Octonionic representations of $GL(8,\mathscr{R})$ and $GL(4,\mathscr{C})$

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Octonionic algebra being nonassociative is difficult to manipulate. We introduce left/right octonionic barred operators which enable us to reproduce the associative $GL(8,\mathscr{R})$ group. Extracting the basis of $GL(4,\mathscr{C})$, we establish an interesting connection between the structure of left/right octonionic barred operators and generic 4×4 complex matrices. As an application we give an octonionic representation of the four-dimensional Clifford algebra. © *1997 American Institute of Physics.* [S0022-2488(97)00701-9]

I. INTRODUCTION

Semi-simple Lie groups, classified in four categories, orthogonal groups, unitary groups, symplectic groups and exceptional groups, were respectively associated with real, complex, quaternionic and octonionic algebras. Thus, such algebras became the core of the classification of possible symmetries in physics.[1–4]

We know that the anti-Hermitian generators of $SU(2,\mathscr{C})$ can be represented by the three quaternionic imaginary units e_1, e_2, e_3:

$$e_1 \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 \leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 \leftrightarrow \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \tag{1}$$

It permits any quaternionic numbers or matrix to be translated into a complex matrix but *not* necessarily vice-versa. In fact, to define the most general 2×2 complex matrix, we need eight real numbers. This problem is solved by introducing the barred quaternion $1|e_1$ $(\leftrightarrow i\mathbb{1}_{2\times 2})$ which allows us to obtain a faithful quaternionic representation of $GL(2,\mathscr{C})$.[5]

Exploiting the barred operator idea, we find the following 16 quaternionic operators

$$1, \quad \mathbf{Q}, \quad 1|\mathbf{Q}, \quad e_1|\mathbf{Q}, \quad e_2|\mathbf{Q}, \quad e_3|\mathbf{Q}, \tag{2}$$

where $\mathbf{Q}\equiv(e_1,e_2,e_3)$. These operators become essential to formulate special relativity with real quaternions,[6] allowing us to overcome the difficulties which in the past did not permit a (real) quaternionic version of special relativity. Besides, they can be used to give a representation of $GL(4,\mathscr{R})$. The situation can be summarized as follows:

$$GL(2,\mathscr{C}) \leftrightarrow q+p|e_1, \quad GL(4,\mathscr{R}) \leftrightarrow q+p|e_1+r|e_2+s|e_3,$$

with q, p, r, s quaternionic numbers.

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Inspired by this sequence we try to extend it and find an isomorphism between octonions and 8×8 real [or 4×4 complex] matrices. Obviously a first difficulty is the following: The octonionic algebra is nonassociative whereas $GL(8,\mathscr{R})$ [or $GL(4,\mathscr{C})$], satisfying the Jacobi identity, is associative. This seems a hopeless situation.

In this paper, we introduce left/right octonionic barred operators which enables us to find translation rules between 8×8 real matrices and octonionic numbers. On our road we also find an interesting isomorphism between the structure of left/right octonionic barred operators, on the one hand, and 4×4 complex matrices, on the other hand.

This article is organized as follows: In section II, we give a brief introduction to the octonionic division algebra. In section III, we discuss octonionic barred operators and explain the need to distinguish between left-barred and right-barred operators. In section IV, we investigate the relation between barred octonions and 8×8 real matrices. In this section, we also give the translation rules between our octonionic barred operators and $GL(4,\mathscr{C})$ and as an application we write down octonionic representations of the four-dimensional Clifford algebra. Two appendices, containing explicit octonionic representation of $GL(8,\mathscr{R})$ and $GL(4,\mathscr{C})$, are included. Our conclusions and future developments are drawn in the final section.

II. OCTONIONIC ALGEBRA

A remarkable theorem of Albert[7] shows that the only algebras, $\mathscr{A}$, over the reals, with unit element and admitting a real modulus function $N(a)$ $(a\in\mathscr{A})$ with the following properties,

$$N(0)=0, \tag{3a}$$

$$N(a)>0 \quad \text{if } a\neq 0, \tag{3b}$$

$$N(ra)=|r|N(a) \quad (r\in\mathscr{R}), \tag{3c}$$

$$N(a_1a_2)\leqslant N(a_1)+N(a_2), \tag{3d}$$

are the reals, $\mathscr{R}$, the complex, $\mathscr{C}$, the quaternions, $\mathscr{H}$ ($\mathscr{H}$ in honor of Hamilton[8]), and the octonions, $\mathscr{O}$ (or Graves–Cayley numbers[9,10]). Albert's theorem generalizes famous nineteenth-century results of Frobenius[11] and Hurwitz,[12] who first reached the same conclusion but with the additional assumption that $N(a)^2$ is a quadratic form.

In addition to Albert's theorem on algebras admitting a modulus function $N(a)$, we can characterize the algebras $\mathscr{R}$, $\mathscr{C}$, $\mathscr{H}$ and $\mathscr{O}$ by the concept of *division algebra* (in which one has no nonzero divisors of zero). A classical theorem[13,14] states that the only division algebras over the reals are algebras of dimensions 1, 2, 4, and 8, the only associative division algebras over the reals are $\mathscr{R}$, $\mathscr{C}$ and $\mathscr{H}$, whereas the *nonassociative* algebras include the octonions $\mathscr{O}$ (an interesting discussion concerning nonassociative algebras is presented in Ref. 15). For a very nice review of aspects of the quaternionic and octonionic algebras see Ref. 16 and the recent book by Adler.[17] In this paper we will deal with octonions and their generalizations.

We now summarize our notation for the octonionic algebra and introduce useful elementary properties to manipulate the nonassociative numbers. There is a number of equivalent ways to represent the octonions multiplication table. Fortunately, it is always possible to choose an orthonormal basis $(e_0,\ldots,e_7)$ such that

$$\mathscr{O}=r_0+\sum_{m=1}^{7} r_m e_m \quad (r_{0,\ldots,7} \text{ reals}), \tag{4}$$

where e_m are elements obeying the noncommutative and nonassociative algebra

$$e_m e_n=-\delta_{mn}+\epsilon_{mnp}e_p \quad (m,n,p=1,\ldots,7), \tag{5}$$

with ϵ_{mnp} totally antisymmetric and equal to unity for the seven combinations

$$123,\ 145,\ 176,\ 246,\ 257,\ 347 \quad \text{and}\ 365$$

(each cycle represents a quaternionic subalgebra). The norm, $N(\mathscr{O})$, for the octonions is defined by

$$N(\mathscr{O})=(\mathscr{O}^\dagger\mathscr{O})^{1/2}=(\mathscr{O}\mathscr{O}^\dagger)^{1/2}=(r_0^2+\ldots+r_7^2)^{1/2}, \tag{6}$$

with the octonionic conjugate $\mathscr{O}^\dagger$ given by

$$\mathscr{O}^\dagger=r_0-\sum_{m=1}^{7} r_m e_m\,. \tag{7}$$

The inverse is then

$$\mathscr{O}^{-1}=\mathscr{O}^\dagger/N(\mathscr{O}) \quad (\mathscr{O}\neq 0). \tag{8}$$

We can define an *associator* (analogous to the usual algebraic commutator) as follows:

$$\{x,y,z\}\equiv(xy)z-x(yz), \tag{9}$$

where, in each term on the right-hand side, we must, first of all, perform the multiplication in brackets. Note that for real, complex and quaternionic numbers the associator is trivially null. For octonionic imaginary units we have

$$\{e_m,e_n,e_p\}\equiv(e_me_n)e_p-e_m(e_ne_p)=2\epsilon_{mnps}e_s\,, \tag{10}$$

with ϵ_{mnps} totally antisymmetric and equal to unity for the seven combinations

$$1247,\ 1265,\ 2345,\ 2376,\ 3146,\ 3157 \quad \text{and}\ 4567.$$

Working with octonionic numbers the associator (9) is in general nonvanishing; however, the ''alternative condition'' is fulfilled:

$$\{x,y,z\}+\{z,y,x\}=0. \tag{11}$$

III. LEFT/RIGHT-BARRED OPERATORS

In 1989, writing a quaternionic Dirac equation,[18] Rotelli introduced a *barred* momentum operator

$$-\boldsymbol{\partial}|i \quad [(-\boldsymbol{\partial}|i)\psi\equiv-\boldsymbol{\partial}\psi i]. \tag{12}$$

In recent papers,[19] *partially barred quaternions*,

$$q+p|i \quad [q,p\in\mathscr{H}], \tag{13}$$

have been used to formulate a quaternionic quantum mechanics and field theory. From the viewpoint of group structure, these barred numbers are very similar to complexified quaternions[20]

$$q+\mathscr{I}p \tag{14}$$

(the imaginary unit $\mathscr{I}$ commutes with the quaternionic imaginary units i, j, k), but in physical problems, like eigenvalue calculations, tensor products, and relativistic equations solutions, they give different results.

A complete generalization for quaternionic numbers is represented by the following barred operators:

$$q_1+q_2|i+q_3|j+q_4|k \quad [q_{1,\dots,4}\in\mathscr{H}], \tag{15}$$

which we call *fully barred quaternions*, or simply barred quaternions. They, with their 16 linearly independent elements, form a basis of $GL(4,\mathscr{R})$. They are successfully used to reformulate Lorentz space–time transformations[6] and write down a one-component Dirac equation.[21]

Thus, it seems to us natural to investigate the existence of *barred octonions*

$$\mathscr{O}_0+\sum_{m=1}^{7}\mathscr{O}_m|e_m \quad [\mathscr{O}_{0,\dots,7}\ \text{octonions}]. \tag{16}$$

Nevertheless, we must observe that an octonionic *barred* operator, **a**|**b**, which acts on octonionic wave functions, ψ,

$$[a|b]\psi\equiv a\psi b,$$

is not a well defined object. For $a\neq b$ the triple product $a\psi b$ could be either $(a\psi)b$ or $a(\psi b)$. So, in order to avoid the ambiguity due to the nonassociativity of the octonionic numbers, we need to define left/right-barred operators. We will indicate *left-barred* operators by **a**)**b**, with a and b which represent octonionic numbers. They act on octonionic functions ψ as follows:

$$[a)b]\psi=(a\psi)b. \tag{17a}$$

In similar way we can introduce *right-barred* operators, defined by **a**(**b**,

$$[a(b]\psi=a(\psi b). \tag{17b}$$

Obviously, there are barred-operators in which the nonassociativity is not of relevance, like

$$1)a=1(a\equiv 1|a.$$

Furthermore, from Eq. (11), we have

$$\{x,y,x\}=0,$$

so

$$a)a=a(a\equiv a|a.$$

At first glance it seems that we must consider the following 106 barred-operators:

$$\begin{array}{lr} 1,e_m,1|e_m & (15\ \text{elements}),\\ e_m|e_m & (7),\\ e_m)e_n\ \ (m\neq n) & (42),\\ e_m(e_n\ \ (m\neq n) & (42),\\ (m,n=1,\dots,7). & \end{array}$$

Nevertheless, it is possible to prove that each right-barred operator can be expressed by a suitable combination of left-barred operators. For example, from Eq. (11), by posing $x=e_m$ and $z=e_n$, we quickly obtain

$$e_m(e_n + e_n(e_m \equiv e_m)e_n + e_n)e_m\,. \tag{18}$$

So we can represent the most general octonionic operator by only left-barred objects,

$$\mathscr{O}_0 + \sum_{m=1}^{7} \mathscr{O}_m)e_m \ \ [\mathscr{O}_{0,\ldots,7} \text{ octonions}], \tag{19}$$

reducing to 64 the previous 106 elements. This suggests a correspondence between our barred octonions (19) and $GL(8,\mathscr{R})$ (a complete discussion about the above-mentioned relationship is given in the following section).

IV. TRANSLATION RULES

The nonassociativity of octonions represents a challenge. We overcome the problems due to the octonions nonassociativity by introducing left/right-barred operators. We discuss in the next subsection their relation to $GL(8,\mathscr{R})$. In that subsection, we present our translation idea and give some explicit examples which allow us to establish the isomorphism between our octonionic left/right-barred operators and $GL(8,\mathscr{R})$. In subsection IV B, we focus our attention on the group $GL(4,\mathscr{C})\subset GL(8,\mathscr{R})$. In doing so, we find that only particular combinations of octonionic barred operators give us suitable candidates for the $GL(4,\mathscr{C})$-translation. Finally, in subsection IV C, we explicitly give two octonionic representations for the Dirac gamma-matrices (and consequently we are able to write down, for the first time, octonionic representations for the four-dimensional Clifford algebra).

A. Relation between barred operators and 8×8 real matrices

In order to explain the idea of translation, let us look explicitly at the action of the operators $1|e_1$ and e_2, on a generic octonionic function φ:

$$\varphi = \varphi_0 + e_1\varphi_1 + e_2\varphi_2 + e_3\varphi_3 + e_4\varphi_4 + e_5\varphi_5 + e_6\varphi_6 + e_7\varphi_7 \ \ [\varphi_{0,\ldots,7} \in \mathscr{R}]. \tag{20}$$

We have

$$[1|e_1]\varphi \equiv \varphi e_1 = e_1\varphi_0 - \varphi_1 - e_3\varphi_2 + e_2\varphi_3 - e_5\varphi_4 + e_4\varphi_5 + e_7\varphi_6 - e_6\varphi_7\,, \tag{21a}$$

$$e_2\varphi = e_2\varphi_0 - e_3\varphi_1 - \varphi_2 + e_1\varphi_3 + e_6\varphi_4 + e_7\varphi_5 - e_4\varphi_6 - e_5\varphi_7\,. \tag{21b}$$

If we represent our octonionic function φ by the following real column vector,

$$\varphi \leftrightarrow \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix}, \tag{22}$$

we can rewrite Eqs. (21a) and (21b) in matrix form,

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix} = \begin{pmatrix} -\varphi_1 \\ \varphi_0 \\ \varphi_3 \\ -\varphi_2 \\ \varphi_5 \\ -\varphi_4 \\ -\varphi_7 \\ \varphi_6 \end{pmatrix}, \tag{23a}$$

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix} = \begin{pmatrix} -\varphi_2 \\ \varphi_3 \\ \varphi_0 \\ -\varphi_1 \\ -\varphi_6 \\ -\varphi_7 \\ \varphi_4 \\ \varphi_5 \end{pmatrix}, \tag{23b}$$

In this way we can immediately obtain a real matrix representation for the octonionic barred operators $1|e_1$ and e_2. Following this procedure we can construct the complete set of translation rules for the imaginary units e_m and the barred operators $1|e_m$ (appendix A). In this paper we will use the notation of Refs. 22–24: L_m and R_m will represent the matrix counterpart of the octonionic operators e_m and $1|e_m$,

$$L_m \leftrightarrow e_m \quad \text{and} \quad R_m \leftrightarrow 1|e_m . \tag{24}$$

At first glance it seems that our translation does not work. If we extract the matrices corresponding to e_1, e_2 and e_3, namely,

$$L_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$L_2=\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$L_3=\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

we find

$$L_1L_2\neq L_3. \tag{25}$$

In obvious contrast with the octonionic relation

$$e_1e_2=e_3. \tag{26}$$

This bluff is soon explained. In deducing our translation rules, we understand octonions as operators, and so they must be applied to a certain octonionic function, φ, and *not* upon another ''operator''. So the octonionic relation

$$e_3\varphi[=(e_1e_2)\varphi] \tag{27a}$$

is translated by

$$L_3\varphi, \tag{27b}$$

whereas

$$e_1(e_2\varphi)[\neq e_3\varphi] \tag{28a}$$

becomes

$$L_1L_2\varphi[\neq L_3\varphi]. \tag{28b}$$

We have to differentiate between two kinds of multiplication, ''$\cdot$'' and ''$\times$''. At the level of octonions, one has

$$e_1\cdot e_2=e_3, \tag{29}$$

but at level of octonionic operators

$$e_1 \times e_2 \neq e_3 . \tag{30}$$

For e_m and $1|e_m$, we have simple ''$\times$''-multiplication rules. In fact, utilizing the associator properties we find

$$e_m(e_n\varphi) = (e_m e_n)\varphi + (e_m\varphi)e_n - e_m(\varphi e_n), \tag{31a}$$

$$(\varphi e_m)e_n = \varphi(e_m e_n) - (e_m\varphi)e_n + e_m(\varphi e_n). \tag{31b}$$

Thus,

$$e_m \times e_n \equiv -\delta_{mn} + \epsilon_{mnp}e_p + e_m)e_n - e_m(e_n , \tag{32a}$$

$$[1|e_n]\times[1|e_m] \equiv -\delta_{mn} + \epsilon_{mnp}e_p - e_m)e_n + e_m(e_n . \tag{32b}$$

The previous relation can be soon rewritten in matrix form as follows:[22]

$$L_m L_n \equiv -\delta_{mn} + \epsilon_{mnp}L_p + [R_n , L_m], \tag{33a}$$

$$R_n R_m \equiv -\delta_{mn} + \epsilon_{mnp}R_p + [L_m , R_n]. \tag{33b}$$

Introducing a new matrix multiplication, ''$\circ$'', related to the standard matrix multiplication (row by column) by

$$L_m \circ L_n \equiv L_m L_n + [R_n , L_m], \tag{34}$$

we can quickly reproduce the nonassociative octonionic algebra

$$L_m \circ L_n = -\delta_{mn} + \epsilon_{mnp}L_p . \tag{35}$$

Working with left/right-barred operators we show how the nonassociativity is inherent in our representation. Such operators enable us to reproduce the octonions nonassociativity by the matrix algebra. Consider, for example,

$$[e_3)e_1]\varphi \equiv (e_3\varphi)e_1 = e_2\varphi_0 - e_3\varphi_1 + \varphi_2 - e_1\varphi_3 - e_6\varphi_4 - e_7\varphi_5 + e_4\varphi_6 + e_5\varphi_7 . \tag{36}$$

This equation will be translated into

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix} = \begin{pmatrix} \varphi_2 \\ -\varphi_3 \\ \varphi_0 \\ -\varphi_1 \\ \varphi_6 \\ \varphi_7 \\ -\varphi_4 \\ -\varphi_5 \end{pmatrix} , \tag{37}$$

whereas

$$[e_3(e_1]\varphi \equiv e_3(\varphi e_1) = e_2\varphi_0 - e_3\varphi_1 + \varphi_2 - e_1\varphi_3 + e_6\varphi_4 + e_7\varphi_5 - e_4\varphi_6 - e_5\varphi_7 \tag{38}$$

will become

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix} = \begin{pmatrix} \varphi_2 \\ -\varphi_3 \\ \varphi_0 \\ -\varphi_1 \\ -\varphi_6 \\ -\varphi_7 \\ \varphi_4 \\ \varphi_5 \end{pmatrix}. \tag{39}$$

The nonassociativity is then reproduced since left- and right-barred operators like

$$e_3)e_1 \quad \text{and} \quad e_3(e_1$$

are represented by different matrices. The complete set of translation rules for left/right-barred operators is given in appendix A.

The matrix representation for left/right-barred operators can be quickly obtained by suitable multiplications of the matrices L_m and R_m. Let us clear up our assertion. By direct calculations we can extract the matrices which correspond to the operators

$$e_m)e_n \quad \text{and} \quad e_m(e_n,$$

which we call, respectively,

$$M^L_{mn} \quad \text{and} \quad M^R_{mn}.$$

Our left/right-barred operators can be represented by an ordered action of the operators e_m and $1|e_m$, and so we can related the matrices M^L_{mn} and M^R_{mn} to the matrices L_m and R_m:

$$M^L_{mn} \equiv R_n L_m, \tag{40a}$$

$$M^R_{mn} \equiv L_m R_n. \tag{40b}$$

The previous discussions concerning the octonions' nonassociativity and the isomorphism between $GL(8,\mathscr{R})$ and barred octonions, can be now, elegantly, presented as follows.

1. Matrix representation for octonions nonassociativity

$$M^L_{mn} \neq M^R_{mn} \quad [R_n L_m \neq L_m R_n \quad \text{for} \quad m \neq n]. \tag{41}$$

2. Isomorphism between GL(8,$\mathscr{R}$) and barred octonions

If we rewrite our 106 barred operators by real matrices,

$$\begin{aligned} &1,\ L_m,\ R_m && (15\ \text{matrices}),\\ &M \equiv L_m R_m = R_m L_m && (7),\\ &M^L_{mn} \equiv R_n L_m\ \ (m \neq n) && (42),\\ &M^R_{mn} \equiv L_n R_m\ \ (m \neq n) && (42), \end{aligned}$$

$$(m,n=1,\dots,7),$$

we have two different basis for $GL(8,\mathscr{R})$:

$$(1) \quad 1,\ L_m,\ R_m,\ R_nL_m,$$

$$(2) \quad 1,\ L_m,\ R_m,\ L_mR_n.$$

We now remark some difficulties deriving from the octonions' nonassociativity. When we translate from barred octonions to 8×8 real matrices there is no problem. For example, in the octonionic equation

$$e_4\{[(e_6\varphi)e_1]e_5\}, \tag{42}$$

we quickly recognize the following left-barred operators,

$$e_4(e_5 \quad \text{and} \quad e_6)e_1.$$

We can translate Eq. (42) into

$$M^L_{45}M^L_{61}\varphi. \tag{43}$$

In going from 8×8 real matrices to octonions we should be careful in ordering. For example,

$$AB\varphi \tag{44}$$

can be understood as

$$(AB)\varphi \tag{45a}$$

or

$$A(B\varphi). \tag{45b}$$

The first choice is related to the "×" multiplication (different from the standard octonionic multiplication). In order to avoid confusion we translate Eq. (44) by Eq. (45b). In general

$$ABC\dots Z\varphi \equiv A(B(C\dots(Z\varphi)\dots)). \tag{46}$$

B. Relation between barred operators and 4×4 complex matrices

Some complex groups play a critical role in physics. No one can deny the importance of $U(1,\mathscr{C})$ or $SU(2,\mathscr{C})$. In relativistic quantum mechanics, $GL(4,\mathscr{C})$ is essential in writing the Dirac equation. Having $GL(8,\mathscr{R})$, we should be able to extract its subgroup $GL(4,\mathscr{C})$. So, we can translate the famous Dirac-gamma matrices and write down a one-component octonionic Dirac equation.[25]

Let us show how we can isolate our 32 basis of $GL(4,\mathscr{C})$: Working with the symplectic decomposition of octonions

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} \leftrightarrow \psi_1+e_2\psi_2+e_4\psi_3+e_6\psi_4 \quad [\psi_{1,\dots,4}\in\mathscr{C}(1,e_1)]. \tag{47}$$

we analyze the action of left-barred operators on our octonionic wave functions ψ. For example, we find

$$[1|e_1]\psi \equiv \psi e_1 = \psi_1 + e_2(e_1\psi_2) + e_4(e_1\psi_3) + e_6(e_1\psi_4),$$

$$e_2\psi = -\psi_2 + e_2\psi_1 - e_4\psi_4^* + e_6\psi_3^*,$$

$$[e_3)e_1]\psi \equiv (e_3\psi)e_1 = \psi_2 + e_2\psi_1 + e_4\psi_4^* - e_6\psi_3^*.$$

Following the same methodology of the previous section, we can immediately note a correspondence between the complex matrix $i\mathbb{1}_{4\times4}$ and the octonionic barred operator $1|e_1$:

$$\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \leftrightarrow 1|e_1. \tag{48}$$

The translation does not work for all barred operators. Let us show it, explicitly. For example, we cannot find a 4×4 complex matrix which, acting on

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix},$$

gives the column vector

$$\begin{pmatrix} -\psi_2 \\ \psi_1 \\ -\psi_4^* \\ \psi_3^* \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \psi_2 \\ \psi_1 \\ \psi_4^* \\ -\psi_3^* \end{pmatrix},$$

and so we have not the possibility to relate

$$e_2 \quad \text{or} \quad e_3)e_1$$

with a complex matrix. Nevertheless, a combined action of such operators gives us

$$e_2\psi + (e_3\psi)e_1 = 2e_2\psi_1,$$

and it allows us to represent the octonionic barred operator

$$e_2 + e_3)e_1 \tag{49a}$$

by the 4×4 complex matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{49b}$$

Following this procedure we can represent a generic 4×4 complex matrix by octonionic barred operators. The explicit correspondence rules are given in appendix B.

We conclude our discussion concerning the relation between barred operators and 4×4 complex matrices, noting that the 32 basis elements of $GL(4,\mathscr{C})$ can be directly extracted from the 64 generators of $GL(8,\mathscr{R})$. It is well known that any complex matrix can be rewritten as a real matrix by the following isomorphism:

$$1 \leftrightarrow \mathbb{1}_{2\times 2} \quad \text{and} \quad i \leftrightarrow -i\sigma_2 .$$

The situation at the lowest order is

$$GL(2,\mathscr{R}) \quad \text{generators:} \quad \mathbb{1}_{2\times 2}, \ \sigma_1, \quad -i\sigma_2, \quad \sigma_3;$$

$$GL(1,\mathscr{C}) \quad \text{isomorphic:} \quad \mathbb{1}_{2\times 2}, \ -i\sigma_2 .$$

In a similar way (choosing appropriate combinations of left-barred octonionic operators, in which only $\pm\mathbb{1}_{2\times 2}$ and $\pm i\sigma_2$ appear) we can extract from $GL(8,\mathscr{R})$ the 32 basis elements of $GL(4,\mathscr{C})$. For further details see appendix B.

C. Octonionic representations of the four-dimensional Clifford algebra

We show explicitly two octonionic representations for the Dirac gamma-matrices.[26]

1. Dirac representation

$$\gamma^0 = \frac{1}{3} - \frac{2}{3}\sum_{m=1}^{3} e_m|e_m + \frac{1}{3}\sum_{n=4}^{7} e_n|e_n , \tag{50a}$$

$$\gamma^1 = -\frac{2}{3}\, e_6 - \frac{1}{3}\, |e_6 + e_5)e_3 - e_3)e_5 - \frac{1}{3}\sum_{p,s=1}^{7} \epsilon_{ps6}e_p)e_s , \tag{50b}$$

$$\gamma^2 = -\frac{2}{3}\, e_7 - \frac{1}{3}\, |e_7 + e_3)e_4 - e_4)e_3 - \frac{1}{3}\sum_{p,s=1}^{7} \epsilon_{ps7}e_p)e_s , \tag{50c}$$

$$\gamma^3 = -\frac{2}{3}\, e_4 - \frac{1}{3}\, |e_4 + e_7)e_3 - e_3)e_7 - \frac{1}{3}\sum_{p,s=1}^{7} \epsilon_{ps4}e_p)e_s . \tag{50d}$$

2. Majorana representation

$$\gamma^0 = \frac{1}{3}\, e_7 - \frac{1}{3}\, |e_7 + e_3)e_4 - e_5)e_2 + e_6)e_1 - \frac{1}{3}\sum_{p,s=1}^{7} \epsilon_{ps7}e_p)e_s , \tag{51a}$$

$$\gamma^1 = \frac{2}{3}\, e_1 + \frac{1}{3}\, |e_1 + e_5)e_4 - e_4)e_5 + \frac{1}{3}\sum_{p,s=1}^{7} \epsilon_{ps1}e_p)e_s , \tag{51b}$$

$$\gamma^2 = \frac{2}{3}\, e_7 + \frac{1}{3}\, |e_7 + e_4)e_3 - e_3)e_4 + \frac{1}{3}\sum_{p,s=1}^{7} \epsilon_{ps7}e_p)e_s , \tag{51c}$$

$$\gamma^3 = \frac{2}{3}\, e_3 + \frac{1}{3}\, |e_3 + e_7)e_4 - e_4)e_7 + \frac{1}{3}\sum_{p,s=1}^{7} \epsilon_{ps3}e_p)e_s . \tag{51d}$$

V. CONCLUSIONS

The modern notion of symmetry in physics heavily depends upon using the associative Lie groups. So, at first glance, it seems that octonions have not any relation with our physical world. Having a nonassociative algebra needs special care. In this work, we introduced a ''trick'' which allowed us to manipulate octonions without useless efforts, namely *barred octonions*.

This paper aimed to give a clear exposition of the potentiality of *barred numbers*. Their possible applications could occur in different fields, like group theory, quantum mechanics, and nuclear physics. We preferred in our work to focus our attention on the mathematical subject. Physical applications are investigated elsewhere.[19,25,27]

We summarize the more important results found in previous sections:

M—Mathematical Contents

(M1) The introduction of barred operators (natural objects if one works with noncommutative numbers) facilitates our job and enables us to formulate a ''friendly'' connection between 8×8 real matrices and octonions.

(M2) The nonassociativity is reproduced by left/right-barred operators. We consider these operators the natural extension of barred quaternions, recently introduced in literature.[5,19]

(M3) We tried to investigate the properties of our barred numbers and studied their special characteristics in order to use them in a proper way. After having established their isomorphism to $GL(8,\mathcal{R})$, life became easier.

(M4) The connection between $GL(8,\mathcal{R})$ and barred octonions gives us the possibility of extracting the octonionic generators corresponding to the complex subgroup $GL(4,\mathcal{C})$. This step represents the main tool to manipulate octonions in quantum mechanics.

(M5) To the best of our knowledge, for the first time, an octonionic representation for the four-dimensional Clifford algebra appears in print.

I—Further Investigations

We conclude with a listing of open questions for future investigations, whose study leads to further insights.

(I1) How may we complete the translation? Note that translation, as presented in this paper, works for $4n\times 4n$ matrices. What about odd-dimensional matrices?

(I2) From the translation rules we can extract the multiplication rules for generic octonionic barred operators. This will allow us to work directly with octonions without translations.

(I3) Inspired from Eq. (34), we could look for a more convenient way to express the new nonassociative multiplication (for example, we can try to modify the standard multiplication rule: row by column).

(I4) A last interesting research topic could be to generalize the group theoretical structure by our barred octonionic operators.

Many of the problems on this list deal with technical details although the answers to some will be important for further development of the subject.

We hope that the work presented in this paper demonstrates that octonions may constitute a coherent and well-defined branch of theoretical research. We are convinced that octonions represent largely uncharted and potentially very interesting terrain for theoretical investigations.

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APPENDIX A: OCTONIONIC REPRESENTATION OF $GL(8,\mathcal{R})$

In this appendix we give the translation rules between octonionic left/right-barred operators and 8×8 real matrices. In order to simplify our presentation we introduce the following notation:

$$\{a,b,c,d\}_{(1)}\equiv\begin{pmatrix} a & 0 & 0 & 0\\ 0 & b & 0 & 0\\ 0 & 0 & c & 0\\ 0 & 0 & 0 & d\end{pmatrix},\quad \{a,b,c,d\}_{(2)}\equiv\begin{pmatrix} 0 & a & 0 & 0\\ b & 0 & 0 & 0\\ 0 & 0 & 0 & c\\ 0 & 0 & d & 0\end{pmatrix}, \tag{A1a}$$

$$\{a,b,c,d\}_{(3)}\equiv\begin{pmatrix} 0 & 0 & a & 0\\ 0 & 0 & 0 & b\\ c & 0 & 0 & 0\\ 0 & d & 0 & 0\end{pmatrix},\quad \{a,b,c,d\}_{(4)}\equiv\begin{pmatrix} 0 & 0 & 0 & a\\ 0 & 0 & b & 0\\ 0 & c & 0 & 0\\ d & 0 & 0 & 0\end{pmatrix}, \tag{A1b}$$

where a, b, c, d and 0 represent 2×2 real matrices.

From now on, with σ_1, σ_2, σ_3 we represent the standard Pauli matrices:

$$\sigma_1=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}. \tag{A2}$$

The only necessary translation rules that we need to know explicitly are the following:

$$\mathbf{e}_1\leftrightarrow\{-i\sigma_2,-i\sigma_2,-i\sigma_2,\ i\sigma_2\}_{(1)},\quad \mathbf{1}|\mathbf{e}_1\leftrightarrow\{-i\sigma_2,i\sigma_2,i\sigma_2,-i\sigma_2\}_{(1)},$$

$$\mathbf{e}_2\leftrightarrow\{-\sigma_3,\sigma_3,-\mathbb{1},\mathbb{1}\}_{(2)},\quad \mathbf{1}|\mathbf{e}_2\leftrightarrow\{-\mathbb{1},\mathbb{1},\mathbb{1},-\mathbb{1}\}_{(2)},$$

$$\mathbf{e}_3\leftrightarrow\{-\sigma_1,\sigma_1,-i\sigma_2,-i\sigma_2\}_{(2)},\quad \mathbf{1}|\mathbf{e}_3\leftrightarrow\{-i\sigma_2,-i\sigma_2,i\sigma_2,i\sigma_2\}_{(2)},$$

$$\mathbf{e}_4\leftrightarrow\{-\sigma_3,\mathbb{1},\sigma_3,-\mathbb{1}\}_{(3)},\quad \mathbf{1}|\mathbf{e}_4\leftrightarrow\{-\mathbb{1},-\mathbb{1},\mathbb{1},\mathbb{1}\}_{(3)},$$

$$\mathbf{e}_5\leftrightarrow\{-\sigma_1,i\sigma_2,\sigma_1,i\sigma_2\}_{(3)},\quad \mathbf{1}|\mathbf{e}_5\leftrightarrow\{-i\sigma_2,-i\sigma_2,-i\sigma_2,-i\sigma_2\}_{(3)},$$

$$\mathbf{e}_6\leftrightarrow\{-\mathbb{1},-\sigma_3,\sigma_3,\mathbb{1}\}_{(4)},\quad \mathbf{1}|\mathbf{e}_6\leftrightarrow\{-\sigma_3,\sigma_3,-\sigma_3,\sigma_3\}_{(4)},$$

$$\mathbf{e}_7\leftrightarrow\{-i\sigma_2,-\sigma_1,\sigma_1,-i\sigma_2\}_{(4)},\quad \mathbf{1}|\mathbf{e}_7\leftrightarrow\{-\sigma_1,\sigma_1,-\sigma_1,\sigma_1\}_{(4)}.$$

The remaining rules can be easily constructed remembering that

$$\mathbf{e}_m\leftrightarrow L_m,$$

$$\mathbf{1}|\mathbf{e}_m\leftrightarrow R_m,$$

$$\mathbf{e}_m|\mathbf{e}_m\leftrightarrow M^L_{mm}\equiv R_mL_m,$$

$$M^R_{mm}\equiv L_mR_m,$$

$$\mathbf{e}_m)\mathbf{e}_n\leftrightarrow M^L_{mn}\equiv R_nL_m,$$

$$\mathbf{e}_m(\mathbf{e}_n\leftrightarrow M^R_{mn}\equiv L_mR_n.$$

For example,

$$\mathbf{e}_1|\mathbf{e}_1\leftrightarrow\begin{pmatrix}-i\sigma_2 & 0 & 0 & 0\\ 0 & -i\sigma_2 & 0 & 0\\ 0 & 0 & -i\sigma_2 & 0\\ 0 & 0 & 0 & i\sigma_2\end{pmatrix}\begin{pmatrix}-i\sigma_2 & 0 & 0 & 0\\ 0 & i\sigma_2 & 0 & 0\\ 0 & 0 & i\sigma_2 & 0\\ 0 & 0 & 0 & -i\sigma_2\end{pmatrix}=\{-\mathbb{1},\mathbb{1},\mathbb{1},\mathbb{1}\}_{(1)},$$

$$\mathbf{e}_3)\mathbf{e}_1\leftrightarrow\begin{pmatrix}-i\sigma_2 & 0 & 0 & 0\\ 0 & i\sigma_2 & 0 & 0\\ 0 & 0 & i\sigma_2 & 0\\ 0 & 0 & 0 & -i\sigma_2\end{pmatrix}\begin{pmatrix}0 & -\sigma_1 & 0 & 0\\ \sigma_1 & 0 & 0 & 0\\ 0 & 0 & 0 & -i\sigma_2\\ 0 & 0 & -i\sigma_2 & 0\end{pmatrix}=\{\sigma_3,\sigma_3,\mathbb{1},-\mathbb{1}\}_{(2)},$$

and

$$\mathbf{e}_3(\mathbf{e}_1\leftrightarrow\begin{pmatrix}0 & -\sigma_1 & 0 & 0\\ \sigma_1 & 0 & 0 & 0\\ 0 & 0 & 0 & -i\sigma_2\\ 0 & 0 & -i\sigma_2 & 0\end{pmatrix}\begin{pmatrix}-i\sigma_2 & 0^0 & 0 & \\ 0 & i\sigma_2 & 0 & 0\\ 0 & 0 & i\sigma_2 & 0\\ 0 & 0 & 0 & -i\sigma_2\end{pmatrix}=\{\sigma_3,\sigma_3,-\mathbb{1},\mathbb{1}\}_{(2)}.$$

Following this procedure any matrix representation of right/left-barred operators can be obtained. Using Mathematica,[28] we have proved the linear independence of the 64 elements which represent the most general octonionic operator

$$\mathscr{O}_0+\sum_{m=1}^{7}\mathscr{O}_m)e_m.$$

So our barred operators form a complete basis for any 8×8 real matrix and this establishes the isomorphism between $GL(8,\mathscr{R})$ and barred octonions.

We conclude this appendix giving a compact notation for the 64 left-barred operators (a similar trick works for the right ones).

For $m,n=1,\ldots,7$ $(m\neq n)$ and $\alpha,\beta=1,\ldots,7$ (labels of the rows and columns of the corresponding matrix X), we have

$$\mathbf{e}_m)\mathbf{e}_n+\mathbf{e}_n)\mathbf{e}_m\leftrightarrow X_{\alpha\beta}=\begin{cases}-2, & \alpha,\beta=m,n;n,m,\\ 0, & \text{otherwise.}\end{cases}\tag{A3a}$$

$$\mathbf{e}_m+\mathbf{1}|\mathbf{e}_m\leftrightarrow X_{\alpha\beta}=\begin{cases}-2, & \alpha,\beta=0,\ m,\\ +2, & \alpha,\beta=m,0,\\ 0, & \text{otherwise.}\end{cases}\tag{A3b}$$

For the minus combination, after introducing the index p defined by $e_me_n=\epsilon_{mnp}$ $(m\neq n)$, we have the following rules:

$$\mathbf{e}_m)\mathbf{e}_n-\mathbf{e}_n)\mathbf{e}_m\leftrightarrow X_{\alpha\beta}=\begin{cases}2\epsilon_{abp}, & \alpha,\beta=a,b(\neq m,n),\\ 2, & \alpha,\beta=0,\ p;p,0,\\ 0, & \text{otherwise;}\end{cases}\tag{A4a}$$

$$\mathbf{e}_m-\mathbf{1}|\mathbf{e}_m\leftrightarrow X_{\alpha\beta}=\begin{cases}-2\epsilon_{abm}, & \alpha,\beta=a,b,\\ 0, & \text{otherwise.}\end{cases}\tag{A4b}$$

APPENDIX B: OCTONIONIC REPRESENTATION OF $GL(4,\mathscr{C})$

We give the action of barred operators on octonionic functions:

$$\psi = \psi_1 + e_2\psi_2 + e_4\psi_3 + e_6\psi_4 \quad [\psi_{1,\dots,4} \in \mathscr{C}(1,e_1)].$$

In the following we will use the notation

$$e_2 \rightarrow \{-\psi_2, \psi_1, -\psi_4^*, \psi_3^*\}$$

to indicate

$$e_2\psi = -\psi_2 + e_2\psi_1 - e_4\psi_4^* + e_6\psi_3^* .$$

As occurred in the previous appendix we need to know only the action of the barred operators $\mathbf{e}_m$ and $\mathbf{1}|\mathbf{e}_m$:

$$\mathbf{e}_1 \rightarrow \{e_1\psi_1, -e_1\psi_2, -e_1\psi_3, -e_1\psi_4\}, \quad \mathbf{1}|\mathbf{e}_1 \rightarrow \{e_1\psi_1, e_1\psi_2, e_1\psi_3, e_1\psi_4\},$$

$$\mathbf{e}_2 \rightarrow \{-\psi_2, \psi_1, -\psi_4^*, \psi_3^*\}, \quad \mathbf{1}|\mathbf{e}_2 \rightarrow \{-\psi_2^*, \psi_1^*, \psi_4^*, -\psi_3^*\},$$

$$\mathbf{e}_3 \rightarrow \{-e_1\psi_2, -e_1\psi_1, -e_1\psi_4^*, e_1\psi_3^*\}, \quad \mathbf{1}|\mathbf{e}_3 \rightarrow \{e_1\psi_2^*, -e_1\psi_1^*, e_1\psi_4^*, -e_1\psi_3^*\},$$

$$\mathbf{e}_4 \rightarrow \{-\psi_3, \psi_4^*, \psi_1, -\psi_2^*\}, \quad \mathbf{1}|\mathbf{e}_4 \rightarrow \{-\psi_3^*, -\psi_4^*, \psi_1^*, \psi_2^*\},$$

$$\mathbf{e}_5 \rightarrow \{-e_1\psi_3, e_1\psi_4^*, -e_1\psi_1, -e_1\psi_2^*\}, \quad \mathbf{1}|\mathbf{e}_5 \rightarrow \{e_1\psi_3^*, -e_1\psi_4^*, -e_1\psi_1^*, e_1\psi_2^*\},$$

$$\mathbf{e}_6 \rightarrow \{-\psi_4, -\psi_3^*, \psi_2^*, \psi_1\}, \quad \mathbf{1}|\mathbf{e}_6 \rightarrow \{-\psi_4^*, \psi_3^*, -\psi_2^*, \psi_1^*\},$$

$$\mathbf{e}_7 \rightarrow \{e_1\psi_4, e_1\psi_3^*, -e_1\psi_2^*, e_1\psi_1\}, \quad \mathbf{1}|\mathbf{e}_7 \rightarrow \{-e_1\psi_4^*, -e_1\psi_3^*, e_1\psi_2^*, e_1\psi_1^*\}.$$

From the previous correspondence rules we immediately obtain the others barred operators. We give, as example, the construction of the operator $\mathbf{e}_4)\ \mathbf{e}_7$. We know that

$$\mathbf{e}_4 \rightarrow \{-\psi_3, \psi_4^*, \psi_1, -\psi_2^*\} \quad \text{and} \quad \mathbf{1}|\mathbf{e}_7 \rightarrow \{-e_1\psi_4^*, -e_1\psi_3^*, e_1\psi_2^*, e_1\psi_1^*\}.$$

Combining these operators we find

$$\{-e_1(-\psi_2^*)^*, -e_1\psi_1^*, e_1(\psi_4^*)^*, e_1(-\psi_3)^*\},$$

and so

$$\mathbf{e}_4)\mathbf{e}_7 \rightarrow \{e_1\psi_2, -e_1\psi_1^*, e_1\psi_4, -e_1\psi_3^*\}.$$

As remarked at the end of subsection IV B, we can extract the 32 basis elements of $GL(4,\mathscr{C})$ directly by suitable combinations of 64 basis elements of $GL(8,\mathscr{R})$. We must choose the combination which have only $\mathbb{1}_{2\times 2}$ and $-i\sigma_2$ as matrix elements. Nevertheless we must take care in manipulating our octonionic barred operators. If we wish to extract from $GL(8,\mathscr{R})$ the 32 elements which characterize $GL(4,\mathscr{C})$ we need to change the octonionic basis of $GL(8,\mathscr{R})$. In fact, the natural choice for the symplectic octonionic representation,

$$\psi = (\varphi_0 + e_1\varphi_1) + e_2(\varphi_2 + e_1\varphi_3) + e_4(\varphi_4 + e_1\varphi_5) + e_6(\varphi_6 + e_1\varphi_7),$$

requires the following real counterpart,

$$\tilde{\varphi} = \varphi_0 + e_1\varphi_1 + e_2\varphi_2 - e_3\varphi_3 + e_4\varphi_4 - e_5\varphi_5 + e_6\varphi_6 + e_7\varphi_7,$$

whereas we used in subsection IV A the following basis:

$$\varphi=\varphi_0+e_1\varphi_1+e_2\varphi_2+e_3\varphi_3+e_4\varphi_4+e_5\varphi_5+e_6\varphi_6+e_7\varphi_7 .$$

The changes in the signs of $e_3\varphi_3$ and $e_5\varphi_5$ imply a modification in the generators of $GL(8,\mathscr{R})$. For example, $\mathbf{e}_2$ and $\mathbf{e}_3)\,\mathbf{e}_1$ now read

$$\mathbf{e}_2\equiv\{-\mathbb{1},\mathbb{1},-\sigma_3,\sigma_3\}_{(2)} \quad \text{and} \quad \mathbf{e}_3)\mathbf{e}_1\equiv\{\mathbb{1},\mathbb{1},\sigma_3,-\sigma_3\}_{(2)} ,$$

i.e., the change of basis induces the following modifications:

$$\mathbb{1}\rightleftharpoons\sigma_3 .$$

Their appropriate combination gives

$$\frac{\mathbf{e}_2+\mathbf{e}_3)\mathbf{e}_1}{2}\equiv\{0,\mathbb{1},0,0\}_{(2)}\xrightarrow{\text{complexifing}}\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

as required by Eq. (49b).

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