Progress of Theoretical Physics, Vol. 96, No. 4, October 1996

Octonionic Dirac Equation

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(Received June 10, 1996)

In order to obtain a consistent formulation of octonionic quantum mechanics (OQM), we introduce left-right barred operators. Such operators enable us to find the translation rules between octonionic numbers and 8×8 real matrices (a translation is also given for 4×4 complex matrices). We develop an octonionic relativistic free wave equation, linear in the derivatives. Even if the wave functions are only one-component we show that four independent solutions, corresponding to those of the Dirac equation, exist.

§ 1. Introduction

Since the sixties, there has been renewed and intense interest in the use of octonions in physics.[1] Octonionic algebra has been in fact linked with a number of interesting subjects: structure of interactions,[2] $SU(3)$ color symmetry and quark confinement,[3),4] standard model gauge group,[5] exceptional GUT groups,[6] Dirac-Clifford algebra,[7] nonassociative Yang-Mills theories,[8),9] space-time symmetries in ten dimensions,[10] and supersymmetry and supergravity theories.[11),12] Moreover, the recent successful application of quaternionic numbers in quantum mechanics,[13)~17] in particular in formulating a quaternionic Dirac equation,[18)~21] suggests going one step further and using octonions as an underlying numerical field.

In this work, we overcome the problems due to the nonassociativity of the octonionic algebra by introducing left-right barred operators (which will be sometimes called barred octonions). Such operators complete the mathematical material introduced in the recent papers of Joshi et al.[8),9] Then, we investigate their relations to $GL(8, \mathfrak{R})$ and $GL(4, \mathcal{C})$. Establishing this relation we find interesting translation rules, which give us the opportunity to formulate a consistent OQM.

The philosophy behind the translation can be concisely expressed by the following statement: "There exists at least one version of octonionic quantum mechanics where the standard quantum mechanics is reproduced". The use of a complex scalar product (complex geometry)[22] will be the main tool to obtain OQM.

We wish to stress that translation rules do not imply that our octonionic quantum world (with complex geometry) is equivalent to the standard quantum world. When translation fails the two worlds are not equivalent. An interesting case can be supersymmetry.[23]

Similar translation rules, between quaternionic quantum mechanics (QQM) with complex geometry and standard quantum mechanics, have recently been found.[16] As an application, such rules can be exploited in reformulating in a natural way of the electroweak sector of the standard model.[17]

In § 2, we discuss octonionic algebra and introduce barred operators. Then, in

§ 3, we investigate the relation between barred octonions and 8×8 real matrices. In this section, we also give the translation rules between octonionic barred operators and $GL(4, \mathcal{C})$, which will be very useful in formulating our OQM (full details of the mathematical material appear elsewhere[24]). In § 4, we explicitly develop an octonionic Dirac equation and suggest a possible difference between complex and octonionic quantum theories. In the final section we draw our conclusions.

§ 2. Octonionic barred operators

We can characterize the algebras $\mathfrak{R}$, $\mathcal{C}$, $\mathcal{H}$ and $\mathcal{O}$ by the concept of **division algebra** (in which one has no nonzero divisors of zero). Octonions, which locate a nonassociative division algebra, can be represented by seven imaginary units ($e_1, \cdots, e_7$) and $e_0 \equiv 1$:

$$\mathcal{O} = r_0 + \sum_{m=1}^{7} r_m e_m . \quad (r_{0,\cdots,7} \text{ real}) \tag{1}$$

These seven imaginary units, e_m, obey the noncommutative and nonassociative algebra

$$e_m e_n = -\delta_{mn} + \epsilon_{mnp} e_p \quad (m, n, p = 1, \cdots, 7) \tag{2}$$

with ϵ_{mnp} totally antisymmetric and equal to unity for the seven combinations 123, 145, 176, 246, 257, 347 and 365. The norm, $N(\mathcal{O})$, for the octonions is defined by

$$N(\mathcal{O}) = (\mathcal{O}^{\dagger} \mathcal{O})^{1/2} = (\mathcal{O} \mathcal{O}^{\dagger})^{1/2} = (r_0^2 + \cdots + r_7^2)^{1/2} \tag{3}$$

with the octonionic conjugate $\mathcal{O}^{\dagger}$ given by

$$\mathcal{O}^{\dagger} = r_0 - \sum_{m=1}^{7} r_m e_m . \tag{4}$$

The inverse is then

$$\mathcal{O}^{-1} = \mathcal{O}^{\dagger} / N(\mathcal{O}) . \quad (\mathcal{O} \neq 0) \tag{5}$$

We can define an **associator** (analogous to the usual algebraic commutator) as follows:

$$\{x, y, z\} \equiv (xy)z - x(yz) , \tag{6}$$

where, in each term on the right hand, we must, first of all, perform the multiplication in brackets. Note that for real, complex and quaternionic numbers, the associator is trivially null. For octonionic imaginary units we have

$$\{e_m, e_n, e_p\} \equiv (e_m e_n) e_p - e_m (e_n e_p) = 2 \epsilon_{mnps} e_s \tag{7}$$

with ϵ_{mnps} totally antisymmetric and equal to unity for the seven combinations

1247, 1265, 2345, 2376, 3146, 3157 and 4567 .

Working with octonionic numbers, the associator (6) is in general non-vanishing. However, the "alternative condition" is fulfilled:

$$\{x, y, z\}+\{z, y, x\}=0 . \tag{8}$$

In 1989, writing a quaternionic Dirac equation,[19] Rotelli introduced a **barred** momentum operator

$$-\partial|i \quad [(-\partial|i)\psi\equiv-\partial\psi i] . \tag{9}$$

In a recent paper,[16] based upon the Rotelli operators, **partially barred quaternions**

$$q+p|i \quad [q, p\in\mathcal{H}] , \tag{10}$$

have been used to formulate a quaternionic quantum mechanics.

A complete generalization for quaternionic numbers is represented by the following barred operators:

$$q_1+q_2|i+q_3|j+q_4|k . \quad [q_{1,\cdots,4}\in\mathcal{H}] \tag{11}$$

We refer to these as **fully barred quaternions**, or simply barred quaternions. They, with their 16 linearly independent elements, form a basis of $GL(4, \mathfrak{R})$ and can be used to reformulate Lorentz space-time transformations[25] and write down a one-component Dirac equation.[21]

Thus, it seems to us natural to investigate the existence of **barred octonions**:

$$\mathcal{O}_0+\sum_{m=1}^{7}\mathcal{O}_m|e_m . \quad [\mathcal{O}_{0,\cdots,7} \text{ octonions}] . \tag{12}$$

Nevertheless, we must observe that an octonionic **barred** operator, $\boldsymbol{a}|\boldsymbol{b}$, which acts on octonionic wave functions, ψ,

$$[a|b]\psi\equiv a\psi b ,$$

is not a well-defined object; for $a\neq b$ the triple product $a\psi b$ could be either $(a\psi)b$ or $a(\psi b)$. So, in order to avoid the ambiguity due to the nonassociativity of the octonionic numbers, we need to define left/right-barred operators. We will indicate **left-barred** operators by $\boldsymbol{a})\boldsymbol{b}$, with a and b representing octonionic numbers. They act on octonionic functions ψ as follows:

$$[a)b]\psi=(a\psi)b . \tag{13a}$$

In a similar way we can introduce **right-barred** operators, defined by $\boldsymbol{a}(\boldsymbol{b}$,

$$[a(b]\psi=a(\psi b) . \tag{13b}$$

Obviously, there are barred-operators in which the nonassociativity is not relevant, for example

$$1)a=1(a\equiv1|a .$$

Furthermore, from Eq. (8), we have

$$\{x, y, x\}=0 ,$$

so

$$a)a=a(a\equiv a|a .$$

In addition, it is possible to prove, by Eq. (8), that each right-barred operator can be expressed by a suitable combination of left-barred operators. For further details, the reader can consult the mathematical paper.[24] So we can represent the most general octonionic operator by only 64 left-barred objects

$$\mathcal{O}_0+\sum_{m=1}^{7}\mathcal{O}_m)e_m\,. \quad [\mathcal{O}_{0,\cdots,7} \text{ octonions}] \tag{14}$$

This suggests a correspondence between our barred octonions and $GL(8,\mathfrak{R})$ (a complete discussion about the above-mentioned relationship is given in the following section).

§ 3. Translation rules

In order to explain the idea of translation, let us look explicitly at the action of the operators $1|e_1$ and e_2, on a generic octonionic function φ

$$\varphi=\varphi_0+e_1\varphi_1+e_2\varphi_2+e_3\varphi_3+e_4\varphi_4+e_5\varphi_5+e_6\varphi_6+e_7\varphi_7\,. \quad [\varphi_{0,\cdots,7}\in\mathfrak{R}] \tag{15}$$

We have

$$[1|e_1]\varphi\equiv\varphi e_1=e_1\varphi_0-\varphi_1-e_3\varphi_2+e_2\varphi_3-e_5\varphi_4+e_4\varphi_5+e_7\varphi_6-e_6\varphi_7\,, \tag{16a}$$

$$e_2\varphi=e_2\varphi_0-e_3\varphi_1-\varphi_2+e_1\varphi_3+e_6\varphi_4+e_7\varphi_5-e_4\varphi_6-e_5\varphi_7\,. \tag{16b}$$

If we represent our octonionic function φ by the following real column vector:

$$\varphi\leftrightarrow\begin{pmatrix}\varphi_0\\ \varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\\ \varphi_5\\ \varphi_6\\ \varphi_7\end{pmatrix}, \tag{17}$$

we can rewrite Eqs. (16a, b) in matrix form,

$$\begin{pmatrix}0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&-1&0&0&0\\ 0&0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&1&0\end{pmatrix}\begin{pmatrix}\varphi_0\\ \varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\\ \varphi_5\\ \varphi_6\\ \varphi_7\end{pmatrix}=\begin{pmatrix}-\varphi_1\\ \varphi_0\\ \varphi_3\\ -\varphi_2\\ \varphi_5\\ -\varphi_4\\ -\varphi_7\\ \varphi_6\end{pmatrix}, \tag{18a}$$

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix} = \begin{pmatrix} -\varphi_2 \\ \varphi_3 \\ \varphi_0 \\ -\varphi_1 \\ -\varphi_6 \\ -\varphi_7 \\ \varphi_4 \\ \varphi_5 \end{pmatrix} . \tag{18b}$$

In this way we can immediately obtain a real matrix representation for the octonionic barred operators $1|e_1$ and e_2. Following this procedure we can construct the complete set of translation rules.[24)]

Let us now discuss of the relation between octonions and complex matrices. Complex groups play a critical role in physics. No one can deny the importance of $U(1, \mathcal{C})$ or $SU(2, \mathcal{C})$. In relativistic quantum mechanics, $GL(4, \mathcal{C})$ is essential in writing the Dirac equation. Having $GL(8, \mathfrak{R})$, we should be able to extract its subgroup $GL(4, \mathcal{C})$. So, we can translate the famous Dirac-gamma matrices and write down a new octonionic Dirac equation.

If we analyse the action of left-barred operators on our octonionic wave functions

$$\psi=\psi_1+e_2\psi_2+e_4\psi_3+e_6\psi_4\,, \quad [\psi_{1,\cdots,4}\in \mathcal{C}(1, e_1)] \tag{19}$$

we find, for example,

$$e_2\psi=-\psi_2+e_2\psi_1-e_4\psi_4^*+e_6\psi_3^*\,,$$

$$[e_3)e_1]\psi\equiv(e_3\psi)e_1=\psi_2+e_2\psi_1+e_4\psi_4^*-e_6\psi_3^*\,.$$

Obviously, neither the previous operators, e_2 or $e_3)e_1$, can be represented by matrices. Nevertheless we note that their combined action gives us

$$e_2\psi+(e_3\psi)e_1=2e_2\psi_1\,,$$

and it allows us to represent the octonionic barred operator

$$e_2+e_3)e_1\,, \tag{20a}$$

by the 4×4 complex matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{20b}$$

Following this procedure we can represent a generic 4×4 complex matrix by octonionic barred operators. In Appendix B we give the full basis of $GL(4, \mathcal{C})$ in terms of octonionic left-barred operators. It is clear that only particular combinations of left-barred operators are allowed to reproduce the associative matrix algebra.

In order to make our discussion smooth, we refer the interested reader to the mathematical paper of Ref. 24). We can quickly relate $1|e_1$ with the complex matrix $i\mathbf{1}_{4\times4}$ which will be relevant to an **appropriate** definition for the octonionic momentum operator.[26] The operator $1|e_1$ (represented by the matrix $i\mathbf{1}_{4\times4}$) commutes with all operators which can be translated by 4×4 complex matrices. This is not generally true for a generic octonionic operator. For example, we can show that the operator $1|e_1$ does not commute with e_2, explicitly:

$$e_2\{[1|e_1]\psi\}\equiv e_2(\psi e_1)=-e_1\psi_2-e_3\psi_1-e_5\psi_4^*-e_7\psi_3^*\,, \tag{21a}$$

$$[1|e_1]\{e_2\psi\}\equiv(e_2\psi)e_1=-e_1\psi_2-e_3\psi_1+e_5\psi_4^*+e_7\psi_3^*\,. \tag{21b}$$

The interpretation is simple: e_2 cannot be represented by a 4×4 complex matrix.

We conclude this section by showing explicitly an octonionic representation for the Dirac gamma-matrices:[27]

Dirac representation,

$$\gamma^0=\frac{1}{3}-\frac{2}{3}\sum_{m=1}^{3}e_m|e_m+\frac{1}{3}\sum_{n=4}^{7}e_n|e_n\,, \tag{22a}$$

$$\gamma^1=-\frac{2}{3}e_6-\frac{1}{3}|e_6+e_5)e_3-e_3)e_5-\frac{1}{3}\sum_{p,s=1}^{7}\epsilon_{ps6}e_p)e_s\,, \tag{22b}$$

$$\gamma^2=-\frac{2}{3}e_7-\frac{1}{3}|e_7+e_3)e_4-e_4)e_3-\frac{1}{3}\sum_{p,s=1}^{7}\epsilon_{ps7}e_p)e_s\,, \tag{22c}$$

$$\gamma^3=-\frac{2}{3}e_4-\frac{1}{3}|e_4+e_7)e_3-e_3)e_7-\frac{1}{3}\sum_{p,s=1}^{7}\epsilon_{ps4}e_p)e_s\,. \tag{22d}$$

§ 4. Octonionic Dirac equation

In the previous section we gave the gamma-matrices in three different octonionic representations. Obviously, we can investigate the possibility of having a more simple representation for our octonionic γ^μ-matrices, without translation.

Why not

$$e_1\,,\quad e_2\,,\quad e_3\quad \text{and}\quad e_4|e_4$$

or

$$e_1\,,\quad e_2\,,\quad e_3\quad \text{and}\quad e_4)e_1\,?$$

Apparently, they represent suitable choices. Nevertheless, the octonionic world is full of hidden traps and so we must proceed with prudence. Let us start from the standard Dirac equation

$$\gamma^\nu p_\nu\psi=m\psi\,, \tag{23}$$

(we discuss the momentum operator in the paper of Ref. 26), where p_ν represents the "real" eigenvalue of the momentum operator) and apply $\gamma^\mu p_\mu$ to our equation

$$\gamma^\mu p_\mu(\gamma^\nu p_\nu\psi)=m\gamma^\mu p_\mu\psi\,. \tag{24}$$

The previous equation can be concisely rewritten as

$$p^{\mu}p_{\nu}\gamma^{\mu}(\gamma^{\nu}\psi)=m^2\psi\,. \tag{25}$$

Requiring that each component of ψ satisfies the standard Klein-Gordon equation, we find the Dirac condition, which becomes in the octonionic world

$$\gamma^{\mu}(\gamma^{\nu}\psi)+\gamma^{\nu}(\gamma^{\mu}\psi)=2g^{\mu\nu}\psi\,, \tag{26}$$

where the parentheses are relevant because of the octonions nonassociative nature. Using octonionic numbers and no barred operators, we can obtain, from (26), the standard Dirac condition

$$\{\gamma^{\mu},\gamma^{\nu}\}=2g^{\mu\nu}\,. \tag{27}$$

In fact, recalling the associator property [which follows from Eq. (7)]

$$\{a,b,\psi\}=-\{b,a,\psi\}\,,\quad [a,b \text{ octonionic numbers}]$$

we quickly find the following correspondence relation:

$$(ab+ba)\psi=a(b\psi)+b(a\psi)\,.$$

We have no problem in writing down three suitable gamma-matrices which satisfy the Dirac condition (27),

$$(\gamma^1,\gamma^2,\gamma^3)\equiv(e_1,e_2,e_3)\,, \tag{28}$$

but, barred operators like

$$e_4|e_4 \quad \text{or} \quad e_4)e_1$$

cannot represent the matrix γ^0. After straightforward algebraic manipulations, one can prove that the barred operator, $e_4|e_4$, does not anticommute with e_1,

$$e_1(e_4\psi e_4)+e_4(e_1\psi)e_4=-2(e_3\psi_2+e_7\psi_4)\neq 0\,,\quad [\psi=\psi_1+e_2\psi_2+e_4\psi_3+e_6\psi_4] \tag{29}$$

whereas $e_4)e_1$ anticommutes with e_1

$$e_1[(e_4\psi)e_1]+[e_4(e_1\psi)]e_1=0\,. \tag{30a}$$

But we know that $\gamma_0{}^2=1$, whereas

$$\{e_4[(e_4\psi)e_1]\}e_1=\psi_1-e_2\psi_2+e_4\psi_3-e_6\psi_4\neq\psi\,. \tag{30b}$$

Thus, we must be satisfied with the octonionic representations given in the previous section.

We recall that the appropriate momentum operator in OQM with complex geometry[26] is

$$\mathcal{P}^{\mu}\equiv\partial^{\mu}|e_1\,.$$

Thus, the octonionic Dirac equation, in covariant form, is given by

$$\gamma^{\mu}(\partial_{\mu}\psi e_1)=m\psi\,, \tag{31}$$

where γ^μ are represented by octonionic barred operators (22a~d). We can now proceed in the standard manner. Plane wave solutions exist $[\boldsymbol{p}\equiv(-\boldsymbol{\partial}|e_1)$ commutes with a generic octonionic Hamiltonian] and are of the form

$$\psi(\boldsymbol{x}, t)=[u_1(\boldsymbol{p})+e_2u_2(\boldsymbol{p})+e_4u_3(\boldsymbol{p})+e_6u_4(\boldsymbol{p})]e^{-\boldsymbol{p}\boldsymbol{x}e_1}. \quad [u_{1,\cdots,4}\in \boldsymbol{C}(1, e_1)] \tag{32}$$

Let us start with

$$\boldsymbol{p}\equiv(0, 0, p_z)$$

from (31), we have

$$E(\gamma^0\psi)-p_z(\gamma^3\psi)=m\psi\,. \tag{33}$$

Using the explicit form of the octonionic operators $\gamma^{0,3}$ and extracting their action (see Appendix A) we find

$$E(u_1+e_2u_2-e_4u_3-e_6u_4)-p_z(u_3-e_2u_4-e_4u_1+e_6u_2)=m(u_1+e_2u_2+e_4u_3+e_6u_4)\,. \tag{34}$$

From (34), we derive four complex equations:

$$(E-m)u_1=+p_zu_3\,,$$

$$(E-m)u_2=-p_zu_4\,,$$

$$(E+m)u_3=+p_zu_1\,,$$

$$(E+m)u_4=-p_zu_2\,.$$

After simple algebraic manipulations, we find the following octonionic Dirac solutions:

$$E=+|E| \quad u^{(1)}=N\left(1+e_4\frac{p_z}{|E|+m}\right), \quad u^{(2)}=N\left(e_2-e_6\frac{p_z}{|E|+m}\right)=u^{(1)}e_2\,;$$

$$E=-|E| \quad u^{(3)}=N\left(\frac{p_z}{|E|+m}-e_4\right), \quad u^{(4)}=N\left(e_2\frac{p_z}{|E|+m}+e_6\right)=u^{(3)}e_2$$

with N real mormalization constant. Setting the norm to $2|E|$, we find

$$N=(|E|+m)^{1/2}\,.$$

We now observe (as for the quaternionic Dirac equation) a difference with respect to the standard Dirac equation. Working in our representation (22a~d) and introducing the octonionic spinor

$$\bar{u}\equiv(\gamma_0u)^+=u_1^*-e_2u_2+e_4u_3+e_6u_4\,, \quad [u=u_1+e_2u_2+e_4u_3+e_6u_4]$$

we have

$$\bar{u}^{(1)}u^{(1)}=u^{(1)}\bar{u}^{(1)}=\bar{u}^{(2)}u^{(2)}=u^{(2)}\bar{u}^{(2)}=2(m+e_4p_z)\,. \tag{35}$$

Thus we find

$$u^{(1)}\bar{u}^{(1)}+u^{(2)}\bar{u}^{(2)}=4(m+e_4p_z)\,, \tag{36a}$$

instead of the expected relation

$$u^{(1)}\bar{u}^{(1)}+u^{(2)}\bar{u}^{(2)}=\gamma^0 E-\gamma^3 p_z+m\,. \tag{36b}$$

Furthermore, the previous difference is compensated for if we compare the complex projection of (36a) with the trace of (36b)

$$[(u^{(1)}\bar{u}^{(1)}+u^{(2)}\bar{u}^{(2)})^{OQM}]_c \equiv \mathrm{Tr}[(u^{(1)}\bar{u}^{(1)}+u^{(2)}\bar{u}^{(2)})^{CQM}]=4m\,, \tag{37}$$

which suggests we redefine the trace as a "complex" trace. We know that spinor relations like (36a, b) are relevant in perturbation calculus, so the previous results suggest us to analyze quantum electrodynamics in order to investigate possible differences between complex and octonionic quantum field. This could represent the aim of a future work.

§ 5. Conclusions

In the physical literature, we find a method to partially overcome the issues relating to the octonions nonassociativity. Some researchers introduce a "new" imaginary unit "$i=\sqrt{-1}$" which commutes with all others octonionic imaginary units, e_m. The new field is often called the **complexified octonionic field**. Different papers have been written in such a formalism: Quark Structure and Octonions,[3] Octonions, Quark and QCD,[4] Dirac-Clifford algebra,[7] Octonions and Isospin,[28] and so on. In the literature we also find a Dirac equation formulation by **complexified** octonions with an embarrassing doubling of solutions: "$\cdots$ *the wave functions* $\tilde{\psi}$ *is not a column matrix, but must be taken as an octonion.* $\tilde{\psi}$ *therefore consists of eight wave functions, rather than the four wave functions of the Dirac equation*".[28] In this paper we have presented an alternative way to look at the octonionic world. No new imaginary unit is necessary to formulate in a consistent way an octonionic quantum mechanics.

Nevertheless, complexified ring division algebras have been used in interesting works of Morita[29] to formulate the entire standard model.

Having a nonassociative algebra needs special care. In this work, we introduced a "trick" which allowed us to manipulate octonions without useless efforts. We summarize the more important results found in previous sections:

P-Physical Content :

P1 - We emphasize that a characteristic of our formalism is the *absolute need of a complex scalar product* (in QQM the use of a complex geometry is not obligatory and thus a question of choice). Using a complex geometry we overcame the hermiticity problem and gave the appropriate and unique definition of momentum operator.

P2 - A positive feature of this octonionic version of quantum mechanics, is the appearance of all four standard Dirac free-particle solutions notwithstanding the one-component structure of the wave functions. We have the following situation for the division algebras:

field : complex , quaternions , octonions ,
Dirac Equation : 4×4 , 2×2 , 1×1 (matrix dimension) .

P3 - Many physical results can be reobtained by translation, so we have one version of octonionic quantum mechanics where the standard quantum mechanics could be reproduced. This represents for the authors a first fundamental step towards an octonionic world. We remark that our translation will not be possible in all situations, so it is only partial, consistent with the fact that the octonionic version could provide additional physical predictions.

I - Further Investigations :

We list some open questions for future investigation, whose study should lead to further insights.

I4 - The reproduction in octonionic calculations of the standard QED results is a nontrivial objective, due to the explicit differences in certain spinorial identities (see § 4). We will study this problem in a forthcoming paper.

I5 - A very attractive point is to try to treat the strong field by octonions, and then to formulate in a suitable manner a standard model, based on our octonionic dynamical Dirac equation.

We conclude by emphasizing that the core of our paper is surely represented by the absolute need of adopting a complex geometry within the quantum octonionic world.

Appendix A
—— $\gamma^{0,3}$-*Action on Octonionic Spinors* ——

In the following tables, we explicitly show the action on the octonionic spinor

$$u = u_1 + e_2 u_2 + e_4 u_3 + e_6 u_4 \,, \quad [u_{1,\dots,4} \in \mathcal{C}(1, e_1)]$$

of the barred operators which appear in γ^0 and γ^3. Using such tables, after straightforward algebraic manipulations we find

$$\gamma^0 u = u_1 + e_2 u_2 - e_4 u_3 - e_6 u_4 \,,$$

$$\gamma^3 u = u_3 - e_2 u_4 - e_4 u_1 + e_6 u_2 \,.$$

γ^0-action	u_1	$e_2 u_2$	$e_4 u_3$	$e_6 u_4$
$e_1 \vert e_1$	$-u_1$	$e_2 u_2$	$e_4 u_3$	$e_6 u_4$
$e_2 \vert e_2$	$-u_1^*$	$-e_2 u_2^*$	$e_4 u_3$	$e_6 u_4$
$e_3 \vert e_3$	$-u_1^*$	$e_2 u_2^*$	$e_4 u_3$	$e_6 u_4$
$e_4 \vert e_4$	$-u_1^*$	$e_2 u_2^*$	$-e_4 u_3^*$	$e_6 u_4$
$e_5 \vert e_5$	$-u_1^*$	$e_2 u_2$	$e_4 u_3^*$	$e_6 u_4$
$e_6 \vert e_6$	$-u_1^*$	$e_2 u_2$	$e_4 u_3$	$-e_6 u_4^*$
$e_7 \vert e_7$	$-u_1^*$.	$e_2 u_2$	$e_4 u_3$	$e_6 u_4^*$

γ^3-action	u_1	e_2u_2	e_4u_3	e_6u_4
e_4	e_4u_1	$-e_6u_2^*$	$-u_3$	e_2u_4
$1\vert e_4$	$e_4u_1^*$	$e_6u_2^*$	$-u_3^*$	$-e_2u_4^*$
$e_7)e_3$	$e_4u_1^*$	e_6u_2	u_3	$-e_2u_4^*$
$e_3)e_7$	$-e_4u_1^*$	$-e_6u_2^*$	$-u_3$	e_2u_4
$e_6)e_2$	$e_4u_1^*$	$-e_6u_2$	u_3	$-e_2u_4^*$
$e_2)e_6$	$-e_4u_1^*$	$-e_6u_2^*$	$-u_3$	$-e_2u_4$
$e_5)e_1$	e_4u_1	$e_6u_2^*$	u_3	$-e_2u_4^*$
$e_1)e_5$	$-e_4u_1^*$	$-e_6u_2^*$	$-u_3^*$	$e_2u_4^*$

Appendix B
—— *An Octonionic Basis of GL(4, C)* ——

In the following charts we establish the connection between 4×4 complex matrices and octonionic left-barred operators. We indicate with $\mathfrak{R}_{mn}(\mathcal{C}_{mn})$ the 4×4 real (complex) matrices by $1(i)$ in mn-element and zeros elsewhere.

4×4 complex matrices and left-barred operators:

$$\mathfrak{R}_{11}\leftrightarrow\frac{1}{2}[1-e_1|e_1]$$

$$\mathfrak{R}_{12}\leftrightarrow\frac{1}{6}[2e_1)e_3+e_3)e_1-2|e_2-e_2+e_4)e_6-e_6)e_4+e_5)e_7-e_7)e_5]$$

$$\mathfrak{R}_{13}\leftrightarrow\frac{1}{6}[2e_1)e_5+e_5)e_1-2|e_4-e_4+e_6)e_2-e_2)e_6+e_7)e_3-e_3)e_7]$$

$$\mathfrak{R}_{14}\leftrightarrow\frac{1}{6}[2e_1)e_7+e_7)e_1-2|e_6-e_6+e_2)e_4-e_4)e_2+e_5)e_3-e_3)e_5]$$

$$\mathfrak{R}_{21}\leftrightarrow\frac{1}{2}[e_2+e_3)e_1]$$

$$\mathfrak{R}_{22}\leftrightarrow\frac{1}{6}[1+e_1|e_1+e_4|e_4+e_5|e_5+e_6|e_6+e_7|e_7]-\frac{1}{3}[e_2|e_2+e_3|e_3]$$

$$\mathfrak{R}_{23}\rightarrow\frac{1}{2}[-e_2)e_4-e_3)e_5]$$

$$\mathfrak{R}_{24}\leftrightarrow\frac{1}{2}[e_3)e_7-e_2)e_6]$$

$$\mathfrak{R}_{31}\leftrightarrow\frac{1}{2}[e_4+e_5)e_1]$$

$$\mathfrak{R}_{32}\leftrightarrow\frac{1}{2}[-e_5)e_3-e_4)e_2]$$

$$\mathcal{R}_{33}\leftrightarrow\frac{1}{6}[1+e_1|e_1+e_2|e_2+e_3|e_3+e_6|e_6+e_7|e_7]-\frac{1}{3}[e_4|e_4+e_5|e_5]$$

$$\mathcal{R}_{34}\leftrightarrow\frac{1}{2}[e_5)e_7-e_4)e_6]$$

$$\mathcal{R}_{41}\leftrightarrow\frac{1}{2}[e_6-e_7)e_1]$$

$$\mathcal{R}_{42}\leftrightarrow\frac{1}{2}[e_7)e_3-e_6)e_2]$$

$$\mathcal{R}_{43}\rightarrow\frac{1}{2}[e_7)e_5-e_6)e_4]$$

$$\mathcal{R}_{44}\leftrightarrow\frac{1}{6}[1+e_1|e_1+e_2|e_2+e_3|e_3+e_4|e_4+e_5|e_5]-\frac{1}{3}[e_6|e_6+e_7|e_7]$$

$$\mathcal{C}_{11}\leftrightarrow\frac{1}{2}[1|e_1+e_1]$$

$$\mathcal{C}_{12}\leftrightarrow\frac{1}{6}[-2e_1)e_2-e_3-2|e_3-e_2)e_1+e_4)e_7+e_6)e_5-e_5)e_6-e_7)e_4]$$

$$\mathcal{C}_{13}\leftrightarrow\frac{1}{6}[-2e_1)e_4-e_5-2|e_5-e_4)e_1-e_6)e_3-e_2)e_7+e_7)e_2+e_3)e_6]$$

$$\mathcal{C}_{14}\leftrightarrow\frac{1}{6}[-2e_1)e_6+e_7+2|e_7-e_6)e_1-e_2)e_5+e_4)e_3+e_5)e_2-e_3)e_4]$$

$$\mathcal{C}_{21}\leftrightarrow\frac{1}{2}[-e_3+e_2)e_1]$$

$$\mathcal{C}_{22}\leftrightarrow\frac{1}{6}[1|e_1-e_1+e_4)e_5-e_5)e_4-e_6)e_7+e_7)e_6]-\frac{1}{3}[e_2)e_3-e_3)e_2]$$

$$\mathcal{C}_{23}\leftrightarrow\frac{1}{2}[-e_2)e_5+e_3)e_4]$$

$$\mathcal{C}_{24}\leftrightarrow\frac{1}{2}[e_3)e_6+e_2)e_7]$$

$$\mathcal{C}_{31}\leftrightarrow\frac{1}{2}[-e_5+e_4)e_1]$$

$$\mathcal{C}_{32}\leftrightarrow\frac{1}{2}[e_5)e_2-e_4)e_3]$$

$$\mathcal{C}_{33}\leftrightarrow\frac{1}{6}[1|e_1-e_1+e_2)e_3-e_3)e_2-e_6)e_7+e_7)e_6]-\frac{1}{3}[e_4)e_5-e_5)e_4]$$

$$\mathcal{C}_{34}\leftrightarrow\frac{1}{2}[e_5)e_6+e_4)e_7]$$

$$\mathcal{C}_{41}\leftrightarrow\frac{1}{2}[e_7+e_6)e_1]$$

$$\mathcal{C}_{42} \leftrightarrow \frac{1}{2}[-e_7)e_2 - e_6)e_3]$$

$$\mathcal{C}_{43} \leftrightarrow \frac{1}{2}[-e_7)e_4 - e_6)e_5]$$

$$\mathcal{C}_{44} \leftrightarrow \frac{1}{6}[1|e_1 - e_1 + e_2)e_3 - e_3)e_2 + e_4)e_5 - e_5)e_4] - \frac{1}{3}[e_7)e_6 - e_6)e_7]$$

References

1) F. Gürsey, *Symmetries in Physics(1600-1980): Proc. of the 1st International Meeting on the History of Scientific Ideas, Seminari d'Història de les Ciènces, Barcelona, Spain,* 1987, p. 557.
2) A. Pais, Phys. Rev. Lett. **7** (1961), 291.
3) M. Günaydin and F. Gürsey, J. Math. Phys. **14** (1973), 1651; Phys. Rev. **D9** (1974), 3387.
4) K. Morita, Prog. Theor. Phys. **65** (1981), 787.
5) G. Dixon, Nuovo Cim. **B105** (1990), 349.
6) F. Gürsey, Yale Preprint C00-3075-178 (1978).
7) J. D. Edmonds, Phys. Essay, **5** (1992), 56.
8) A. Waldron and G. C. Joshi, Melbourne Preprint UM-P-92/60 (1992).
9) G. C. Lassig and G. C. Joshi, Melbourne Preprint UM-P-95/09 (1995).
A. Ritz and G. C. Joshi, Melbourne Preprint UM-P-95/69 (1995).
10) A. J. Davies and G. C. Joshi, J. Math. Phys. **27** (1986), 3036.
11) T. Kugo and P. Townsend, Nucl. Phys. **B221** (1987), 357.
12) B. Julia, Leptens Preprint 82/14 (1982).
13) S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields* (Oxford, New York, 1995).
14) S. L. Adler, Nucl. Phys. **B415** (1994), 195.
15) S. De Leo and P. Rotelli, Phys. Rev. **D45** (1992), 575; Nuovo Cim. **B110** (1995), 33.
S. De Leo, Prog. Theor. Phys. **94** (1995), 11; Int. J. Theor. Phys. **35** (1996), 1821.
16) S. De Leo and P. Rotelli, Prog. Theor. Phys. **92** (1994), 917; **96** (1996), 247.
17) S. De Leo and P. Rotelli, Int. J. Mod. Phys. **A10** (1995), 4359; Mod. Phys. Lett. **A11** (1996), 357.
S. De Leo and P. Rotelli, J. of Phys. **G22** (1996), 1137.
18) S. L. Adler, Phys. Lett. **221B** (1989), 39.
19) P. Rotelli, Mod. Phys. Lett. **A4** (1989), 933.
20) A. J. Davies, Phys. Rev. **D41** (1990), 2628.
21) S. De Leo, Int. J. Mod. Phys. **A11** (1996), 3973.
22) L. P. Horwitz and L. C. Biedenharn, Ann. of Phys. **157** (1984), 432.
J. Rembieliński, J. of Phys. **A11** (1978), 2323.
23) K. Abdel-Khalek and P. Rotelli, *Quaternionic Supersymmetry,* in preparation.
24) S. De Leo and K. Abdel-Khalek, J. Math. Phys. (to be published), hep-th/9607140.
25) S. De Leo, J. Math. Phys. **37** (1996), 2955.
26) S. De Leo and K. Abdel-Khalek, Prog. Theor. Phys. **96** (1996), 823.
27) Equations (22a ~ d) represent the octonionic counterpart of the complex matrices given on page 49 of the book:
C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1985).
28) R. Penney, Nouvo Cim. **B3** (1971), 95.
29) K. Morita, Prog. Theor. Phys. **67** (1981), 1860; **68** (1982), 2159; **70** (1983), 1648; **72** (1984), 1056; **73** (1984), 999; **75** (1985), 220; **90** (1993), 219.