4567

Working with octonionic numbers, the associator (7) is in general nonvanishing; however, the "alternative condition" is fulfilled

$$\{x, y, z\} + \{z, y, x\} = 0 \tag{9}$$

3. LEFT/RIGHT-BARRED OPERATORS

In 1989, writing a quaternionic Dirac equation,[26] Rotelli introduced a *barred* momentum operator

$$-\partial \mid i \qquad [(-\partial \mid i)\psi \equiv -\partial\psi i] \tag{10}$$

In a recent paper[23] based upon the Rotelli operators, *partially generalized quaternions*

$$q + p \mid i \qquad [q, p \in \mathcal{H}] \tag{11}$$

have been used to formulate a quaternionic quantum mechanics. From the viewpoint of group structure, these barred numbers are very similar to complexified quaternions[41]

$$q + \mathcal{I}p \tag{12}$$

(the imaginary unit $\mathcal{I}$ commutes with the quaternionic imaginary units i, j, k), but in physical problems like eigenvalue calculations, tensor products, or relativistic equation solutions they give different results.

A complete generalization for quaternionic numbers is represented by the following barred operators:

$$q_1 + q_2 \mid i + q_3 \mid j + q_4 \mid k \qquad [q_{1,\ldots,4} \varepsilon \mathcal{H}] \tag{13}$$

which we call *fully generalized quaternions*, or simply generalized quaternions. Fully generalized quaternions, with their 16 linearly independent elements, form a basis of $GL(4, \mathcal{R})$. They are successfully used to reformulate Lorentz space-time transformations[42] and write down a one-component Dirac equation.[28]

Thus, it seems to us natural to investigate the existence of *generalized octonions*

$$o_0 + \sum_{m=1}^{7} o_m \mid e_m \tag{14}$$

Nevertheless, we must observe that an octonionic *barred* operator **a | b** which acts on octonionic wave functions ψ,

$$[a \mid b]\psi \equiv a\psi b$$

is not a well-defined object. For $a \neq b$ the triple product $a\psi b$ could be either $(a\psi)\, b$ or $a\,(\psi b)$. So, in order to avoid the ambiguity due to the nonassociativity of the octonionic numbers, we need to define left/right-barred operators. We will indicate *left-barred* operators by **a) b**, with a and b representing octonionic numbers. They act on octonionic functions ψ as follows:

$$[a\ )\ b]\psi = (a\psi)b \tag{15a}$$

In similar way we can introduce *right-barred* operators, defined by **a (b**,

$$[a\ (\ b]\psi = a\,(\psi b) \tag{15b}$$

Obviously, there are barred operators in which the nonassociativity is not of relevance, like

$$1\ )\ a = 1\ (\ a \equiv 1 \mid a$$

Furthermore, from equation (9), we have

$$\{x, y, x\} = 0$$

so

$$a\ )\ a = a\ (\ a \equiv a \mid a$$

At first glance it seems that we must consider the following 106 barred operators:

$$1, \quad e_m, \quad 1 \mid e_m \qquad \text{(15 elements)}$$

$$e_m \mid e_m \qquad (7)$$

$$e_m\ )\ e_n \quad (m \neq n) \qquad (42)$$

$$e_m\ (\ e_n \quad (m \neq n) \qquad (42)$$

$$(m, n = 1, \ldots, 7)$$

Nevertheless, it is possible to prove that each right-barred operator can be expressed by a suitable combination of left-barred operators. For example, from equation (9), by posing $x = e_m$ and $z = e_n$, we quickly obtain

$$e_m\,(\,e_n + e_n\,(\,e_m \equiv e_m\,)\,e_n + e_n\,)\,e_m \tag{16}$$

So we can represent the most general octonionic operator by only left-barred objects

$$o_0 + \sum_{m=1}^{7} o_m\,)\;\; e_m \qquad [o_{0,\ldots,7} \text{ octonions}] \tag{17}$$

reducing to 64 the previous 106 elements. This suggests a correspondence between our generalized octonions (17) and $GL(8, \mathcal{R})$ (a complete discussion about the above-mentioned relationship is given in the following section).

4. TRANSLATION RULES

The nonassociativity of octonions represents a challenge. We overcome the problems due to octonion nonassociativity by introducing left/right-barred operators. We discuss in the next subsection their relation to $GL(8, \mathcal{R})$. In that subsection, we present our translation idea and give some explicit examples which allow us to establish the isomorphism between our octonionic left/right-barred operators and $GL(8, \mathcal{R})$. In Section 4.2 we focus our attention on the group $GL(4, \mathcal{C}) \subset GL(8, \mathcal{R})$. In doing so, we find that only particular combinations of octonionic barred operators give us suitable candidates for the $GL(4, \mathcal{C})$ translation. Finally, in Section 4.3 we explicitly give three octonionic representations for the gamma matrices.

4.1. Relation between Barred Operators and 8 × 8 Real Matrices

In order to explain the idea of translation, let us look explicitly at the action of the operators $1 \mid e_1$ and e_2 on a generic octonionic function φ:

$$\varphi = \varphi_0 + e_1\varphi_1 + e_2\,\varphi_2 + e_3\varphi_3 + e_4\varphi_4 + e_5\,\varphi_5 + e_6\varphi_6 + e_7\varphi_7 \qquad [\varphi_{0,\ldots,7}\ \varepsilon\ \mathcal{R}] \tag{18}$$

We have

$$[1 \mid e_1]\varphi \equiv \varphi e_1 = e_1\varphi_0 - \varphi_1 - e_3\varphi_2 + e_2\varphi_3 - e_5\varphi_4 + e_4\varphi_5 + e_7\varphi_6 - e_6\varphi_7 \tag{19a}$$

$$e_2\varphi = e_2\varphi_0 - e_3\varphi_1 - \varphi_2 + e_1\varphi_3 + e_6\varphi_4 + e_7\varphi_5 - e_4\varphi_6 - e_5\varphi_7 \tag{19b}$$

If we represent our octonionic function φ by the real column vector

$$\varphi \leftrightarrow \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix} \tag{20}$$

we can rewrite the equations (19a) and (19b) in matrix form,

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix} = \begin{pmatrix} -\varphi_1 \\ \varphi_0 \\ \varphi_3 \\ -\varphi_2 \\ \varphi_5 \\ -\varphi_4 \\ -\varphi_7 \\ \varphi_6 \end{pmatrix} \tag{21a}$$

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix} = \begin{pmatrix} -\varphi_2 \\ \varphi_3 \\ \varphi_0 \\ -\varphi_1 \\ -\varphi_6 \\ -\varphi_7 \\ \varphi_4 \\ \varphi_5 \end{pmatrix} \tag{21b}$$

In this way we can immediately obtain a real matrix representation for the octonionic barred operators $1 \mid e_1$ and e_2. Following this procedure, we can construct the complete set of translation rules for the imaginary units e_m and the barred operators $1 \mid e_m$ (Table III in Appendix A1). In this paper we will use the Joshi notation[15]: L_m and R_m will represent the matrix counterparts of the octonionic operators e_m and $1 \mid e_m$,

$$L_m \leftrightarrow e_m \qquad \text{and} \qquad R_m \leftrightarrow 1 \mid e_m \tag{22}$$

At first glance it seems that our translation does not work. If we extract from Table III the matrices corresponding to e_1, e_2, and e_3, namely,

$$L_i = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$L_3 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

we find

$$L_1 L_2 \neq L_3 \tag{23}$$

in obvious contrast with the octonionic relation

$$e_1 e_2 = e_3 \tag{24}$$

This bluff is soon explained. In deducing our translation rules, we understand octonions as operators, and so they must be applied to a certain octonionic function, φ. If we have the octonionic relation

$$(e_1 e_2)\varphi = e_3\varphi \tag{25a}$$

the matrix counterpart will be

$$L_3\varphi \tag{25b}$$

since the matrix counterparts are defined by their action upon the "wave function" and *not* upon another "operator." On the other hand,

$$e_1(e_2\varphi) \neq e_3\varphi \tag{26a}$$

will be translated by

$$L_1 L_2\varphi \neq L_3\varphi \tag{26b}$$

We have to differentiate between two kinds of multiplication, "·" and "×." At the level of octonions, one has

$$e_1 \cdot e_2 = e_3 \tag{27}$$

but at the level of octonionic operators

$$e_1 \times e_2 \neq e_3 \tag{28}$$

$$[e_1 \times e_2 \equiv e_3 + e_1\,)\,e_2 - e_1\,(\,e_2 \rightarrow \text{see below}]$$

After completing our translation rules we will return to this point and discuss the multiplication rules for octonionic barred operators.

Working with left/right-barred operators, we show how the nonassociativity is inherent in our representation. Such operators enable us to reproduce the octonion nonassociativity by the matrix algebra. Consider, for example,

$$[e_3\ e_1]\varphi \equiv (e_3\varphi)e_1 = e_2\varphi_0 - e_3\varphi_1 + \varphi_2 - e_1\varphi_3 - e_6\varphi_4 - e_7\varphi_5 + e_4\varphi_6 + e_5\varphi_7 \tag{29}$$

This equation will be translated into

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix} = \begin{pmatrix} \varphi_2 \\ -\varphi_3 \\ \varphi_0 \\ -\varphi_1 \\ \varphi_6 \\ \varphi_7 \\ -\varphi_4 \\ -\varphi_5 \end{pmatrix} \tag{30}$$

whereas

$$[e_3 \,(\, e_1]\varphi \equiv e_3(\varphi e_1) = e_2\varphi_0 - e_3\varphi_1 + \varphi_2 - e_1\varphi_3 + e_5\varphi_4 + e_7\varphi_5 - e_4\varphi_6 - e_5\varphi_7 \tag{31}$$

will become

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix} = \begin{pmatrix} \varphi_2 \\ -\varphi_3 \\ \varphi_0 \\ -\varphi_1 \\ -\varphi_6 \\ -\varphi_7 \\ \varphi_4 \\ \varphi_5 \end{pmatrix} \tag{32}$$

The nonassociativity is then reproduced since left- and right-barred operators like

$$e_3 \,)\, e_1 \quad \text{and} \quad e_3 \,(\, e_1$$

are represented by different matrices. The complete set of translation rules for left/right-barred operators is given in Tables V and VI of Appendix A1. Using Mathematica,[43] we have proved the linear independence of the 64 elements which appear in Tables III–VI. So our barred operators form a complete basis for any 8×8 real matrix and this establishes the isomorphism between $GL(8, \mathcal{R})$ and generalized octonions. We provide the necessary tables for translating any generic 8×8 real matrix into left/right-barred operators in Appendix A2.

The matrix representation for left/right-barred operators can be quickly obtained by suitable multiplications of the matrices L_m and R_m. Let us clear up our assertion. From Tables V and VI we can extract the matrices which correspond to the operators

$$e_m \,)\, e_n \quad \text{and} \quad e_m \,(\, e_n$$

which we call, respectively,

$$M^L_{mn} \quad \text{and} \quad M^R_{mn}$$

Our left/right-barred operators can be represented by an ordered action of the operators e_m and $1 \mid e_m$, and so we can relate the matrices M^L_{mn} and

M^R_{mn} quoted in Tables V and VI to the matrices L_m and R_m given in Table III. Explicitly,

$$M^L_{mn} \equiv R_n \, L_m \tag{33a}$$

$$M^R_{mn} \equiv L_m \, R_n \tag{33b}$$

The previous discussions concerning octonion nonassociativity and the isomorphism between $GL(8, \mathcal{R})$ and generalized octonions can be now elegantly, presented as follows.

1. *Matrix representation for octonion nonassociativity*:

$$M^L_{mn} \neq M^R_{mn} \qquad [R_n L_m \neq L_m R_n \quad \text{for } m \neq n] \tag{34}$$

2. *Isomorphism between* $GL(8, \mathcal{R})$ *and generalized octonions.*

If we rewrite our 106 barred operators by real matrices,

$$1, \quad L_m, \quad R_m \qquad (15 \text{ matrices})$$

$$M \equiv L_m R_m = R_m L_m \qquad (7)$$

$$M^L_{mn} \equiv R_n L_m \quad (m \neq n) \qquad (42)$$

$$M^R_{mn} \equiv L_n R_m \quad (m \neq n) \qquad (42)$$

$$(m, n = 1, \ldots, 7)$$

we have two different bases for $GL(8, \mathcal{R})$:

$$(1) \qquad 1, \, L_m, \, R_m, \, R_n L_m$$

$$(2) \qquad 1, \, L_m, \, R_m, \, L_m R_n$$

We now remark some difficulties deriving from octonion nonassociativity. When we translate from generalized octonions to 8×8 real matrices there is no problem. For example, in the octonionic equation

$$e_4\{[(e_6\varphi)e_1]e_5\} \tag{35}$$

we quickly recognize the following left-barred operators,

$$e_4 \, (\, e_5 \quad \text{and} \quad e_6 \,) \, e_1$$

Using our tables, we can translate equation (35) into

$$M^L_{45} M^L_{61} \varphi \tag{36}$$

Nevertheless, in going from 8×8 real matrices to octonions we should be careful in ordering. For example,

$$AB \, \varphi \tag{37}$$

can be understood as

$$(AB)\varphi \tag{38a}$$

or

$$A(B\varphi) \tag{38b}$$

Which is the right equation? To find the solution, let us explicitly use particular matrices. Defining

$$A \rightarrow L_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \leftrightarrow e_6 \tag{39a}$$

$$B \rightarrow M^L_{31} \tag{39b}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \leftrightarrow e_3 \,)\, e_1$$

we find that the previous matrix equations (38a) and (38b), respectively, become

$$e_6 \times [e_3 \,)\; e_1]\varphi \tag{40a}$$

and

$$e_6[(e_3\varphi)e_1] \tag{40b}$$

We know that "×" multiplication is different from the standard octonionic multiplication, so

$$e_6 \times [e_3 \,)\; e_1] \neq -e_3 \,)\, e_1$$

Using Appendix A2 and translating the matrix AB, we can obtain the octonionic operator which corresponds to

$$e_6 \times [e_3 \,)\, e_1]$$

Explicitly, we have

$$\{e_4 - 1 \mid e_4 - 2e_1 \,)\, e_5 - e_5 \,)\, e_1 - e_6 \,)\, e_2 + e_2 \,)\, e_6$$
$$+ 2e_7 \,)\, e_3 - e_3 \,)\, e_7\}/3$$

Its complicated form suggests that we choose (38b) for translating (37). In general

$$ABC\ldots Z\varphi \equiv A(B(C \ldots (Z\varphi) \ldots)) \tag{41}$$

Only for e_m and $1 \mid e_m$ do we have simple "$\times$" multiplication rules. In fact, utilizing the associator properties, we find

$$e_m(e_n\varphi) = (e_m e_n)\varphi + (e_m\varphi)e_n - e_m(\varphi e_n) \tag{42a}$$

$$(\varphi e_m)e_n = \varphi(e_m e_n) - (e_m\varphi)e_n + e_m(\varphi e_n) \tag{42b}$$

Thus,

$$e_m \times e_n \equiv -\delta_{mn} + \varepsilon_{mnp}e_p + e_m \,)\, e_n - e_m \,(\, e_n \tag{43a}$$

$$[1 \mid e_n] \times [1 \mid e_m] \equiv -\delta_{mn} + \varepsilon_{mnp}e_p - e_m \,)\, e_n + e_m \,(\, e_n \tag{43b}$$

At the beginning of this subsection, we noted that the correspondence between the matrices L_m and the octonionic imaginary units e_m is in contrast with the standard octonionic relations

$$e_m e_n = -\delta_{mn} + \varepsilon_{mnp}e_p \tag{44}$$

For example, consider

$$L_1 L_2 \neq L_3$$

Introducing a new matrix multiplication "$\circ$," we can quickly reproduce the nonassociative octonionic algebra. From equation (42a), we find

$$L_m L_n \varphi = L_m \circ L_n \varphi + [R_n, L_m]\varphi \tag{45}$$

so we can relate the new matrix multiplication "$\circ$" to the standard matrix multiplication (row by column) as follows:

$$L_m \circ L_n \equiv L_m L_n + [R_n, L_m] \tag{46}$$

Equation (44) is then translated by

$$L_m \circ L_n = -\delta_{mn} + \varepsilon_{mnp} L_p \tag{47}$$

4.2. Relation between Barred Operators and 4 × 4 Complex Matrices

Some complex groups play a critical role in physics. No one can deny the importance of $U(1, \mathcal{C})$ or $SU(2, \mathcal{C})$. In relativistic quantum mechanics, $GL(4, \mathcal{C})$ is essential in writing the Dirac equation. Having $GL(8, \mathcal{R})$, we should be able to extract its subgroup $GL(4, \mathcal{C})$. So we can translate the famous Dirac gamma matrices and write down a new octonionic Dirac equation.

Let us show how we can isolate our 32 bases of $GL(4, \mathcal{C})$: If we analyze the action of left-barred operators on our octonionic wave functions

$$\psi = \psi_1 + e_2\psi_2 + e_4\psi_3 + e_6\psi_4 \qquad [\psi_{1,..,4} \in \mathcal{C}(1, e_1)] \tag{48}$$

we find, for example,

$$[1 \mid e_1]\psi \equiv \psi e_1 = \psi_1 + e_2(e_1\psi_2) + e_4(e_1\psi_3) + e_6(e_1\psi_4)$$

$$e_2\varphi = -\psi_2 + e_2\psi_1 - e_4\psi_4^* + e_6\psi_3^*$$

$$[e_3\,)\; e_1]\psi \equiv (e_3\psi)e_1 = \psi_2 + e_2\varphi_1 + e_4\psi_4^* - e_6\psi_3^*$$

The action of our barred operators is quoted in Tables X–XIII in Appendix B1.

Following the same methodology as in the previous section, we can immediately note a correspondence between the complex matrix $i\mathbf{1}_{4\times4}$ and the octonionic barred operator $1 \mid e_1$,

$$\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \leftrightarrow 1 \mid e_1 \tag{49}$$

Observe that we are working with the symplectic decomposition of octonions

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \leftrightarrow \psi_1 + e_2\psi_2 + e_4\psi_3 + e_6\psi_4 \tag{50}$$

Such an identification will be much clearer when we introduce a *complex geometry*. In fact, choosing a complex projection for our scalar products,

$$\psi_1, \quad e_2\psi_2, \quad e_4\psi_3, \quad e_6\psi_4$$

will represent complex-orthogonal states.

The translation does not work for all barred operators. Let us show this explicitly. For example, we cannot find a 4 × 4 complex matrix which, acting on

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

gives the column vector

$$\begin{pmatrix} -\psi_2 \\ \psi_1 \\ -\psi_4^* \\ \psi_3^* \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \psi_2 \\ \psi_1 \\ \psi_4^* \\ -\psi_3^* \end{pmatrix}$$

and so we do not have the possibility to relate

$$e_2 \quad \text{or} \quad e_3 \,)\, e_1$$

with a complex matrix. Nevertheless, the combined action of such operators gives

$$e_2\psi + (e_3\psi)e_1 = 2e_2\psi_1$$

which allows us to represent the octonionic barred operator

$$e_2 + e_3 \,)\, e_1 \tag{51a}$$

by the 4 × 4 complex matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{51b}$$

Following this procedure, we can represent a generic 4 × 4 complex matrix by octonionic barred operators. The explicit correspondence tables are given in Appendix B2.

We conclude this subsection by discussing a point which will be relevant to an appropriate definition for the octonionic momentum operator (Section 5.2): The operator $1 \mid e_1$ (represented by the matrix $i\mathbf{1}_{4\times 4}$) commutes with all operators which can be translated by 4 × 4 complex matrices (see Appendix

B2). This is not generally true for a generic octonionic operator. For example, we can show that the operator $1 \mid e_1$ does not commute with e_2; explicitly,

$$e_2\{[1 \mid e_1]\psi\} \equiv e_2(\psi e_1) = -e_1\psi_2 - e_3\psi_1 - e_5\psi_4^* - e_7\psi_3^* \qquad (52a)$$

$$[1 \mid e_1]\{e_2\psi\} \equiv (e_2\psi)e_1 = -e_1\psi_2 - e_3\psi_1 + e_5\psi_4^* + e_7\psi_3^* \qquad (52b)$$

The interpretation is simple: e_2 cannot be represented by a 4×4 complex matrix.

4.3. Octonionic Representations of the Gamma Matrices

We conclude this section by showing explicitly three octonionic representations for the Dirac gamma matrices[(44),4]:

1. *Dirac Representation*:

$$\gamma^0 = \frac{1}{3} - \frac{2}{3}\sum_{m=1}^{3} e_m \mid e_m + \frac{1}{3}\sum_{n=4}^{7} e_n \mid e_n \qquad (53a)$$

$$\gamma^1 = -\frac{2}{3}e_6 - \frac{1}{3} \mid e_6 + e_5\,)\, e_3 - e_3\,)\, e_5 - \frac{1}{3}\sum_{p,s=1}^{7} \varepsilon_{ps6} e_p\,)\, e_s \qquad (53b)$$

$$\gamma^2 = -\frac{2}{3}e_7 - \frac{1}{3} \mid e_7 + e_3\,)\, e_4 - e_4\,)\, e_3 - \frac{1}{3}\sum_{p,s=1}^{7} \varepsilon_{ps7} e_p\,)\, e_s \qquad (53c)$$

$$\gamma^3 = -\frac{2}{3}e_4 - \frac{1}{3} \mid e_4 + e_7\,)\, e_3 - e_3\,)\, e_7 - \frac{1}{3}\sum_{p,s=1}^{7} \varepsilon_{ps4} e_p\,)\, e_s \qquad (53d)$$

2. *Majorana Representation*:

$$\gamma^0 = \frac{1}{3}e_7 - \frac{1}{3} \mid e_7 + e_3\,)\, e_4 - e_5\,)\, e_2 + e_6\,)\, e_1 - \frac{1}{3}\sum_{p,s=1}^{7} \varepsilon_{ps7} e_p\,)\, e_s \qquad (54a)$$

$$\gamma^1 = \frac{2}{3}e_1 + \frac{1}{3} \mid e_1 + e_5\,)\, e_4 - e_4\,)\, e_5 + \frac{1}{3}\sum_{p,s=1}^{7} \varepsilon_{ps1} e_p\,)\, e_s \qquad (54b)$$

$$\gamma^2 = \frac{2}{3}e_7 + \frac{1}{3} \mid e_7 + e_4\,)\, e_3 - e_3\,)\, e_4 + \frac{1}{3}\sum_{p,s=1}^{7} \varepsilon_{ps7} e_p\,)\, e_s \qquad (54c)$$

$$\gamma^3 = \frac{2}{3}e_3 + \frac{1}{3} \mid e_3 + e_7\,)\, e_4 - e_4\,)\, e_7 + \frac{1}{3}\sum_{p,s=1}^{7} \varepsilon_{ps3} e_p\,)\, e_s \qquad (54d)$$

[4] Equations (53a)–(53d) [(54a)–(54d), (55a)–(55d)] represent the octonionic counterpart of the complex matrixes given on p. 49 [694] of ref. 44.

3. *Chiral Representation*:

$$\gamma^0 = \frac{1}{3}e_4 - \frac{1}{3} \mid e_4 + e_7 \,)\, e_3 - e_2 \,)\, e_6 + e_5 \,)\, e_1 - \frac{1}{3}\sum_{p,s=1}^{7} \varepsilon_{ps4} e_p \,)\, e_s \tag{55a}$$

$$\gamma^1 = -\frac{2}{3}e_6 - \frac{1}{3} \mid e_6 + e_5 \,)\, e_3 - e_3 \,)\, e_5 - \frac{1}{3}\sum_{p,s=1}^{7} \varepsilon_{ps6} e_p \,)\, e_s \tag{55b}$$

$$\gamma^2 = -\frac{2}{3}e_7 - \frac{1}{3} \mid e_7 + e_3 \,)\, e_4 - e_4 \,)\, e_3 - \frac{1}{3}\sum_{p,s=1}^{7} \varepsilon_{ps7} e_p \,)\, e_s \tag{55c}$$

$$\gamma^3 = -\frac{2}{3}e_4 - \frac{1}{3} \mid e_4 + e_7 \,)\, e_3 - e_3 \,)\, e_7 - \frac{1}{3}\sum_{p,s=1}^{7} \varepsilon_{4} e_p \,)\, e_s \tag{55d}$$

5. OCTONIONIC PHYSICAL WORLD

We organize this section into three subsections. Section 5.1 we discuss the Dirac algebra and its problems related to the nonassociativity of the octonionic numbers. In Section 5.2 we introduce the concept of complex geometry and define an appropriate momentum operator. In the final subsection we present the octonionic completeness relations.

5.1. Dirac Algebra

In the previous section we gave the gamma matrices in three different octonionic representations. Obviously, we can investigate the possibility of having a simpler representation for our octonionic γ^μ-matrices without translation.

Why not

$$e_1, \quad e_2, \quad e_3, \quad \text{and} \quad e_4 \mid e_4$$

or

$$e_1, \quad e_2, \quad e_3, \quad \text{and} \quad e_4 \,)\, e_1?$$

Apparently, they represent suitable choices. Nevertheless, the octonionic world is full of hidden traps and so we must proceed with caution. Let us start from the standard Dirac equation

$$\gamma^\mu p_\nu \psi = m\psi \tag{56}$$

(we will discuss the momentum operator in the following subsection; for the moment, p_ν represents the "real" eigenvalue of the momentum operator) and apply $\gamma^\mu p_\mu$ to our equation

$$\gamma^\mu p_\mu (\gamma^\nu p_\nu \psi) = m\gamma^\mu p_\mu \psi \tag{57}$$

The previous equation can be concisely rewritten as

$$p^{\mu} p_{\nu} \gamma^{\mu}(\gamma^{\nu}\psi) = m^2 \psi \tag{58}$$

Requiring that each component of ψ satisfy the standard Klein–Gordon equation, we find the Dirac condition, which becomes in the octonionic world

$$\gamma^{\mu}(\gamma^{\nu}\psi) + \gamma^{\nu}(\gamma^{\mu}\psi) = 2g^{\mu\nu}\psi \tag{59}$$

(where the parentheses are relevant because of the octonion nonassociativity). Using octonionic numbers and no barred operators, we can obtain from (59) the standard Dirac condition

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu} \tag{60}$$

In fact, recalling the associator property [which follows from equation (8)]

$$\{a, b, \psi\} = -\{b, a, \psi\} \qquad [a, b \text{ octonionic numbers}]$$

we quickly find the following correspondence relation:

$$(ab + ba)\psi = a(b\psi) + b(a\psi)$$

We have no problem writing down three suitable gamma matrices which satisfy the Dirac condition (60),

$$(\gamma^1, \gamma^2, \gamma^3) \equiv (e_1, e_2, e_3) \tag{61}$$

but barred operators like

$$e_4 \mid e_4 \quad \text{or} \quad e_4\,)\,e_1$$

cannot represent the matrix γ^0. From Tables X–XIII (Appendix B1), after straightforward algebraic manipulations, one can prove that the barred operator $e_4 \mid e_4$, does not anticommute with e_1,

$$e_1(e_4\psi e_4) + e_4\,(e_1\psi)e_4 = -2(e_3\psi_2 + e_7\psi_4) \neq 0 \tag{62}$$

whereas $e_4\,)\,e_1$ anticommutes with e_1,

$$e_1[(e_4\psi)e_1] + [e_4(e_1\psi)]e_1 = 0 \tag{63a}$$

but

$$\{e_4[(e_4\psi)e_1]\}e_1 = \psi_1 - e_2\psi_2 + e_4\psi_5 - e_6\psi_4 \neq \psi \tag{63b}$$

Thus, we must be satisfied with the octonionic representations given in the previous section. In the following subsection, we discuss two interesting questions: Do the octonionic imaginary units e_1, e_2, e_3 satisfy all the gamma matrix properties? What about their hermiticity?

5.2. Complex Geometry and Octonionic Momentum Operator

We begin this subsection by presenting an apparently hopeless problem related to the nonassociativity of the octonionic field. Working in quantum mechanics, we require that an antihermitian operator satisfies the following relation:

$$\int d\mathbf{x}\ \psi^{\dagger}(A\phi) = -\int d\mathbf{x}\ (A\psi)^{\dagger}\phi \tag{64}$$

In octonionic quantum mechanics (OQM) we can immediately verify that $\boldsymbol{\partial}$ represents an antihermitian operator with all the properties of a translation operator. Nevertheless, while in complex (CQM) and quaternionic (QQM) quantum mechanics we can define a corresponding hermitian operator multiplying by an imaginary unit the operator $\boldsymbol{\partial}$, one encounters in OQM the following problem:

no imaginary unit $\mathbf{e}_m$ *represents an antihermitian operator*

In fact, the nonassociativity of the octonionic algebra implies in general (for arbitrary ψ and ϕ)

$$\int d\mathbf{x}\,\psi^{\dagger}(e_m\phi) \neq -\int d\mathbf{x}\,(e_m\psi)^{\dagger}\phi = \int d\mathbf{x}\,(\psi^{\dagger}e_m)\phi \qquad (m=1,\ldots,7) \tag{65}$$

This contrasts with the situation within complex and quaternionic quantum mechanics. Such a difficulty is overcome by using a complex projection of the scalar product (complex geometry) with respect to one of our imaginary units. We break the symmetry between the seven imaginary units $e_1,\ldots, e_7$ and choose as projection plane the one characterized by $(1,\ e_1)$. The new scalar product is quickly obtained by performing, in the standard definition, the following substitution:

$$\int d\mathbf{x} \to \int_c d\mathbf{x} \equiv \frac{1 - e_1 \mid e_1}{2}\int d\mathbf{x}$$

Working in OQM with *complex geometry*, e_1 represents an antihermitian operator. In order to simplify the proof, we write the octonionic functions ψ and ϕ as follows:

$$\psi = \psi_1 + e_2\psi_2 + e_4\psi_3 + e_6\psi_4$$
$$\phi = \phi_1 + e_2\phi_2 + e_4\phi_3 + e_6\phi_4$$

$[\psi_{1,\ldots,4}$ and $\phi_{1,\ldots,4} \in \mathscr{C}(1, e_1)]$. The antihermiticity of e_1 is shown if

$$\int_c d\mathbf{x}\ \psi^\dagger(e_1\phi) = -\int_c d\mathbf{x}\ (e_1\psi)^\dagger\phi \tag{66}$$

In the previous equation the only nonvanishing terms are represented by *diagonal* terms ($\sim \psi_1^\dagger\phi_1, \psi_2^\dagger\phi_2, \psi_3^\dagger\phi_3, \psi_4^\dagger\phi_4$). In fact, *off-diagonal* terms like $\psi_2^\dagger\phi_3, \psi_3^\dagger\phi_4$ are killed by the complex projection,

$$(\psi_2^\dagger e_2)[e_1(e_4\phi_3)] \sim (\alpha_2 e_2 + \alpha_3 e_3)(\alpha_4 e_4 + \alpha_5 e_5) \sim \alpha_6 e_6 + \alpha_7 e_7$$

$$[(\psi_5^\dagger e_4)e_1](e_6\phi_4) \sim (\beta_4 e_4 + \beta_3 e_5)(\beta_6 e_6 + \beta_7 e_7) \sim \beta_2 e_2 + \beta_3 e_3$$

$(\alpha_{2,\ldots,7}$ and $\beta_{2,\ldots,7} \in \mathscr{R})$. The diagonal terms give

$$\int_c d\mathbf{x}\ \psi^\dagger(e_1\phi) = \psi_1^\dagger(e_1\phi_1) - (\psi_2^\dagger e_2)[e_1(e_2\phi_1)] - (\psi_3^\dagger e_4)[e_1(e_4\phi_3)] - (\psi_4^\dagger e_6)[e_1(e_6\phi_4)] \tag{67a}$$

$$-\int_c d\mathbf{x}\ (e_1\psi)^\dagger\phi = (\psi_1^\dagger e_1)\phi_1 - [(\psi_2^\dagger e_2)e_1](e_2\phi_2) - [(\psi_3^\dagger e_4)e_1](e_4\phi_3) - [(\psi_4^\dagger e_6)e_1](e_6\phi_4) \tag{67b}$$

The parentheses in (67a) and (67b) are not relevant since

$\psi_1^\dagger e_1\phi_1$	$(1, e_1)$	is a complex number
$\psi_2^\dagger e_2 e_1 e_2\phi_2$	(subalgebra 123)	
$\psi_3^\dagger e_4 e_1 e_4\phi_3$	(subalgebra 145)	
$\psi_4^\dagger e_6 e_1 e_6\phi_4$	(subalgebra 176)	are quaternionic numbers

The above-mentioned demonstration does not work for the imaginary units $e_2, \ldots, e_7$ (breaking the symmetry between the seven octonionic imaginary units).

Now, we can define a hermitian operator multiplying by e_1 the operator ∂. However, such an operator is not expected to commute with the Hamiltonian, which will be, in general, an octonionic quantity. The final step toward an appropriate definition of the momentum operator is represented by choosing as imaginary unit the barred operator $1 \mid e_1$ (the antihermiticity proof is very similar to the previous one). In OQM with complex geometry the appropriate momentum operator is then given by

$$\mathbf{p} \equiv -\boldsymbol{\partial} \mid e_1 \tag{68}$$

Obviously, in order to write equations that are relativistically covariant, we must treat the space components and time in the same way, hence we are obliged to modify the standard QM operator $i\partial_t$ by the following substitution:

$$i\partial_t \rightarrow \partial_t \mid e_1$$

and so the octonionic Dirac equation becomes

$$\partial_t \psi e_1 = \boldsymbol{\alpha} \cdot (\mathbf{p}\psi) + m\beta\psi \qquad (\mathbf{p} \equiv -\boldsymbol{\partial}|e_1) \tag{69}$$

The possibility to write a consistent momentum operator represents for us an impressive argument in favor of the use of a complex geometry in the formulation of OQM. Besides, such a complex geometry gives us a welcome *quadrupling* of solutions. In fact,

$$\psi, \quad e_2\psi, \quad e_4\psi, \quad e_6\psi \qquad \psi \in \mathcal{C}(1, e_1)$$

now represent complex-orthogonal solutions. Therefore, we have the possibility to write a one-component octonionic Dirac equation in which all four standard Dirac free-particle solutions appear.

5.3. Octonionic Completeness Relations

We observe that the dimensionality of a complete set of states for complex inner product $\langle\psi|\phi\rangle_c$ is *four times* that for the octonionic inner product $\langle\psi|\phi\rangle$. Specifically, if $|\eta_l\rangle$ are a complete set of intermediate states for the octonionic inner product, so that

$$\langle\psi|\phi\rangle = \sum_l \langle\psi|\eta_l\rangle\langle\eta_l|\phi\rangle$$

$|\eta_l\rangle$, $|\eta_l e_2\rangle$, $|\eta_l e_4\rangle$, and $|\eta_l e_6\rangle$ form a complete set of states for the complex inner product,

$$\begin{aligned}|\phi\rangle &= \sum_l (|\eta_l\rangle\langle\eta_l|\phi\rangle_c + |\eta_l e_2\rangle\langle\eta_l e_2|\phi\rangle_c \\ &\quad + |\eta_l e_4\rangle\langle\eta_l e_4|\phi\rangle_c + |\eta_l e_6\rangle\langle\eta_l e_6|\phi\rangle_c) \\ &= \sum_m |\chi_m\rangle\langle\chi_m|\phi\rangle_c\end{aligned}$$

where χ_m represents *complex* orthogonal states. Thus the completeness relation can be written as (for further details on the completeness relation, see the interesting work of Horwitz and Biedenharn, ref. 30, p. 455)

$$\overrightarrow{\mathbf{1}} = \sum_m |\chi_m\rangle\langle\langle\chi_m|$$

$$\overleftarrow{\mathbf{1}} = \sum_m |\chi_m\rangle\rangle\langle\chi_m|$$

so in our formalism we generalize Dirac's notation by the definitions

$$\langle\!\langle\chi_m|\phi\rangle = \langle\chi_m|\phi\rangle_c$$

$$\langle\phi|\chi_m\rangle\!\rangle = \langle\phi|\chi_m\rangle_c$$

so in our formalism we generalize Dirac's notation by the definitions

$$\langle\!\langle\chi_m|\phi\rangle = \langle\chi_m|\phi\rangle_c$$

$$\langle\phi|\chi_m\rangle\!\rangle = \langle\phi|\chi_m\rangle_c$$

6. OCTONIONIC DIRAC EQUATION

As remarked in Section 5, the appropriate momentum operator in OQM is

$$\mathbf{p} \equiv -\partial \mid e_1$$

Thus, the octonionic Dirac equation in covariant form is given by

$$\gamma^\mu(\partial_\mu\psi e_1) = m\psi \tag{70}$$

where γ^μ are represented by octonionic barred operators (53a)–(53d). We can now proceed in the standard manner. Plane wave solutions exist [$\mathbf{p}$ ($\equiv -\partial \mid e_1$) commutes with a generic octonionic Hamiltonian] and are of the form

$$\psi(\mathbf{x}, t) = [u_1(\mathbf{p}) + e_2u_2(\mathbf{p}) + e_4u_3(\mathbf{p}) + e_6u_4(\mathbf{p})]e^{-pxe^1} \quad [u_{1,\ldots,4} \in \mathcal{C}(1, e_1)] \tag{71}$$

Let us start with

$$\mathbf{p} \equiv (0, 0, p_2)$$

From (70), we have

$$E(\gamma^0\psi) - p_z(\gamma^3\psi) = m\psi \tag{72}$$

Using the explicit form of the octonionic operators $\gamma^{0,3}$ and extracting their action (see Section 6.1) from the tables in Appendix B1, we find

$$E(u_1 + e_2u_2 - e_4u_3 - e_6u_4) - p_z(u_3 - e_2u_4 - e_4u_1 + e_6u_2)$$
$$= m(u_1 + e_2u_2 + e_4u_3 + e_6u_4) \tag{73}$$

From (73), we derive four complex equations:

$$(E - m)u_1 = +p_zu_3$$

$$(E - m)u_2 = -p_zu_4$$

$$(E + m)u_3 = +p_zu_1$$

$$(E + m)u_4 = -p_zu_2$$

After simple algebraic manipulations we find the following octonionic Dirac solutions:

$$E=+|E|, \qquad u^{(1)}=N\left(1+e_4\frac{p_z}{|E|+m}\right), u^{(2)}=N\left(e_2-e_6\frac{p_z}{|E|+m}\right)=u^{(1)}e_2$$

$$E=-|E|, u^{(3)}=N\left(\frac{p_z}{|E|+m}-e_4\right), u^{(4)}=N\left(e_2\frac{p_z}{|E|+m}+e_6\right)=u^{(3)}e_2$$

with N a real normalization constant. Setting the norm to $2|E|$, we find

$$N=(|E|+m)^{1/2}$$

We now observe (as for the quaternionic Dirac equation) a difference with respect to the standard Dirac equation. Working in our representation (53a)–(53d) and introducing the octonionic spinor

$$\overline{u} \equiv (\gamma_0 u)^+ = u_1^* - e_2 u_2 + e_4 u_3 + e_6 u_4 \qquad [u = u_1 + e_2 u_2 + e_4 u_3 + e_6 u_4]$$

we have

$$\overline{u}^{(1)} u^{(1)} = u^{(1)} \overline{u}^{(1)} = \overline{u}^{(2)} u^{(2)} = u^{(2)} \overline{u}^{(2)} = 2(m + e_4 p_z) \tag{74}$$

Thus we find

$$u^{(1)} \overline{u}^{(1)} + u^{2} \, \overline{u}^{(2)} = 4(m + e_4 p_z) \tag{75a}$$

instead of the expected relation

$$u^{(1)} \overline{u}^{(1)} + u^{(2)} \overline{u}^{(2)} = \gamma^0 E - \gamma^3 p_z + m \tag{75b}$$

Furthermore, the previous difference is compensated if we compare the complex projection of (75a) with the trace of (75b),

$$[(u^{(1)} \overline{u}^{(1)} + u^{(2)} \overline{u}^{(2)})^{\text{OQM}}]_c \equiv \text{Tr}[(u^{(1)} \overline{u}^{(1)} + u^{(2)} \overline{u}^{(2)})^{\text{CQM}}] = 4m \tag{76}$$

We know that spinor relations like (75a) and (75b) are relevant in perturbation calculus, so the previous results suggest that we analyze quantum electrodynamics in order to investigate possible differences between complex and octonionic quantum mechanics. This could represent the aim of future work.

6.1. $\gamma^{0,3}$ Action on Octonionic Spinors

In Tables I and II we explicitly show the action on the octonionic spinor

$$u = u_1 + e_2 u_2 + e_4 u_3 + e_6 u_4 \qquad [u_{1,\ldots,4} \in \mathcal{C}(1, e_1)]$$

Table I.

γ^0-action	u_1	e_2u_2	e_4u_3	e_6u_4
$e_1 \mid e_1$	$-u_1$	e_2u_2	e_4u_3	e_6u_4
$e_2 \mid e_2$	$-u_1^*$	$-e_2u_2^*$	e_4u_3	e_6u_4
$e_3 \mid e_3$	$-u_1^*$	$e_2u_2^*$	e_4u_3	e_6u_4
$e_4 \mid e_4$	$-u_1^*$	$e_2u_2^*$	$-e_4u_3^*$	e_6u_4
$e_5 \mid e_5$	$-u_1^*$	e_2u_2	$e_4u_3^*$	e_6u_4
$e_6 \mid e_6$	$-u_1^*$	e_2u_2	e_4u_3	$-e_6u_4^*$
$e_7 \mid e_7$	$-u_1^*$	e_2u_2	e_4u_3	$e_6u_4^*$

Table II.

γ^3-action	u_1	e_2u_2	e_4u_3	e_6u_4
e_4	e_4u_1	$-e_6u_2^*$	$-u_3$	e_2u_4
$1 \mid e_4$	$e_4u_1^*$	$-e_6u_2^*$	$-u_3^*$	$-e_2u_4^*$
$e_7 \mid e_3$	$e_4u_1^*$	e_6u_2	u_3	$-e_2u_4^*$
$e_3 \mid e_7$	$-e_4u_1^*$	$-e_6u_2^*$	$-u_3$	e_2u_4
$e_6 \mid e_2$	$e_4u_1^*$	$-e_6u_2$	u_3	$-e_2u_4^*$
$e_2 \mid e_6$	$-e_4u_1^*$	$-e_6u_2^*$	$-u_3$	$-e_2u_4$
$e_5 \mid e_1$	e_4u_1	$e_6u_2^*$	u_3	$-e_2u_4^*$
$e_1 \mid e_5$	$-e_4u_1^*$	$-e_6u_2^*$	$-u_3^*$	$e_2u_4^*$

of the barred operators which appear in γ^0 and γ^3. Using such tables, after straightforward algebraic manipulations we find

$$\gamma^0 u = u_1 + e_2u_2 - e_4u_3 - e_6u_4$$

$$\gamma^3 u = u_3 - e_2u_4 - e_4u_1 + e_5u_2$$

7. CONCLUSIONS

This paper has aimed at giving a clear exposition of the potentiality of generalized numbers in quantum mechanics. We know that quantum mechanics is the basic tool for different physical applications. Many physicists believe that imaginary numbers are related to the deep secret of quantization. Penrose[(45)] thinks that the quantization is completely based on complex numbers. Trying to overcome the problem of quantum gravity, he proposed to complexify the Minkowskian space-time. This represents the main assumption behind the twistor program. Adler[(20)] believes that quantization processes should not be limited to complex numbers, but should be extended to another member of the division algebras rank, the quaternionic field. He postulates that a successful unification of the fundamental forces will require a generalization beyond complex quantum mechanics. Adler envisages a two-level correspondence principle:

	classical physics and fields
distance scale ↓	complex quantum mechanics and fields
	quaternionic quantum field dynamics (preonic level[46])

with quaternionic quantum dynamics interfacing with complex quantum theory, and then with complex quantum theory interfacing in the familiar manner with classical physics (ref. 20, p. 498).

Following this approach, we are tempted to postulate that octonionic quantum field theory may play an essential role in an even deeper fundamental level of physical structure.

Quaternionic quantum mechanics, using complex geometry[22–24] or quaternionic geometry,[20,41,46] seems to be consistent from the mathematical point of view. Due to the octonion nonassociativity property, octonionic quantum mechanics seems to be a puzzle. In the physical literature, we find a method to partially overcome the issues relating to octonion nonassociativity. Some people introduce a "new" imaginary unit "$i = \sqrt{-1}$" which commutes with all other octonionic imaginary units e_m. The new field is often called the *complexified octonionic field.* Papers written in such a formalism, have dealt with, e.g., quark structure and octonions,[10] octonions, quarks, and QCD,[11] octonions and isospin,[29] Dirac–Clifford algebra,[14] and so on. Nevertheless, we do not like complexifying the octonionic field and so we have presented in this paper an alternative way to look at the octonionic world. No new imaginary unit is necessary to formulate in a consistent way an octonionic quantum mechanics.

A first motivation in using octonion numbers in physics can be concisely stated as follows: We hope to get a better understanding of standard theories if we have more than one concrete realization. In this way we can recognize the fundamental postulates which hold for any generic numerical field.

Having a nonassociative algebra calls for special care. In this work, we introduced a "trick" which allowed us to manipulate octonions. We summarize the more important results found in previous sections:

M. *Mathematical Contents*

M1. The introduction of barred operators (natural objects if one works with noncommutative numbers) facilitates our job and enables us to formulate a "friendly" connection between 8×8 real matrices and octonions.

M2. The nonassociativity is reproduced by left/right-barred operators. We consider these operators the natural extension of generalized quaternions, recently introduced in literature.[23]

M3. We tried to investigate the properties of our generalized numbers and studied their special characteristics in order to use them in a proper way. After having established their isomorphism to $GL(8, \mathcal{R})$, life became easier.

M4. The connection between $GL(8, \mathcal{R})$ and generalized octonions gives us the possibility to extract the octonionic generators corresponding to the complex subgroup $GL(4, \mathcal{C})$. This step represents the main tool to manipulate octonions in quantum mechanics.

M5. To the best of our knowledge, this is the first appearance in the literature of an octonionic representation for the 4-dimensional Clifford algebra.

P. *Physical Contents*

P1. We emphasize that a characteristic of our formalism is the *absolute need for a complex scalar product* (in QQM the use of a complex geometry is not obligatory and thus a question of choice). Using a complex geometry, we overcame the hermiticity problem and gave the appropriate and unique definition of momentum operator.

P2. A positive feature of this octonionic version of quantum mechanics is the appearance of all four standard Dirac free-particle solutions notwithstanding the one-component structure of the wave functions. We have the following situation for the division algebras:

field	complex	quaternions	octonions	
Diarac equation	4×4	2×2	1×1	(matrix dimension)

P3. Many physical results can be reobtained by translation, so we have one version of octonionic quantum mechanics where part of the standard quantum mechanics is reproduced. This represents for the authors a first fundamental step toward an octonionic world. We remark that our translation will not be possible in all situations, so it is only partial, consistent with the fact that the octonionic version could provide additional physical predictions.

I. *Further Investigations*

We conclude with a listing of open questions for future investigations, whose study may lead to further insights.

I1. How may we complete the translation? Note that translation, as presented in this paper, works for $4n \times 4n$ matrices. What about odd-dimensional matrices?

I2. From the translation tables we can extract the multiplication rules for the octonionic barred operators. This will allow us to work directly with octonions without translations.

I3. Inspired by equation (46), we could look for a more convenient way to express the new nonassociative multiplication (for example, we could try to modify the standard multiplication rule: row by column).

I4. The reproduction in octonionic calculations of the standard QED results will be a nontrivial objective, due to the explicit differences in certain spinorial identities (see Section 5.3). We are going to study this problem in a forthcoming paper.

I5. A very attractive point is to try to treat the strong field by octonions and then to formulate in a suitable manner a standard model based on our octonionic dynamical Dirac equation.

I6. A last interesting research topic could be to generalize the group-theoretic structure by our barred octonionic operators.

Many of the problems on this list deal with technical details, although the answers to some will be important for further development of the subject.

We hope that the work presented in this paper demonstrates that octonionic quantum mechanics may constitute a coherent and well-defined branch of theoretical physics. We are convinced that octonionic quantum mechanics represents largely uncharted and potentially very interesting terrain.

We conclude by emphasizing that the core of our paper is represented by the absolute need to adopt a complex geometry within a quantum octonionic world.

APPENDIX A1

In this Appendix we give the translation rules between octonionic left-right barred operators and 8 × 8 real matrices. In order to simplify our translation tables, we introduce the following notation:

$$\{a,b,c,d\}_{(1)} \equiv \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}, \qquad \{a,b,c,d\}_{(2)} \equiv \begin{pmatrix} 0 & a & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & d & 0 \end{pmatrix} \tag{77a}$$

$$\{a,b,c,d\}_{(3)} \equiv \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ c & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{pmatrix}, \qquad \{a,b,c,d\}_{(4)} \equiv \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix} \tag{77b}$$

Table III. Translation Rules between 8×8 Real Matrices and Octonionic Barred Operators $e_m, 1 \mid e_m$

$\mathbf{e}_1 \leftrightarrow \{-i\sigma_2, -i\sigma_2, -i\sigma_2, i\sigma_2\}_{(1)}$	$\mathbf{1} \mid \mathbf{e}_1 \leftrightarrow \{-i\sigma_2, i\sigma_2, i\sigma_2, -i\sigma_2\}_{(1)}$
$\mathbf{e}_2 \leftrightarrow \{-\sigma_3, \sigma_3, -1, 1\}_{(2)}$	$\mathbf{1} \mid \mathbf{e}_2 \leftrightarrow \{-1, 1, 1, -1\}_{(2)}$
$\mathbf{e}_3 \leftrightarrow \{-\sigma_1, \sigma_1, -i\sigma_2, -i\sigma_2\}_{(2)}$	$\mathbf{1} \mid \mathbf{e}_3 \leftrightarrow \{-i\sigma_2, -i\sigma_2, i\sigma_2, i\sigma_2\}_{(2)}$
$\mathbf{e}_4 \leftrightarrow \{-\sigma_3, 1, \sigma_3, -1\}_{(3)}$	$\mathbf{1} \mid \mathbf{e}_4 \leftrightarrow \{-1, -1, 1, 1\}_{(3)}$
$\mathbf{e}_5 \leftrightarrow \{-\sigma_1, i\sigma_2, \sigma_1, i\sigma_2\}_{(3)}$	$\mathbf{1} \mid \mathbf{e}_5 \leftrightarrow \{-i\sigma_2, -i\sigma_2, -i\sigma_2, -i\sigma_2\}_{(3)}$
$\mathbf{e}_6 \leftrightarrow \{-1, -\sigma_3, \sigma_3, 1\}_{(4)}$	$\mathbf{1} \mid \mathbf{e}_6 \leftrightarrow \{-\sigma_3, \sigma_3, -\sigma_3, \sigma_3\}_{(4)}$
$\mathbf{e}_7 \leftrightarrow \{-i\sigma_2, -\sigma_1, \sigma_1, -i\sigma_2\}_{(4)}$	$\mathbf{1} \mid \mathbf{e}_7 \leftrightarrow \{-\sigma_1, \sigma_1, -\sigma_1, \sigma_1\}_{(4)}$

Table IV. Translation Rules between 8×8 Real Matrices and Octonionic Barred Operators $e_m \mid e_m$

$\mathbf{e}_1 \mid \mathbf{e}_1 \leftrightarrow \{-1, 1, 1, 1\}_{(1)}$	$\mathbf{e}_2 \mid \mathbf{e}_2 \leftrightarrow \{-\sigma_3, -\sigma_3, 1, 1\}_{(1)}$
$\mathbf{e}_3 \mid \mathbf{e}_3 \leftrightarrow \{-\sigma_3, \sigma_3, 1, 1\}_{(1)}$	$\mathbf{e}_4 \mid \mathbf{e}_4 \leftrightarrow \{-\sigma_3, 1, -\sigma_3, 1\}_{(1)}$
$\mathbf{e}_5 \mid \mathbf{e}_5 \leftrightarrow \{-\sigma_3, 1, \sigma_3, 1\}_{(1)}$	$\mathbf{e}_6 \mid \mathbf{e}_6 \leftrightarrow \{-\sigma_3, 1, 1, -\sigma_3\}_{(1)}$
$\mathbf{e}_7 \mid \mathbf{e}_7 \leftrightarrow \{-\sigma_3, 1, 1, \sigma_3\}_{(1)}$	

where a, b, c, d, and 0 represent 2×2 real matrices.

In Tables III–VI, $\sigma_1, \sigma_2, \sigma_3$ represent the standard Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{78}$$

APPENDIX A2

In this Appendix we explicitly give the rules which enable us, given a generic 8×8 real matrix, to quickly obtain its octonionic counterpart (Tables VII and VIII). We have

$$M_{8\times 8} = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 & \varepsilon_1 & \varphi_1 & \eta_1 & \lambda_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 & \varepsilon_2 & \varphi_2 & \eta_2 & \lambda_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 & \varepsilon_3 & \varphi_3 & \eta_3 & \lambda_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \delta_4 & \varepsilon_4 & \varphi_4 & \eta_4 & \lambda_4 \\ \alpha_5 & \beta_5 & \gamma_5 & \delta_5 & \varepsilon_5 & \varphi_5 & \eta_5 & \lambda_5 \\ \alpha_6 & \beta_6 & \gamma_6 & \delta_6 & \varepsilon_6 & \varphi_6 & \eta_6 & \lambda_6 \\ \alpha_7 & \beta_7 & \gamma_7 & \delta_7 & \varepsilon_7 & \varphi_7 & \eta_7 & \lambda_7 \\ \alpha_8 & \beta_8 & \gamma_8 & \delta_8 & \varepsilon_8 & \varphi_8 & \eta_8 & \lambda_8 \end{pmatrix} \leftrightarrow O = \sum_{m=1}^{64} x_m \rho_m \tag{79}$$

Table V. Translation Rules between 8 × 8 Real Matrices and Octonionic Left-Barred Operators

$\mathbf{e}_1\,)\,\mathbf{e}_2 \leftrightarrow \{i\sigma_2, -i\sigma_2, i\sigma_2, i\sigma_2\}_{(2)}$	$\mathbf{e}_1\,)\,\mathbf{e}_3 \leftrightarrow \{-1, -1, -1, 1\}_{(2)}$
$\mathbf{e}_1\,)\,\mathbf{e}_4 \leftrightarrow \{i\sigma_2, -i\sigma_2, -i\sigma_2, -i\sigma_2\}_{(3)}$	$\mathbf{e}_1\,)\,\mathbf{e}_5 \leftrightarrow \{-1, 1, -1, -1\}_{(3)}$
$\mathbf{e}_1\,)\,\mathbf{e}_6 \leftrightarrow \{-\sigma_1, -\sigma_1, \sigma_1, -\sigma_1\}_{(4)}$	$\mathbf{e}_1\,)\,\mathbf{e}_7 \leftrightarrow \{\sigma_3, \sigma_3, -\sigma_3, \sigma_3\}_{(4)}$
$\mathbf{e}_2\,)\,\mathbf{e}_1 \leftrightarrow \{-\sigma_1, -\sigma_1, -i\sigma_2, -i\sigma_2\}_{(2)}$	$\mathbf{e}_2\,)\,\mathbf{e}_3 \leftrightarrow \{\sigma_1, -\sigma_1, i\sigma_2, -i\sigma_2\}_{(1)}$
$\mathbf{e}_2\,)\,\mathbf{e}_4 \leftrightarrow \{1, -1, -\sigma_3, \sigma_3\}_{(4)}$	$\mathbf{e}_2\,)\,\mathbf{e}_5 \leftrightarrow \{i\sigma_2, -i\sigma_2, -\sigma_1, \sigma_1\}_{(4)}$
$\mathbf{e}_2\,)\,\mathbf{e}_6 \leftrightarrow \{-\sigma_3, -\sigma_3, -1, -1\}_{(3)}$	$\mathbf{e}_2\,)\,\mathbf{e}_7 \leftrightarrow \{-\sigma_1, -\sigma_1, i\sigma_2, i\sigma_2\}_{(3)}$
$\mathbf{e}_3\,)\,\mathbf{e}_1 \leftrightarrow \{\sigma_3, \sigma_3, 1, -1\}_{(2)}$	$\mathbf{e}_3\,)\,\mathbf{e}_2 \leftrightarrow \{-\sigma_1, -\sigma_1, -i\sigma_2, i\sigma_2, \}_{(1)}$
$\mathbf{e}_3\,)\,\mathbf{e}_4 \leftrightarrow \{i\sigma_2, i\sigma_2, -\sigma_1, \sigma_1\}_{(4)}$	$\mathbf{e}_3\,)\,\mathbf{e}_5 \leftrightarrow \{-1, -1, \sigma_3, -\sigma_3\}_{(4)}$
$\mathbf{e}_3\,)\,\mathbf{e}_6 \leftrightarrow \{\sigma_1, -\sigma_1, -i\sigma_2, -i\sigma_2\}_{(3)}$	$\mathbf{e}_3\,)\,\mathbf{e}_7 \leftrightarrow \{-\sigma_3, \sigma_3, -1, -1\}_{(3)}$
$\mathbf{e}_4\,)\,\mathbf{e}_1 \leftrightarrow \{-\sigma_1, i\sigma_2, -\sigma_1, i\sigma_2\}_{(3)}$	$\mathbf{e}_4\,)\,\mathbf{e}_2 \leftrightarrow \{-1, -\sigma_3, -1, -1\}_{(4)}$
$\mathbf{e}_4\,)\,\mathbf{e}_3 \leftrightarrow \{-i\sigma_2, -\sigma_1, -i\sigma_2, -\sigma_1\}_{(4)}$	$\mathbf{e}_4\,)\,\mathbf{e}_5 \leftrightarrow \{\sigma_1, i\sigma_2, -\sigma_1, -i\sigma_2\}_{(1)}$
$\mathbf{e}_4\,)\,\mathbf{e}_6 \leftrightarrow \{\sigma_3, 1, -\sigma_3, -1\}_{(2)}$	$\mathbf{e}_4\,)\,\mathbf{e}_7 \leftrightarrow \{\sigma_1, -i\sigma_2, -\sigma_1, i\sigma_2\}_{(2)}$
$\mathbf{e}_5\,)\,\mathbf{e}_1 \leftrightarrow \{\sigma_3, -1, \sigma_3, 1\}_{(3)}$	$\mathbf{e}_5\,)\,\mathbf{e}_2 \leftrightarrow \{-i\sigma_2, -\sigma_1, i\sigma_2, -\sigma_1\}_{(4)}$
$\mathbf{e}_5\,)\,\mathbf{e}_3 \leftrightarrow \{1, \sigma_3, -1, \sigma_3\}_{(4)}$	$\mathbf{e}_5\,)\,\mathbf{e}_4 \leftrightarrow \{-\sigma_1, - i\sigma_2, -\sigma_1, i\sigma_2\}_{(1)}$
$\mathbf{e}_5\,)\,\mathbf{e}_6 \leftrightarrow \{-\sigma_1, i\sigma_2, -\sigma_1, -i\sigma_2\}_{(2)}$	$\mathbf{e}_5\,)\,\mathbf{e}_7 \leftrightarrow \{\sigma_3, 1, \sigma_3, -1\}_{(2)}$
$\mathbf{e}_6\,)\,\mathbf{e}_1 \leftrightarrow \{i\sigma_2, \sigma_1, -\sigma_1, -i\sigma_2\}_{(4)}$	$\mathbf{e}_6\,)\,\mathbf{e}_2 \leftrightarrow \{\sigma_3, -1, 1, -\sigma_3\}_{(3)}$
$\mathbf{e}_6\,)\,\mathbf{e}_3 \leftrightarrow \{-\sigma_1, i\sigma_2, i\sigma_2, -\sigma_1\}_{(3)}$	$\mathbf{e}_6\,)\,\mathbf{e}_4 \leftrightarrow \{-\sigma_3, -1, -1, -\sigma_3\}_{(2)}$
$\mathbf{e}_6\,)\,\mathbf{e}_5 \leftrightarrow \{\sigma_1, -i\sigma_2, i\sigma_2, -\sigma_1\}_{(2)}$	$\mathbf{e}_6\,)\,\mathbf{e}_7 \leftrightarrow \{-\sigma_1, -i\sigma_2, -i\sigma_2, -\sigma_1\}_{(1)}$
$\mathbf{e}_7\,)\,\mathbf{e}_1 \leftrightarrow \{-1, -\sigma_3, \sigma_3, -1\}_{(4)}$	$\mathbf{e}_7\,)\,\mathbf{e}_2 \leftrightarrow \{\sigma_1, -i\sigma_2, -i\sigma_2, -\sigma_1\}_{(3)}$
$\mathbf{e}_7\,)\,\mathbf{e}_3 \leftrightarrow \{\sigma_3, -1, 1, \sigma_3\}_{(3)}$	$\mathbf{e}_7\,)\,\mathbf{e}_4 \leftrightarrow \{-\sigma_1, \sigma_2, -i\sigma_2, -\sigma_1\}_{(2)}$
$\mathbf{e}_7\,)\,\mathbf{e}_5 \leftrightarrow \{-\sigma_3, -1, -1, \sigma_3\}_{(2)}$	$\mathbf{e}_7\,)\,\mathbf{e}_6 \leftrightarrow \{\sigma_1, i\sigma_2, i\sigma_2, -\sigma_1\}_{(1)}$

where x_m are real numbers and

$\rho_1 = \mathbf{1}$,	$\rho_2 = \mathbf{e}_1$,	$\rho_3 = \mathbf{e}_2$,	$\rho_4 = \mathbf{e}_3$
$\rho_5 = \mathbf{e}_4$,	$\rho_5 = \mathbf{e}_5$,	$\rho_7 = \mathbf{e}_6$,	$\rho_8 = \mathbf{e}_7$
$\rho_9 = \mathbf{1} \mid \mathbf{e}_1$,	$\rho_{10} = \mathbf{1} \mid \mathbf{e}_2$,	$\rho_{11} = \mathbf{1} \mid \mathbf{e}_3$,	$\rho_{12} = \mathbf{1} \mid \mathbf{e}_4$
$\rho_{13} = \mathbf{1} \mid \mathbf{e}_5$,	$\rho_{14} = \mathbf{1} \mid \mathbf{e}_6$,	$\rho_{15} = \mathbf{1} \mid \mathbf{e}_7$,	$\rho_{16} = \mathbf{e}_1 \mid \mathbf{e}_1$
$\rho_{17} = \mathbf{e}_2 \mid \mathbf{e}_2$,	$\rho_{18} = \mathbf{e}_3 \mid \mathbf{e}_3$,	$\rho_{19} = \mathbf{e}_4 \mid \mathbf{e}_4$,	$\rho_{20} = \mathbf{e}_5 \mid \mathbf{e}_5$
$\rho_{21} = \mathbf{e}_6 \mid \mathbf{e}_6$,	$\rho_{22} = \mathbf{e}_7 \mid \mathbf{e}_7$,	$\rho_{23} = \mathbf{e}_1\,)\,\mathbf{e}_2$,	$\rho_{24} = \mathbf{e}_1\,)\,\mathbf{e}_3$
$\rho_{25} = \mathbf{e}_1\,)\,\mathbf{e}_4$,	$\rho_{26} = \mathbf{e}_1\,)\,\mathbf{e}_5$,	$\rho_{27} = \mathbf{e}_1\,)\,\mathbf{e}_6$,	$\rho_{28} = \mathbf{e}_1\,)\,\mathbf{e}_7$
$\rho_{29} = \mathbf{e}_2\,)\,\mathbf{e}_1$,	$\rho_{30} = \mathbf{e}_2\,)\,\mathbf{e}_3$,	$\rho_{31} = \mathbf{e}_2\,)\,\mathbf{e}_4$,	$\rho_{32} = \mathbf{e}_2\,)\,\mathbf{e}_5$
$\rho_{33} = \mathbf{e}_2\,)\,\mathbf{e}_6$,	$\rho_{34} = \mathbf{e}_2\,)\,\mathbf{e}_7$,	$\rho_{35} = \mathbf{e}_3\,)\,\mathbf{e}_1$,	$\rho_{36} = \mathbf{e}_3\,)\,\mathbf{e}_2$
$\rho_{37} = \mathbf{e}_3\,)\,\mathbf{e}_4$,	$\rho_{38} = \mathbf{e}_3\,)\,\mathbf{e}_5$,	$\rho_{39} = \mathbf{e}_3\,)\,\mathbf{e}_6$,	$\rho_{40} = \mathbf{e}_3\,)\,\mathbf{e}_7$
$\rho_{41} = \mathbf{e}_4\,)\,\mathbf{e}_1$,	$\rho_{42} = \mathbf{e}_4\,)\,\mathbf{e}_2$,	$\rho_{43} = \mathbf{e}_4\,)\,\mathbf{e}_3$,	$\rho_{44} = \mathbf{e}_4\,)\,\mathbf{e}_5$
$\rho_{45} = \mathbf{e}_4\,)\,\mathbf{e}_6$,	$\rho_{46} = \mathbf{e}_4\,)\,\mathbf{e}_7$,	$\rho_{47} = \mathbf{e}_5\,)\,\mathbf{e}_1$,	$\rho_{48} = \mathbf{e}_5\,)\,\mathbf{e}_2$
$\rho_{49} = \mathbf{e}_5\,)\,\mathbf{e}_3$,	$\rho_{50} = \mathbf{e}_5\,)\,\mathbf{e}_4$,	$\rho_{51} = \mathbf{e}_5\,)\,\mathbf{e}_6$,	$\rho_{52} = \mathbf{e}_5\,)\,\mathbf{e}_7$
$\rho_{53} = \mathbf{e}_6\,)\,\mathbf{e}_1$,	$\rho_{54} = \mathbf{e}_6\,)\,\mathbf{e}_2$,	$\rho_{55} = \mathbf{e}_6\,)\,\mathbf{e}_3$,	$\rho_{56} = \mathbf{e}_6\,)\,\mathbf{e}_4$
$\rho_{57} = \mathbf{e}_6\,)\,\mathbf{e}_5$,	$\rho_{58} = \mathbf{e}_6\,)\,\mathbf{e}_7$,	$\rho_{59} = \mathbf{e}_7\,)\,\mathbf{e}_1$,	$\rho_{60} = \mathbf{e}_7\,)\,\mathbf{e}_2$
$\rho_{61} = \mathbf{e}_7\,)\,\mathbf{e}_3$,	$\rho_{62} = \mathbf{e}_7\,)\,\mathbf{e}_4$,	$\rho_{63} = \mathbf{e}_7\,)\,\mathbf{e}_5$,	$\rho_{64} = \mathbf{e}_7\,)\,\mathbf{e}_6$

We also give the translation rules by right-barred operators (Table IX). Obviously we must modify equation (79) ($p_{23,\ldots,64}$ will represent right-barred operators).

Table VI. Translation Rules between 8 × 8 Real Matrices and Octonionic Right-Barred Operators

$\mathbf{e}_1 \,(\, \mathbf{e}_2 \leftrightarrow \{i\sigma_2, -i\sigma_2, -i\sigma_2, -i\sigma_2\}_{(2)}$	$\mathbf{e}_1 \,(\, \mathbf{e}_3 \leftrightarrow \{-1, -1, 1, -1\}_{(2)}$
$\mathbf{e}_1 \,(\, \mathbf{e}_4 \leftrightarrow \{i\sigma_2, i\sigma_2, -i\sigma_2, i\sigma_2\}_{(3)}$	$\mathbf{e}_1 \,(\, \mathbf{e}_5 \leftrightarrow \{-1, -1, -1, 1\}_{(3)}$
$\mathbf{e}_1 \,(\, \mathbf{e}_6 \leftrightarrow \{-\sigma_1, \sigma_1, -\sigma_1, -\sigma_1\}_{(4)}$	$\mathbf{e}_1 \,(\, \mathbf{e}_7 \leftrightarrow \{\sigma_3, -\sigma_3, \sigma_3, \sigma_3\}_{(4)}$
$\mathbf{e}_2 \,(\, \mathbf{e}_1 \leftrightarrow \{-\sigma_1, -\sigma_1, i\sigma_2, i\sigma_2\}_{(2)}$	$\mathbf{e}_2 \,(\, \mathbf{e}_3 \leftrightarrow \{\sigma_1, -\sigma_1, -i\sigma_2, i\sigma_2\}_{(1)}$
$\mathbf{e}_2 \,(\, \mathbf{e}_4 \leftrightarrow \{\sigma_3, -\sigma_3, -1, 1\}_{(4)}$	$\mathbf{e}_2 \,(\, \mathbf{e}_5 \leftrightarrow \{\sigma_1, -\sigma_1, i\sigma_2, -i\sigma_2\}_{(4)}$
$\mathbf{e}_2 \,(\, \mathbf{e}_6 \leftrightarrow \{-1, -\sigma_3, -\sigma_3, -\sigma_3\}_{(3)}$	$\mathbf{e}_2 \,(\, \mathbf{e}_7 \leftrightarrow \{-i\sigma_2, -i\sigma_2, -\sigma_1, -\sigma_1\}_{(3)}$
$\mathbf{e}_3 \,(\, \mathbf{e}_1 \leftrightarrow \{\sigma_3, \sigma_3, -1, 1\}_{(2)}$	$\mathbf{e}_3 \,(\mathbf{e}_2 \leftrightarrow \{-\sigma_1, -\sigma_1, i\sigma_2, -i\sigma_2\}_{(1)}$
$\mathbf{e}_3 \,(\, \mathbf{e}_4 \leftrightarrow \{\sigma_1, -\sigma_1, -i\sigma_2, -i\sigma_2\}_{(4)}$	$\mathbf{e}_3 \,(\, \mathbf{e}_5 \leftrightarrow \{-\sigma_3, \sigma_3, -1, -1\}_{(4)}$
$\mathbf{e}_3 \,(\, \mathbf{e}_6 \leftrightarrow \{i\sigma_2, i\sigma_2, \sigma_1, -\sigma_1\}_{(3)}$	$\mathbf{e}_3 \,(\, \mathbf{e}_7 \leftrightarrow \{-1, -1, -\sigma_3, \sigma_3\}_{(3)}$
$\mathbf{e}_4 \,(\, \mathbf{e}_1 \leftrightarrow \{-\sigma_1, -i\sigma_2, -\sigma_1, -i\sigma_2\}_{(3)}$	$\mathbf{e}_4 \,(\, \mathbf{e}_2 \leftrightarrow \{-\sigma_3, -1, -\sigma_3, -1\}_{(4)}$
$\mathbf{e}_4 \,(\, \mathbf{e}_3 \leftrightarrow \{-\sigma_1, i\sigma_2, -\sigma_1, i\sigma_2\}_{(4)}$	$\mathbf{e}_4 \,(\, \mathbf{e}_5 \leftrightarrow \{\sigma_3, -i\sigma_2, -\sigma_1, i\sigma_1\}_{(1)}$
$\mathbf{e}_4 \,(\, \mathbf{e}_6 \leftrightarrow \{1, \sigma_3, -1, -\sigma_3\}_{(2)}$	$\mathbf{e}_4 \,(\, \mathbf{e}_7 \leftrightarrow \{i\sigma_2, \sigma_2, -i\sigma_2, -\sigma_2\}_{(2)}$
$\mathbf{e}_5 \,(\, \mathbf{e}_1 \leftrightarrow \{\sigma_3, 1, \sigma_3, -1\}_{(3)}$	$\mathbf{e}_5 \,(\, \mathbf{e}_2 \leftrightarrow \{-\sigma_1, -i\sigma_2, -\sigma_1, i\sigma_2\}_{(4)}$
$\mathbf{e}_5 \,(\, \mathbf{e}_3 \leftrightarrow \{\sigma_3, -1, \sigma_3, 1\}_{(4)}$	$\mathbf{e}_5 \,(\, \mathbf{e}_4 \leftrightarrow \{-\sigma_1, i\sigma_2, -\sigma_1, -i\sigma_2\}_{(1)}$
$\mathbf{e}_5 \,(\, \mathbf{e}_6 \leftrightarrow \{-i\sigma_2, -\sigma_1, i\sigma_2, -\sigma_1\}_{(2)}$	$\mathbf{e}_5 \,(\, \mathbf{e}_7 \leftrightarrow \{1, \sigma_3, -1, \sigma_3\}_{(2)}$
$\mathbf{e}_6 \,(\, \mathbf{e}_1 \leftrightarrow \{i\sigma_2, -\sigma_1, \sigma_1, -i\sigma_2\}_{(4)}$	$\mathbf{e}_6 \,(\, \mathbf{e}_2 \leftrightarrow \{1, -\sigma_3, \sigma_3, -1\}_{(3)}$
$\mathbf{e}_6 \,(\, \mathbf{e}_3 \leftrightarrow \{-i\sigma_2, -\sigma_1, -\sigma_1, -i\sigma_2\}_{(3)}$	$\mathbf{e}_6 \,(\, \mathbf{e}_4 \leftrightarrow \{-1, -\sigma_3, -\sigma_3, -1\}_{(2)}$
$\mathbf{e}_6 \,(\, \mathbf{e}_5 \leftrightarrow \{i\sigma_2, \sigma_1, -\sigma_1, -i\sigma_2\}_{(2)}$	$\mathbf{e}_6 \,(\, \mathbf{e}_7 \leftrightarrow \{-\sigma_1, i\sigma_2, i\sigma_2, -\sigma_1\}_{(1)}$
$\mathbf{e}_7 \,(\, \mathbf{e}_1 \leftrightarrow \{-1, \sigma_3, -\sigma_3, -1\}_{(4)}$	$\mathbf{e}_7 \,(\, \mathbf{e}_2 \leftrightarrow \{i\sigma_2, -\sigma_1, \sigma_1, i\sigma_2\}_{(3)}$
$\mathbf{e}_7 \,(\, \mathbf{e}_3 \leftrightarrow \{1, \sigma_3, \sigma_3, -1\}_{(3)}$	$\mathbf{e}_7 \,(\, \mathbf{e}_4 \leftrightarrow \{-i\sigma_2, -\sigma_1, -\sigma_1, i\sigma_2\}_{(2)}$
$\mathbf{e}_7 \,(\, \mathbf{e}_5 \leftrightarrow \{-1, -\sigma_3, \sigma_3, -1\}_{(2)}$	$\mathbf{e}_7 \,(\, \mathbf{e}_6 \leftrightarrow \{\sigma_1, -i\sigma_2, -i\sigma_2, -\sigma_1\}_{(1)}$

APPENDIX B1

We give in Tables X–XIII the action of barred operators on octonionic functions

$$\psi = \psi_1 + e_2\psi_2 + e_4\psi_3 + e_6\psi_4 \qquad [\psi_{1,\ldots,4} \; \varepsilon \; \mathcal{C}(1, e_1)]$$

In these tables we use the notation

$$e_2 \rightarrow \{-\psi_2, \psi_1, -\psi_4^*, \psi_3^*\}$$

to indicate

$$e_2\psi = -\psi_2 + e_2\psi_1 - e_4\psi_4^* + e_6\psi_5^*$$

APPENDIX B2

In the following charts we establish the connection between 4 × 4 complex matrices and octonionic left/right-barred operators. We indicate with

Table VII. Real Coefficients for the Octonionic Barred Operators

$x_1 = (5\alpha_1 + \beta_2 + \gamma_3 + \delta_4 + \varepsilon_5 + \varphi_6 + \eta_7 + \lambda_8)/12$
$x_2 = (4\alpha_1 - \beta_1 - \gamma_4 - \varepsilon_6 + \varphi_5 + \eta_8 - \lambda_7)/10$
$x_3 = (5\alpha_3 + \beta_4 - \gamma_1 - \delta_2 - \varepsilon_7 - \varphi_8 + \eta_5 + \lambda_6)/12$
$x_4 = (5\alpha_4 - \beta_3 + \gamma_2 - \delta_1 - \varepsilon_8 + \varphi_7 - \eta_6 + \lambda_5)/12$
$x_5 = (5\alpha_5 + \beta_6 + \gamma_7 + \delta_8 - \varepsilon_1 - \varphi_2 - \eta_3 - \lambda_4)/12$
$x_6 = (5\alpha_6 - \beta_5 + \gamma_8 - \delta_7 + \varepsilon_2 - \varphi_1 + \eta_4 - \lambda_3)/12$
$x_7 = (5\alpha_7 - \beta_8 - \gamma_5 + \delta_6 + \varepsilon_3 - \varphi_4 - \eta_1 + \lambda_2)/12$
$x_8 = (3\alpha_8 - \beta_7 - \gamma_6 - \delta_5 + \varepsilon_4 + \varphi_3 - \eta_2 - \lambda_1)/12$
$x_9 = (\alpha_2 - 4\beta_1 + \gamma_4 + \varepsilon_6 - \varphi_5 - \eta_8 + \lambda_7)/10$
$x_{10} = (\alpha_3 - \beta_4 - 5\gamma_1 + \delta_2 + \varepsilon_7 + \varphi_8 - \eta_5 - \lambda_6)/12$
$x_{11} = (\alpha_4 + \beta_3 - \gamma_2 - 5\delta_1 + \varepsilon_8 - \varphi_7 + \eta_6 - \lambda_5)/12$
$x_{12} = (\alpha_5 - \beta_6 - \gamma_7 - \delta_8 - 5\varepsilon_1 + \varphi_2 + \eta_3 + \lambda_4)/12$
$x_{13} = (\alpha_6 + \beta_5 - \gamma_8 + \delta_7 - \varepsilon_2 - 5\varphi_1 - \eta_4 + \lambda_3)/12$
$x_{14} = (\alpha_7 + \beta_8 + \gamma_5 - \delta_6 - \varepsilon_3 + \varphi_4 - 5\eta_1 - \lambda_2)/12$
$x_{15} = (\alpha_8 + \beta_7 + \gamma_6 + \delta_5 - \varepsilon_4 - \varphi_3 + \eta_2 - 3\lambda_1)/8$
$x_{16} = (-\alpha_1 - 5\beta_2 + \gamma_3 + \delta_4 + \varepsilon_5 + \varphi_6 + \eta_7 + \lambda_8)/12$
$x_{17} = (-\alpha_1 + \beta_2 - 5\gamma_3 + \delta_4 + \varepsilon_5 + \varphi_6 + \eta_7 + \lambda_8)/12$
$x_{18} = (-\alpha_1 + \beta_2 + \gamma_3 - 5\delta_4 + \varepsilon_5 + \varphi_6 + \eta_7 + \lambda_8)/12$
$x_{19} = (-\alpha_1 + \beta_2 + \gamma_3 + \delta_4 - 5\varepsilon_5 + \varphi_6 + \eta_7 + \lambda_8)/12$
$x_{20} = (-\alpha_1 + \beta_2 + \gamma_3 + \delta_4 + \varepsilon_5 - 5\varphi_6 + \eta_7 + \lambda_8)/12$
$x_{21} = (-\alpha_1 + \beta_2 + \gamma_3 + \delta_4 + \varepsilon_5 + \varphi_6 - 5\eta_7 + \lambda_8)/12$
$x_{22} = (-\alpha_1 + \beta_2 + \gamma_3 + \delta_4 + \varepsilon_5 + \varphi_6 + \eta_7 - 5\lambda_8)/12$

$\mathcal{R}_{mn}$ ($\mathcal{C}_{mn}$) the 4×4 real (complex) matrices with 1 (i) in the mn-element and zeros elsewhere.

4×4 *Complex Matrices and Left-Barred Operators*:

$$\mathcal{R}_{11} \leftrightarrow \frac{1}{2}[1 - e_1 \mid e_1]$$

$$\mathcal{R}_{12} \leftrightarrow \frac{1}{6}[2e_1\,)\,e_3 + e_3\,)\,e_1 - 2 \mid e_2 - e_2 + e_4\,)\,e_6 - e_6\,)\,e_4 + e_5\,)\,e_7 - e_7\,)\,e_5]$$

$$\mathcal{R}_{13} \leftrightarrow \frac{1}{6}[2e_1\,)\,e_5 + e_5\,)\,e_1 - 2 \mid e_4 - e_4 + e_6\,)\,e_2 - e_2\,)\,e_6 + e_7\,)\,e_3 - e_3\,)\,e_7]$$

$$\mathcal{R}_{14} \leftrightarrow \frac{1}{6}[2e_1\,)\,e_7 + e_7\,)\,e_1 - 2 \mid e_6 - e_6 + e_2\,)\,e_4 - e_4\,)\,e_2 + e_5\,)\,e_3 - e_3\,)\,e_5]$$

$$\mathcal{R}_{21} \leftrightarrow \frac{1}{2}[e_2 + e_3\,)\,e_1]$$

$$\mathcal{R}_{22} \leftrightarrow \frac{1}{6}[1 + e_1 \mid e_1 + e_4 \mid e_4 + e_5 \mid e_5 + e_6 \mid e_6 + e_7 \mid e_7] - \frac{1}{3}[e_2 \mid e_2 + e_3 \mid e_3]$$

Table VIII. Real Coefficients for the Octonionic Left-Barred Operators

$x_{23} = (-\alpha_4 - \beta_3 - 5\gamma_2 - \delta_1 - \varepsilon_8 + \varphi_7 - \eta_6 + \lambda_5)/12$
$x_{24} = (\alpha_3 - \beta_4 + \gamma_1 - 5\delta_2 + \varepsilon_7 + \varphi_8 - \eta_5 - \lambda_6)/12$
$x_{25} = (-\alpha_6 - \beta_5 + \gamma_8 - \delta_7 - 5\varepsilon_2 - \varphi_1 + \eta_4 - \lambda_3)/12$
$x_{26} = (\alpha_5 - \beta_6 - \gamma_7 - \delta_8 + \varepsilon_1 - 5\varphi_2 + \eta_3 + \lambda_4)/12$
$x_{27} = (\alpha_8 + \beta_7 + \gamma_6 + \delta_5 - \varepsilon_4 - \varphi_3 - 3\eta_2 + \lambda_1)/12$
$x_{28} = (-\alpha_7 - \beta_8 - \gamma_5 + \delta_6 + \varepsilon_3 - \varphi_4 - \eta_1 - 5\lambda_2)/12$
$x_{29} = (\alpha_4 - 5\beta_3 - \gamma_2 + \delta_1 + \varepsilon_8 - \varphi_7 + \eta_6 - \lambda_5)/12$
$x_{30} = (-\alpha_2 - \beta_1 - \gamma_4 - 5\delta_3 - \varepsilon_6 + \varphi_5 + \eta_8 - \lambda_7)/12$
$x_{31} = (-\alpha_7 - \beta_8 - \gamma_5 + \delta_6 - 5\varepsilon_3 - \varphi_4 - \eta_1 + \lambda_2)/12$
$x_{32} = (\beta_7 - \varphi_3)/2$
$x_{33} = (\alpha_5 - \beta_6 - \gamma_7 - \delta_8 + \varepsilon_1 + \varphi_2 - 5\eta_3 + \lambda_4)/12$
$x_{34} = (\alpha_6 + \beta_5 - \gamma_8 + \delta_7 - \varepsilon_2 + \varphi_1 - \eta_4 - 5\lambda_3)/12$
$x_{35} = (-\alpha_3 - 5\beta_4 - \gamma_1 - \delta_2 - \varepsilon_7 - \varphi_8 + \eta_5 + \lambda_6)/12$
$x_{36} = (\alpha_2 + \beta_1 - 4\gamma_4 + \varepsilon_6 - \varphi_5 - \eta_8 + \lambda_7)/10$
$x_{37} = (-\alpha_8 - \beta_7 - \gamma_6 - \delta_5 - 3\varepsilon_4 + \varphi_3 - \eta_2 - \lambda_1)/8$
$x_{38} = (\alpha_7 + \beta_8 + \gamma_5 - \delta_6 - \varepsilon_3 - 5\varphi_4 + \eta_1 - \lambda_2)/12$
$x_{39} = (-\alpha_6 - \beta_5 + \gamma_8 - \delta_7 + \varepsilon_2 - \varphi_1 - 5\eta_4 - \lambda_3)/12$
$x_{40} = (\alpha_5 - \beta_6 - \gamma_7 - \delta_8 + \varepsilon_1 + \varphi_2 + \eta_3 - 5\lambda_4)/12$
$x_{41} = (\alpha_6 - 5\beta_5 - \gamma_8 + \delta_7 - \varepsilon_2 + \varphi_1 - \eta_4 + \lambda_3)/12$
$x_{42} = (\alpha_7 + \beta_8 - 5\gamma_5 - \delta_6 - \varepsilon_3 + \varphi_4 + \eta_1 - \lambda_2)/12$
$x_{43} = (-\beta_7 - \delta_5)/2$
$x_{44} = (-\alpha_2 - \beta_1 - \gamma_4 - \varepsilon_6 - 4\varphi_5 + \eta_8 - \lambda_7)/10$
$x_{45} = (-\alpha_3 + \beta_4 - \gamma_1 - \delta_2 - \varepsilon_7 - \varphi_8 - 5\eta_5 + \lambda_6)/12$
$x_{46} = (-\alpha_4 - \beta_3 + \gamma_2 - \delta_1 - \varepsilon_8 + \varphi_7 - \eta_6 - 5\lambda_5)/12$
$x_{47} = (-\alpha_5 - 5\beta_6 + \gamma_7 + \delta_8 - \varepsilon_1 - \varphi_2 - \eta_3 - \lambda_4)/12$
$x_{48} = (\alpha_8 + \beta_7 - 3\gamma_6 + \delta_5 - \varepsilon_4 - \varphi_3 + \eta_2 + \lambda_1)/8$
$x_{49} = (-\alpha_7 - \beta_8 - \gamma_5 - 5\delta_6 + \varepsilon_3 - \varphi_4 - \eta_1 + \lambda_2)/12$
$x_{50} = (\alpha_2 + \beta_1 + \gamma_4 - 4\varepsilon_6 - \varphi_5 - \eta_8 + \lambda_7)/10$
$x_{51} = (\alpha_4 + \beta_3 - \gamma_2 + \delta_1 + \varepsilon_8 - \varphi_7 - 5\eta_6 - \lambda_5)/12$
$x_{52} = (-\alpha_3 + \beta_4 - \gamma_1 - \delta_2 - \varepsilon_7 - \varphi_8 + \eta_5 - 5\lambda_6)/12$
$x_{53} = (-\alpha_8 - 5\beta_7 - \gamma_6 - \delta_5 + \varepsilon_4 + \varphi_3 - \eta_2 - \lambda_1)/8$
$x_{54} = (-\alpha_5 + \beta_6 - 5\gamma_7 + \delta_8 - \varepsilon_1 - \varphi_2 - \eta_3 - \lambda_4)/12$
$x_{55} = (\alpha_6 + \beta_5 - \gamma_8 - 5\delta_7 - \varepsilon_2 + \varphi_1 - \eta_4 + \lambda_3)/12$
$x_{56} = (\alpha_3 - \beta_4 + \gamma_1 + \delta_2 - 5\varepsilon_7 + \varphi_8 - \eta_5 - \lambda_6)/12$
$x_{57} = (-\alpha_4 - \beta_3 + \gamma_2 - \delta_1 - \varepsilon_8 - 5\varphi_7 - \eta_6 + \lambda_5)/12$
$x_{58} = (\alpha_2 + \beta_1 + \gamma_4 + \varepsilon_6 - \varphi_5 - \eta_8 - 4\lambda_7)/10$
$x_{59} = (\alpha_7 - 5\beta_8 + \gamma_5 - \delta_6 - \varepsilon_3 + \varphi_4 + \eta_1 - \lambda_2)/12$
$x_{60} = (-\alpha_6 - \beta_5 - 5\gamma_8 - \delta_7 + \varepsilon_2 - \varphi_1 + \eta_4 - \lambda_3)/12$
$x_{61} = (-\alpha_5 + \beta_6 + \gamma_7 - 5\delta_8 - \varepsilon_1 - \varphi_2 - \eta_3 - \lambda_4)/12$
$x_{62} = (\alpha_4 + \beta_3 - \gamma_2 + \delta_1 - 5\varepsilon_8 - \varphi_7 + \eta_6 - \lambda_5)/12$
$x_{63} = (\alpha_3 - \beta_4 + \gamma_1 + \delta_2 + \varepsilon_7 - 5\varphi_8 - \eta_5 - \lambda_6)/12$
$x_{64} = (\delta_3 - \eta_8)/2$

Table IX. Real Coefficients for the Octonionic Right-Barred Operators

$x_{23} = (-\alpha_4 - 5\beta_3 - \gamma_2 - \delta_1 + \varepsilon_8 - \varphi_7 + \eta_6 - \lambda_5)/12$
$x_{24} = (\alpha_3 - 5\beta_4 + \gamma_1 - \delta_2 - \varepsilon_7 - \varphi_8 + \eta_5 + \lambda_6)/12$
$x_{25} = (-\alpha_6 - 5\beta_5 - \gamma_8 + \delta_7 - \varepsilon_2 - \varphi_1 - \eta_4 + \lambda_3)/12$
$x_{26} = (\alpha_5 - 5\beta_6 + \gamma_7 + \delta_8 + \varepsilon_1 - \varphi_2 - \eta_3 - \lambda_4)/12$
$x_{27} = (\alpha_8 - 5\beta_7 - \gamma_6 - \delta_5 + \varepsilon_4 + \varphi_3 - \eta_2 + \lambda_1)/12$
$x_{28} = (-\alpha_7 - 5\beta_8 + \gamma_5 - \delta_6 - \varepsilon_3 + \varphi_4 - \eta_1 - \lambda_2)/12$
$x_{29} = (\alpha_4 - \beta_3 - 5\gamma_2 + \delta_1 - \varepsilon_8 + \varphi_7 - \eta_6 + \lambda_5)/12$
$x_{30} = (-\alpha_2 - \beta_1 - 5\gamma_4 - \delta_3 + \varepsilon_6 - \varphi_5 - \eta_8 + \lambda_7)/12$
$x_{31} = (-\alpha_7 + \beta_8 - 5\gamma_5 - \delta_6 - \varepsilon_3 + \varphi_4 - \eta_1 - \lambda_2)/12$
$x_{32} = (-\alpha_8 - \beta_7 - 5\gamma_6 + \delta_5 - \varepsilon_4 - \varphi_3 + \eta_2 - \lambda_1)/12$
$x_{33} = (\alpha_5 + \beta_6 - 5\gamma_7 + \delta_8 + \varepsilon_1 - \varphi_2 - \eta_3 - \lambda_4)/12$
$x_{34} = (\alpha_6 - \beta_5 - 5\gamma_8 - \delta_7 + \varepsilon_2 + \varphi_1 + \eta_4 - \lambda_3)/12$
$x_{35} = (-\alpha_3 - \beta_4 - \gamma_1 - 5\delta_2 + \varepsilon_7 + \varphi_8 - \eta_5 - \lambda_6)/12$
$x_{36} = (\alpha_2 + \beta_1 - \gamma_4 - 5\delta 3 - \varepsilon_6 + \varphi_5 + \eta_8 - \lambda_7)/12$
$x_{37} = (-\alpha_8 - \beta_7 + \gamma_6 - 5\delta_5 - \varepsilon_4 - \varphi_3 + \eta_2 - \lambda_1)/12$
$x_{38} = (\alpha_7 - \beta_8 - \gamma_5 - 5\delta_6 + \varepsilon_3 - \varphi_4 + \eta_1 + \lambda_2)/12$
$x_{39} = (-\alpha_6 + \beta_5 - \gamma_8 - 5\delta_7 - \varepsilon_2 - \varphi_1 - \eta_4 + \lambda_3)/12$
$x_{40} = (\alpha_5 + \beta_6 + \gamma_7 - 5\delta_8 + \varepsilon_1 - \varphi_2 - \eta_3 - \lambda_4)/12$
$x_{41} = (\alpha_6 - \beta_5 + \gamma_8 - \delta_7 - 5\varepsilon_2 + \varphi_1 + \eta_4 - \lambda_3)/12$
$x_{42} = (\alpha_7 - \beta_8 - \gamma_5 + \delta_6 - 5\varepsilon_3 - \varphi_4 + \eta_1 + \lambda_2)/12$
$x_{43} = (\alpha_8 + \beta_7 - \gamma_6 - \delta_5 - 5\varepsilon_4 + \varphi_3 - \eta_2 + \lambda_1)/12$
$x_{44} = (-\alpha_2 - \beta_1 + \gamma_4 - \delta 3 - 5\varepsilon_6 - \varphi_5 - \eta_8 + \lambda_7)/12$
$x_{45} = (-\alpha_3 - \beta_4 - \gamma_1 + \delta_2 - 5\varepsilon_7 + \varphi_8 - \eta_5 - \lambda_6)/12$
$x_{46} = (-\alpha_4 + \beta_3 - \gamma_2 - \delta_1 - 5\varepsilon_8 - \varphi_7 + \eta_6 - \lambda_5)/12$
$x_{47} = (-\alpha_5 - \beta_6 - \gamma_7 - \delta_8 - \varepsilon_1 - 5\varphi_2 + \eta_3 + \lambda_4)/12$
$x_{48} = (\alpha_8 + \beta_7 - \gamma_6 - \delta_5 + \varepsilon_4 - 5\varphi_3 - \eta_2 + \lambda_1)/12$
$x_{49} = (-\alpha_7 + \beta_8 + \gamma_5 - \delta_6 - \varepsilon_3 - 5\varphi_4 - \eta_1 - \lambda_2)/12$
$x_{50} = (\alpha_2 + \beta_1 - \gamma_4 + \delta 3 - \varepsilon_6 - 5\varphi_5 + \eta_8 - \lambda_7)/12$
$x_{51} = (\alpha_4 - \beta_3 + \gamma_2 + \delta_1 - \varepsilon_8 - 5\varphi_7 - \eta_6 + \lambda_5)/12$
$x_{52} = (-\alpha_3 - \beta_4 - \gamma_1 + \delta_2 + \varepsilon_7 - 5\varphi_8 - \eta_5 - \lambda_6)/12$
$x_{53} = (-\alpha_8 - \beta_7 + \gamma_6 + \delta_5 - \varepsilon_4 - \varphi_3 - 5\eta_2 - \lambda_1)/12$
$x_{54} = (-\alpha_5 - \beta_6 - \gamma_7 - \delta_8 - \varepsilon_1 + \varphi_2 - 5\eta_3 + \lambda_4)/12$
$x_{55} = (\alpha_6 - \beta_5 + \gamma_8 - \delta_7 + \varepsilon_2 + \varphi_1 - 5\eta_4 - \lambda_3)/12$
$x_{56} = (\alpha_3 + \beta_4 + \gamma_1 - \delta_2 - \varepsilon_7 - \varphi_8 - 5\eta_5 + \lambda_6)/12$
$x_{57} = (-\alpha_4 + \beta_3 - \gamma_2 - \delta_1 + \varepsilon_8 - \varphi_7 - 5\eta_6 - \lambda_5)/12$
$x_{58} = (\alpha_2 + \beta_1 - \gamma_4 + \delta_3 - \varepsilon_6 + \varphi_5 - 5\eta_8 - \lambda_7)/12$
$x_{59} = (\alpha_7 - \beta_8 - \gamma_5 + \delta_6 + \varepsilon_3 - \varphi_4 + \eta_1 - 5\lambda_2)/12$
$x_{60} = (-\alpha_6 + \beta_5 - \gamma_8 + \delta_7 - \varepsilon_2 - \varphi_1 - \eta_4 - 5\lambda_3)/12$
$x_{61} = (-\alpha_5 - \beta_6 - \gamma_7 - \delta_8 - \varepsilon_1 + \varphi_2 + \eta_3 - 5\lambda_4)/12$
$x_{62} = (\alpha_4 - \beta_3 + \gamma_2 + \delta_1 - \varepsilon_8 + \varphi_7 - \eta_6 - 5\lambda_5)/12$
$x_{63} = (\alpha_3 + \beta_4 + \gamma_1 - \delta_2 - \varepsilon_7 - \varphi_8 + \eta_5 - 5\lambda_6)/12$
$x_{64} = (-\alpha 2 - \beta 1 + \gamma 4 - \delta_3 + \varepsilon 6 - \varphi 5 - \eta_8 - 5\lambda_7)/12$

Table X. Action on ψ of the Octonionic Barred Operators e_m and $1 \mid e_m$

$\mathbf{e}_1 \rightarrow \{e_1\psi_1,\ -e_1\psi_2,\ -e_1\psi_3,\ -e_1\psi_4\}$	$\mathbf{1} \mid \mathbf{e}_1 \rightarrow \{e_1\psi_1,\ e_1\psi_2,\ e_1\psi_3,\ e_1\psi_4\}$
$\mathbf{e}_2 \rightarrow \{-\psi_2,\ \psi_1,\ -\psi_4^*,\ \psi_3^*\}$	$\mathbf{1} \mid \mathbf{e}_2 \rightarrow \{-\psi_2^*,\ \psi_1^*,\ \psi_4^*,\ -\psi_3^*\}$
$\mathbf{e}_3 \rightarrow \{-e_1\psi_2,\ -e_1\psi_1,\ -e_1\psi_4^*,\ e_1\psi_3^*\}$	$\mathbf{1} \mid \mathbf{e}_3 \rightarrow \{e_1\psi_2^*,\ -e_1\psi_1^*,\ e_1\psi_4^*,\ -e_1\psi_3^*\}$
$\mathbf{e}_4 \rightarrow \{-\psi_3,\ \psi_4^*,\ \psi_1,\ -\psi_2^*\}$	$\mathbf{1} \mid \mathbf{e}_4 \rightarrow \{-\psi_3^*,\ -\psi_4^*,\ \psi_1^*,\ \psi_2^*\}$
$\mathbf{e}_5 \rightarrow \{-e_1\psi_3,\ e_1\psi_4^*,\ -e_1\psi_1,\ -e_1\psi_2^*\}$	$\mathbf{1} \mid \mathbf{e}_5 \rightarrow \{e_1\psi_3^*,\ -e_1\psi_4^*,\ -e_1\psi_1^*,\ e_1\psi_2^*\}$
$\mathbf{e}_6 \rightarrow \{-\psi_4,\ -\psi_3^*,\ \psi_2^*,\ \psi_1\}$	$\mathbf{1} \mid \mathbf{e}_6 \rightarrow \{-\psi_4^*,\ \psi_3^*,\ -\psi_2^*,\ \psi_1^*\}$
$\mathbf{e}_7 \rightarrow \{e_1\psi_4,\ e_1\psi_3^*,\ -e_1\psi_2^*,\ e_1\psi_1\}$	$\mathbf{1} \mid \mathbf{e}_7 \rightarrow \{-e_1\psi_4^*,\ -e_1\psi_3^*,\ e_1\psi_2^*,\ e_1\psi_1^*\}$

Table XI. Action on ψ of the Octonionic Barred Operators $e_m\ e_m$

$\mathbf{e}_1 \mid \mathbf{e}_1 \rightarrow \{-\psi_1,\ \psi_2,\ \psi_3,\ \psi_4\}$	$\mathbf{e}_2 \mid \mathbf{e}_2 \rightarrow \{-\psi_1^*,\ -\psi_2^*,\ \psi_3,\ \psi_4\}$
$\mathbf{e}_3 \mid \mathbf{e}_3 \rightarrow \{-\psi_1^*,\ \psi_2^*,\ \psi_3,\ \psi_4\}$	$\mathbf{e}_4 \mid \mathbf{e}_4 \rightarrow \{-\psi_1^*,\ \psi_2,\ -\psi_3^*,\ \psi_4\}$
$\mathbf{e}_5 \mid \mathbf{e}_5 \rightarrow \{-\psi_1^*,\ \psi_2,\ \psi_3^*,\ \psi_4\}$	$\mathbf{e}_6 \mid \mathbf{e}_6 \rightarrow \{-\psi_1^*,\ \psi_2,\ \psi_3,\ -\psi_4^*\}$
$\mathbf{e}_7 \mid \mathbf{e}_7 \rightarrow \{-\psi_1^*,\ \psi_2,\ \psi_3,\ \psi_4^*\}$	

Table XII. Octonionic Left-Barred Operator Action on ψ

$\mathbf{e}_1\ )\ \mathbf{e}_2 \rightarrow \{-e_1\psi_2^*,\ -e_1\psi_1^*,\ e_1\psi_4^*,\ -e_1\psi_3^*\}$	$\mathbf{e}_1\ )\ \mathbf{e}_3 \rightarrow \{-\psi_2^*,\ -\psi_1^*,\ -\psi_4^*,\ \psi_3^*\}$
$\mathbf{e}_1\ )\ \mathbf{e}_4 \rightarrow \{-e_1\psi_3^*,\ -e_1\psi_4^*,\ -e_1\psi_1^*,\ e_1\psi_2^*\}$	$\mathbf{e}_1\ )\ \mathbf{e}_5 \rightarrow \{-\psi_3^*,\ \psi_4^*,\ -\psi_1^*,\ -\psi_2^*\}$
$\mathbf{e}_1\ )\ \mathbf{e}_6 \rightarrow \{-e_1\psi_4^*,\ e_1\psi_3^*,\ -e_1\psi_2^*,\ -e_1\psi_1^*\}$	$\mathbf{e}_1\ )\ \mathbf{e}_7 \rightarrow \{\psi_4^*,\ \psi_3^*,\ -\psi_2^*,\ \psi_1^*\}$
$\mathbf{e}_2\ )\ \mathbf{e}_1 \rightarrow \{-e_1\psi_2,\ e_1\psi_1\ -e_1\psi_4^*,\ e_1\psi_3^*\}$	$\mathbf{e}_2\ )\ \mathbf{e}_3 \rightarrow \{e_1\psi_1^*,\ e_1\psi_2^*,\ e_1\psi_3,\ e_1\psi_4\}$
$\mathbf{e}_2\ )\ \mathbf{e}_4 \rightarrow \{\psi_4,\ -\psi_3,\ -\psi_2^*,\ \psi_1^*\}$	$\mathbf{e}_2\ )\ \mathbf{e}_5 \rightarrow \{-e_1\psi_4,\ -e_1\psi_3,\ e_1\psi_2^*,\ e_1\psi_1^*\}$
$\mathbf{e}_2\ )\ \mathbf{e}_6 \rightarrow \{-\psi_3,\ -\psi_4,\ -\psi_1^*,\ -\psi_2^*\}$	$\mathbf{e}_2\ )\ \mathbf{e}_7 \rightarrow \{-e_1\psi_3,\ e_1\psi_4,\ e_1\psi_1^*,\ -e_1\psi_2^*\}$
$\mathbf{e}_3\ )\ \mathbf{e}_1 \rightarrow \{\psi_2,\ \psi_1,\ \psi_4^*,\ -\psi_3^*\}$	$\mathbf{e}_3\ )\ \mathbf{e}_2 \rightarrow \{-e_1\psi_1^*,\ e_1\psi_2^*,\ -e_1\psi_3,\ -e_1\psi_4\}$
$\mathbf{e}_3\ )\ \mathbf{e}_4 \rightarrow \{-e_1\psi_4,\ e_1\psi_3,\ e_1\psi_2^*,\ e_1\psi_1^*\}$	$\mathbf{e}_3\ )\ \mathbf{e}_5 \rightarrow \{-\psi_4,\ -\psi_3,\ \psi^*{}_2,\ -\psi_1^*\}$
$\mathbf{e}_3\ )\ \mathbf{e}_6 \rightarrow \{e_1\psi_3,\ e_1\psi_4,\ -e_1\psi_1^*,\ e_1\psi_2^*\}$	$\mathbf{e}_3\ )\ \mathbf{e}_7 \rightarrow \{-\psi_3,\ \psi_4,\ -\psi_1^*,\ -\psi_2^*\}$
$\mathbf{e}_4\ )\ \mathbf{e}_1 \rightarrow \{-e_1\psi_3,\ e_1\psi_4^*,\ e_1\psi_1,\ -e_1\psi_2^*\}$	$\mathbf{e}_4\ )\ \mathbf{e}_2 \rightarrow \{-\psi_4,\ -\psi_3^*,\ -\psi_2,\ -\psi_1^*\}$
$\mathbf{e}_4\ )\ \mathbf{e}_3 \rightarrow \{e_1\psi_4,\ e_1\psi_3^*,\ -e_1\psi_2,\ -e_1\psi_1^*\}$	$\mathbf{e}_4\ )\ \mathbf{e}_5 \rightarrow \{e_1\psi_1^*,\ e_1\psi_2,\ e_1\psi_3^*,\ e_1\psi_4\}$
$\mathbf{e}_4\ )\ \mathbf{e}_6 \rightarrow \{\psi_2,\ \psi_1^*,\ -\psi_4,\ -\psi_3^*\}$	$\mathbf{e}_4\ )\ \mathbf{e}_7 \rightarrow \{e_1\psi_2,\ -e_1\psi_1^*,\ e_1\psi_4,\ -e_1\psi_3^*\}$
$\mathbf{e}_5\ )\ \mathbf{e}_1 \rightarrow \{\psi_3,\ -\psi_4^*,\ \psi_1,\ \psi_2^*\}$	$\mathbf{e}_5\ )\ \mathbf{e}_2 \rightarrow \{e_1\psi_4,\ e_1\psi_3^*,\ e_1\psi_2,\ -e_1\psi_1^*\}$
$\mathbf{e}_5\ )\ \mathbf{e}_3 \rightarrow \{\psi_4,\ \psi_3^*,\ -\psi_2,\ \psi_1^*\}$	$\mathbf{e}_5\ )\ \mathbf{e}_4 \rightarrow \{-e_1\psi_1^*,\ -e_1\psi_2,\ e_1\psi_3^*,\ -e_1\psi_4\}$
$\mathbf{e}_5\ )\ \mathbf{e}_6 \rightarrow \{-e_1\psi_2,\ e_1\psi_1^*,\ e_1\psi_4,\ e_1\psi_3^*\}$	$\mathbf{e}_5\ )\ \mathbf{e}_7 \rightarrow \{\psi_2,\ \psi_1^*,\ \psi_4,\ -\psi_3^*\}$
$\mathbf{e}_6\ )\ \mathbf{e}_1 \rightarrow \{-e_1\psi_4,\ -e_1\psi_3^*,\ e_1\psi_2^*,\ e_1\psi_1\}$	$\mathbf{e}_6\ )\ \mathbf{e}_2 \rightarrow \{\psi_3,\ -\psi_4^*,\ \psi_1^*,\ -\psi_2\}$
$\mathbf{e}_6\ )\ \mathbf{e}_3 \rightarrow \{-e_1\psi_3,\ e_1\psi_4^*,\ e_1\psi_1^*,\ -e_1\psi_2\}$	$\mathbf{e}_6\ )\ \mathbf{e}_4 \rightarrow \{-\psi_2,\ -\psi_1^*,\ -\psi_4^*,\ -\psi_3\}$
$\mathbf{e}_6\ )\ \mathbf{e}_5 \rightarrow \{e_1\psi_2,\ -e_1\psi_1^*,\ e_1\psi_4^*,\ -e_1\psi_3\}$	$\mathbf{e}_6\ )\ \mathbf{e}_7 \rightarrow \{-e_1\psi_1^*,\ -e_1\psi_2,\ -e_1\psi_3,\ -e_1\psi_4^*\}$
$\mathbf{e}_7\ )\ \mathbf{e}_1 \rightarrow \{-\psi_4,\ -\psi_3^*,\ \psi_2^*,\ -\psi_1\}$	$\mathbf{e}_7\ )\ \mathbf{e}_2 \rightarrow \{e_1\psi_3,\ -e_1\psi_4^*,\ -e_1\psi_1^*,\ -e_1\psi_2\}$
$\mathbf{e}_7\ )\ \mathbf{e}_3 \rightarrow \{\psi_3,\ -\psi_4^*,\ \psi_1^*,\ \psi_2\}$	$\mathbf{e}_7\ )\ \mathbf{e}_4 \rightarrow \{-e_1\psi_2,\ e_1\psi_1^*,\ -e_1\psi_4^*,\ -e_1\psi_3\}$
$\mathbf{e}_7\ )\ \mathbf{e}_5 \rightarrow \{-\psi_2,\ -\psi_1^*,\ -\psi_4^*,\ \psi_3\}$	$\mathbf{e}_7\ )\ \mathbf{e}_6 \rightarrow \{e_1\psi_1^*,\ e_1\psi_2,\ e_1\psi_3,\ -e_1\psi_4^*\}$

Table XIII. Octonionic Right-Barred Operator Action on ψ

$\mathbf{e}_1 (\mathbf{e}_2 \to \{-e_1\psi_2^*, -e_1\psi_1^*, -e_1\psi_4^*, e_1\psi_3^*\}$	$\mathbf{e}_1 (\mathbf{e}_3 \to \{-\psi_2^*, -\psi_1^*, -\psi_3^*, \psi_4^*\}$
$\mathbf{e}_1 (\mathbf{e}_4 \to \{-e_1\psi_3^*, e_1\psi_4^*, -e_1\psi_1^*, -e_1\psi_2^*\}$	$\mathbf{e}_1 (\mathbf{e}_5 \to \{-\psi_3^*, -\psi_4^*, -\psi_1^*, \psi_2^*\}$
$\mathbf{e}_1 (\mathbf{e}_6 \to \{-e_1\psi_4^*, -e_1\psi_3^*, e_1\psi_2^*, -e_1\psi_1^*\}$	$\mathbf{e}_1 (\mathbf{e}_7 \to \{\psi_4^*, -\psi_3^*, \psi_2^*, \psi_1^*\}$
$\mathbf{e}_2 (\mathbf{e}_1 \to \{-e_1\psi_2, e_1\psi_1, e_1\psi_4^*, -e_1\psi_3^*\}$	$\mathbf{e}_2 (\mathbf{e}_3 \to \{e_1\psi_1^*, e_1\psi_2^*, -e_1\psi_3, -e_1\psi_4\}$
$\mathbf{e}_2 (\mathbf{e}_4 \to \{\psi_4^*, -\psi_3^*, -\psi_2, \psi_1\}$	$\mathbf{e}_2 (\mathbf{e}_5 \to \{e_1\psi_4^*, e_1\psi_3^*, e_1\psi_2, \psi_1\}$
$\mathbf{e}_2 (\mathbf{e}_6 \to \{-\psi_3^*, -\psi_4^*, -\psi_1, -\psi_2\}$	$\mathbf{e}_2 (\mathbf{e}_7 \to \{e_1\psi_3^*, -e_1\psi_4^*, e_1\psi_1, -e_1\psi_2\}$
$\mathbf{e}_3 (\mathbf{e}_1 \to \{\psi_2, \psi_1, -\psi_4^*, \psi_3^*\}$	$\mathbf{e}_3 (\mathbf{e}_2 \to \{-e_1\psi_1^*, e_1\psi_2^*, e_1\psi_3, e_1\psi_4\}$
$\mathbf{e}_3 (\mathbf{e}_4 \to \{e_1\psi_4^*, e_1\psi_3^*, -e_1\psi_2, e_1\psi_1\}$	$\mathbf{e}_3 (\mathbf{e}_5 \to \{-\psi_4^*, \psi_3^*, -\psi_2, -\psi_1\}$
$\mathbf{e}_3 (\mathbf{e}_6 \to \{-e_1\psi_3^*, e_1\psi_4^*, -e_1\psi_2, -e_1\psi_1\}$	$\mathbf{e}_3 (\mathbf{e}_7 \to \{-\psi_3^*, -\psi_4^*, -\psi_1, \psi_2\}$
$\mathbf{e}_4 (\mathbf{e}_1 \to \{-e_1\psi_3, -e_1\psi_4^*, e_1\psi_1, e_1\psi_2^*\}$	$\mathbf{e}_4 (\mathbf{e}_2 \to \{-\psi_4^*, -\psi_3, -\psi_2^*, -\psi_1\}$
$\mathbf{e}_4 (\mathbf{e}_3 \to \{-e_1\psi_4^*, e_1\psi_3, e_1\psi_2^*, -e_1\psi_1\}$	$\mathbf{e}_4 (\mathbf{e}_5 \to \{e_1\psi_1^*, -e_1\psi_2, e_1\psi_3^*, -e_1\psi_4\}$
$\mathbf{e}_4 (\mathbf{e}_6 \to \{\psi_2^*, \psi_1, -\psi_3, -\psi_4^*\}$	$\mathbf{e}_4 (\mathbf{e}_7 \to \{-e_1\psi_2^*, -e_1\psi_1, -e_1\psi_4^*, -e_1\psi_3\}$
$\mathbf{e}_5 (\mathbf{e}_1 \to \{\psi_3, \psi_4^*, \psi_1, -\psi_2^*\}$	$\mathbf{e}_5 (\mathbf{e}_2 \to \{-e_1\psi_4^*, -e_1\psi_3, e_1\psi_2^*, -e_1\psi_1\}$
$\mathbf{e}_5 (\mathbf{e}_3 \to \{\psi_4^*, -\psi_3, \psi_2^*, \psi_1\}$	$\mathbf{e}_5 (\mathbf{e}_4 \to \{-e_1\psi_1^*, e_1\psi_2, e_1\psi_3^*, e_1\psi_4\}$
$\mathbf{e}_5 (\mathbf{e}_6 \to \{e_1\psi_2^*, e_1\psi_1, e_1\psi_4^*, -e_1\psi_3\}$	$\mathbf{e}_5 (\mathbf{e}_7 \to \{\psi_2^*, \psi_1, -\psi_4^*, \psi_3\}$
$\mathbf{e}_6 (\mathbf{e}_1 \to \{-e_1\psi_4, e_1\psi_3^*, -e_1\psi_2^*, e_1\psi_1\}$	$\mathbf{e}_6 (\mathbf{e}_2 \to \{\psi_3^*, -\psi_4, \psi_1, -\psi_2^*\}$
$\mathbf{e}_6 (\mathbf{e}_3 \to \{e_1\psi_3^*, e_1\psi_4, e_1\psi_1, e_1\psi_2^*\}$	$\mathbf{e}_6 (\mathbf{e}_4 \to \{-\psi_2^*, -\psi_1, -\psi_4, -\psi_3^*\}$
$\mathbf{e}_6 (\mathbf{e}_5 \to \{-e_1\psi_2^*, -e_1\psi_1, e_1\psi_4, e_1\psi_3^*\}$	$\mathbf{e}_6 (\mathbf{e}_7 \to \{-e_1\psi_1^*, e_1\psi_2, e_1\psi_3, -e_1\psi_4^*\}$
$\mathbf{e}_7 (\mathbf{e}_1 \to \{-\psi_4, \psi_3^*, -\psi_2^*, -\psi_1\}$	$\mathbf{e}_7 (\mathbf{e}_2 \to \{-e_1\psi_3^*, e_1\psi_4, -e_1\psi_1, -e_1\psi_2^*\}$
$\mathbf{e}_7 (\mathbf{e}_3 \to \{\psi_3^*, \psi_4, \psi_1, -\psi_2^*\}$	$\mathbf{e}_7 (\mathbf{e}_4 \to \{e_1\psi_2^*, e_1\psi_1, e_1\psi_4, -e_1\psi_3^*\}$
$\mathbf{e}_7 (\mathbf{e}_5 \to \{-\psi_2^*, -\psi_1, \psi_4, -\psi_3^*\}$	$\mathbf{e}_7 (\mathbf{e}_6 \to \{e_1\psi_1^*, -e_1\psi_2, -e_1\psi_3, -e_1\psi_4^*\}$

$$\mathcal{R}_{23} \leftrightarrow \frac{1}{2}[-e_2 \,)\, e_4 - e_3 \,)\, e_5]$$

$$\mathcal{R}_{24} \leftrightarrow \frac{1}{2}[e_3 \,)\, e_7 - e_2 \,)\, e_6]$$

$$\mathcal{R}_{31} \leftrightarrow \frac{1}{2}[e_4 + e_5 \,)\, e_1]$$

$$\mathcal{R}_{32} \leftrightarrow \frac{1}{2}[-e_5 \,)\, e_3 - e_4 \,)\, e_2]$$

$$\mathcal{R}_{33} \leftrightarrow \frac{1}{6}[1 + e_1 \mid e_1 + e_2 \mid e_2 + e_3 \mid e_3 + e_6 \mid e_6 + e_7 \mid e_7]$$

$$-\frac{1}{3}[e_4 \mid e_4 + e_5 \mid e_5]$$

$$\mathcal{R}_{34} \leftrightarrow \frac{1}{2}[e_5 \,)\, e_7 - e_4 \,)\, e_6]$$

$$\mathcal{R}_{41} \leftrightarrow \frac{1}{2}[e_6 - e_7 \,)\, e_1]$$

$$\mathcal{R}_{42} \leftrightarrow \frac{1}{2}[e_7\,)\,e_3 - e_6\,)\,e_2]$$

$$\mathcal{R}_{43} \leftrightarrow \frac{1}{2}[e_7\,)\,e_5 - e_6\,)\,e_4]$$

$$\mathcal{R}_{44} \leftrightarrow \frac{1}{6}[1 + e_1 \mid e_1 + e_2 \mid e_2 + e_3 \mid e_3 + e_4 \mid e_4 + e_5 \mid e_5]$$

$$- \frac{1}{3}[e_6 \mid e_6 + e_7 \mid e_7]$$

$$\mathcal{C}_{11} \leftrightarrow \frac{1}{2}[1 \mid e_1 + e_1]$$

$$\mathcal{C}_{12} \leftrightarrow \frac{1}{6}[-2e_1\,)\,e_2 - e_3 - 2 \mid e_3 - e_2\,)\,e_1 + e_4\,)\,e_7 + e_6\,)\,e_5$$

$$- e_5\,)\,e_6 - e_7\,)\,e_4]$$

$$\mathcal{C}_{13} \leftrightarrow \frac{1}{6}[-2e_1\,)\,e_4 - e_5 - 2 \mid e_5 - e_4\,)\,e_1 - e_6\,)\,e_3 - e_2\,)\,e_7$$

$$+ e_7\,)\,e_2 + e_3\,)\,e_6]$$

$$\mathcal{C}_{14} \leftrightarrow \frac{1}{6}[-2e_1\,)\,e_6 + e_7 + 2 \mid e_7 - e_6\,)\,e_1 - e_2\,)\,e_5 + e_4\,)\,e_3$$

$$+ e_5\,)\,e_2 - e_3\,)\,e_4]$$

$$\mathcal{C}_{21} \leftrightarrow \frac{1}{2}[-e_3 + e_2\,)\,e_1]$$

$$\mathcal{C}_{22} \leftrightarrow \frac{1}{6}[1 \mid e_1 - e_1 + e_4\,)\,e_5 - e_5\,)\,e_4 - e_6\,)\,e_7 + e_7\,)\,e_6]$$

$$- \frac{1}{3}[e_2\,)\,e_3 - e_5\,)\,e_2]$$

$$\mathcal{C}_{23} \leftrightarrow \frac{1}{2}[-e_2\,)\,e_5 + e_3\,)\,e_4]$$

$$\mathcal{C}_{24} \leftrightarrow \frac{1}{2}[e_3\,)\,e_6 + e_2\,)\,e_7]$$

$$\mathcal{C}_{31} \leftrightarrow \frac{1}{2}[-e_5 + e_4\,)\,e_1]$$

$$\mathcal{C}_{32} \leftrightarrow \frac{1}{2}[e_5\,)\,e_2 - e_4\,)\,e_3]$$

$$\mathcal{C}_{33} \leftrightarrow \frac{1}{6}[1 \mid e_1 - e_1 + e_2\,)\,e_3 - e_3\,)\,e_2 - e_6\,)\,e_7 + e_7\,)\,e_6]$$

$$- \frac{1}{3}[e_4\,)\,e_5 - e_5\,)\,e_4]$$

$$\mathscr{C}_{34} \leftrightarrow \frac{1}{2}[e_5\ )\ e_6 + e_4\ )\ e_7]$$

$$\mathscr{C}_{41} \leftrightarrow \frac{1}{2}[e_7 + e_6\ )\ e_1]$$

$$\mathscr{C}_{42} \leftrightarrow \frac{1}{2}[-e_7\ )\ e_2 - e_6\ )\ e_3]$$

$$\mathscr{C}_{43} \leftrightarrow \frac{1}{2}[-e_7\ )\ e_4 - e_6\ )\ e_5]$$

$$\mathscr{C}_{44} \leftrightarrow \frac{1}{6}[1\ |\ e_1 - e_1 + e_2\ )\ e_3 - e_3\ )\ e_2 + e_4\ )\ e_5 - e_5\ )\ e_4]$$
$$-\frac{1}{3}[e_7\ )\ e_6 - e_6\ )\ e_7]$$

4 × 4 *Complex Matrices and Right-Barred Operators*:

$$\mathscr{R}_{11} \leftrightarrow \frac{1}{2}[1 - e_1\ |\ e_1]$$

$$\mathscr{R}_{12} \leftrightarrow \frac{1}{2}[-e_2 + e_3\ (\ e_1]$$

$$\mathscr{R}_{13} \leftrightarrow \frac{1}{2}[-e_4 + e_5\ (\ e_1]$$

$$\mathscr{R}_{14} \leftrightarrow \frac{1}{2}[-e_6 - e_7\ (\ e_1]$$

$$\mathscr{R}_{21} \leftrightarrow \frac{1}{6}[2e_1\ (\ e_3 + e_3\ (\ e_1 + 2\ |\ e_2 + e_2 + e_4\ (\ e_6 - e_6\ (\ e_4 + e_5\ (\ e_7 - e_7\ (\ e_5]$$

$$\mathscr{R}_{22} \leftrightarrow \frac{1}{6}[1 + e_1\ |\ e_1 + e_4\ |\ e_4 + e_5\ |\ e_5 + e_6\ |\ e_6 + e_7\ |\ e_7] - \frac{1}{3}[e_2\ |\ e_2 + e_3\ |\ e_3]$$

$$\mathscr{R}_{23} \leftrightarrow \frac{1}{2}[-e_5\ (\ e_3 - e_4\ (\ e_2]$$

$$\mathscr{R}_{24} \leftrightarrow \frac{1}{2}[e_7\ (\ e_3 - e_6\ (\ e_2]$$

$$\mathscr{R}_{31} \leftrightarrow \frac{1}{6}[2e_1\ (\ e_5 + e_5\ (\ e_1 + 2\ |\ e_4 + e_4 + e_6\ (\ e_2 - e_2\ (\ e_6 + e_7\ (\ e_3 - e_3\ (\ e_7]$$

$$\mathscr{R}_{32} \leftrightarrow \frac{1}{2}[-e_2\ (\ e_4 - e_3\ (\ e_5]$$

$$\mathcal{R}_{33} \leftrightarrow \frac{1}{6}[1+e_1 \mid e_1+e_2 \mid e_2+e_3 \mid e_3+e_6 \mid e_6+e_7 \mid e_7]-\frac{1}{3}[e_4 \mid e_4+e_5 \mid e_5]$$

$$\mathcal{R}_{34} \leftrightarrow \frac{1}{2}[e_7 (e_5-e_6 (e_4]$$

$$\mathcal{R}_{41} \leftrightarrow \frac{1}{6}[2e_1 (e_7+e_7 (e_1+2 \mid e_6+e_6+e_2 (e_4-e_4 (e_2+e_5 (e_3-e_3 (e_5]$$

$$\mathcal{R}_{42} \leftrightarrow \frac{1}{2}[e_3 (e_7-e_2 (e_6]$$

$$\mathcal{R}_{43} \leftrightarrow \frac{1}{2}[e_5 (e_7-e_4 (e_6]$$

$$\mathcal{R}_{44} \leftrightarrow \frac{1}{6}[1+e_1 \mid e_1+e_2 \mid e_2+e_3 \mid e_3+e_4 \mid e_4+e_5 \mid e_5]-\frac{1}{3}[e_6 \mid e_6+e_7 \mid e_7]$$

$$\mathcal{C}_{11} \leftrightarrow \frac{1}{2}[1 \mid e_1+e_1]$$

$$\mathcal{C}_{12} \leftrightarrow \frac{1}{2}[-e_2 (e_1-e_3]$$

$$\mathcal{C}_{13} \leftrightarrow \frac{1}{2}[-e_4 (e_1-e_5]$$

$$\mathcal{C}_{14} \leftrightarrow \frac{1}{2}[-e_6 (e_1+e_7]$$

$$\mathcal{C}_{21} \leftrightarrow \frac{1}{6}[2e_1 (e_2-e_3+-2 \mid e_3+e_2 (e_1+e_4 (e_7+e_6 (e_5-e_5 (e_6-e_7 (e_4]$$

$$\mathcal{C}_{22} \leftrightarrow \frac{1}{6}[1 \mid e_1-e_1-e_4 (e_5+e_5 (e_4+e_6 (e_7-e_7 (e_6]$$
$$-\frac{1}{3}[-e_2 (e_3+e_3 (e_2]$$

$$\mathcal{C}_{23} \leftrightarrow \frac{1}{2}[-e_5 (e_2+e_4 (e_3]$$

$$\mathcal{C}_{24} \leftrightarrow \frac{1}{2}[e_7 (e_2+e_6 (e_3]$$

$$\mathcal{C}_{31} \leftrightarrow \frac{1}{6}[2e_1 (e_4-e_5-2 \mid e_5+e_4 (e_1-e_6 (e_3-e_2 (e_7+e_7 (e_2+e_3 (e_6]$$

$$\mathcal{C}_{32} \leftrightarrow \frac{1}{2}[e_2 (e_5-e_3 (e_4]$$

$$\mathcal{C}_{33} \leftrightarrow \frac{1}{6}[1 \mid e_1 - e_1 - e_2 (e_3 + e_3 (e_2 + e_6 (e_7 - e_7 (e_6]$$

$$- \frac{1}{3}[-e_4 (e_5 + e_5 \mid e_4]$$

$$\mathcal{C}_{34} \leftrightarrow \frac{1}{2}[e_7 (e_4 + e_6 (e_5]$$

$$\mathcal{C}_{41} \leftrightarrow \frac{1}{6}[-2e_1 (e_6 - e_7 + 2 \mid e_7 + e_6 (e_1 - e_2 (e_5 + e_4 (e_3 + e_5 (e_2 - e_3 (e_4]$$

$$\mathcal{C}_{42} \leftrightarrow \frac{1}{2}[-e_3 (e_6 - e_2 (e_7]$$

$$\mathcal{C}_{43} \leftrightarrow \frac{1}{2}[-e_5 (e_6 - e_4 (e_7]$$

$$\mathcal{C}_{44} \leftrightarrow \frac{1}{6}[1 \mid e_1 - e_1 - e_2 (e_3 + e_3 (e_2 - e_4 (e_5 + e_5 \mid e_4]$$

$$- \frac{1}{3}[e_6 (e_7 - e_7 ((e_6]$$

The previous tables could be very useful in order to extract octonionic operator multiplication rules or connections between left/right- barred operators. For example, we quickly find

$$[e_2) e_7 + e_3) e_6] \leftrightarrow 2\mathcal{C}_{24} \leftrightarrow [e_7 (e_2 + e_6 (e_3], \quad \text{and so on}$$

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