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Octonionic Quantum Mechanics and Complex Geometry

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The use of complex geometry allows us to obtain a consistent formulation of octonionic quantum mechanics (OQM). In our octonionic formulation we solve the hermiticity problem and define an appropriate momentum operator within OQM. The nonextendability of the completeness relation and the norm conservation is also discussed in detail.

§ 1. Introduction

In the early thirties, in order to explain the novel phenomena of that time, namely β-decay and the strong interactions, Jordan[1] introduced a nonassociative but commutative algebra as a basic element for a new quantum theory. With the discovery that 3×3 hermitian octonionic matrices realize the Jordan postulate,[2),3)] octonions appeared, for the first time, in quantum mechanics. The hope of applying nonassociative algebras to physics was soon dashed with the Fermi theory of the β-decay and with the Yukawa model of nuclear force. Octonions disappeared from physics soon after being introduced.

During the early sixties, Finkelstein et al.[4),5)] attempted to extend the standard quantum mechanics by using the quaternionic field. The main difficulty, within their framework, was the non-existence of proper tensor products for quaternionic states. Then, with the quark revolution, new problems, such as quark confinement, appeared. Gürsey and Günaydin[6] tried to explain this problem on a fundamental level by investigating the possibility of constructing a consistent octonionic quantum mechanics.

An important step towards a generalization of standard quantum theories is the use of complex scalar products[7] (or complex geometry as referred to by Rembieliński[8]). Without these we cannot define a consistent tensor product.

Nonassociative numbers are difficult to manipulate, and so the use of the octonionic field within OQM is non-trivial. Obviously, if we are not able to construct a suitable OQM, octonions will remain beautiful ghosts in search of a physical incarnation.

In this work, we overcome the problems due to the nonassociativity of the octonionic algebra. Both the postulates of quantum mechanics and the nonassociativity property of octonions will be respected.

Is there an acceptable generic octonionic quantum theory? Do octonionic quantum theories necessitate complex geometry? At this stage these questions lack answers and the aim of our work is to clarify these points.

This article is organized as follows: In § 2, we give a brief introduction to the octonionic division algebra and introduce barred octonions. After this mathematical discussion, in § 3, we show how the complex geometry, the main tool to obtain a suitable formulation of OQM, allows us to overcome the hermiticity problem. In this section we also introduce the appropriate definition for the momentum operator. In § 4, we discuss the octonionic Hilbert space and disprove the standard objections concerning the nonextendability of the completeness relation and norm conservation to octonionic quantum mechanics. Future developments are discussed in the final section.

§ 2. Octonionic algebra and barred octonions

A classical theorem[9] states that the only division algebra over the reals are algebras of dimensions 1, 2, 4 and 8, and only associative division algebras over the reals are $\mathcal{R}$, $\mathcal{C}$ and $\mathcal{H}$ (quaternions). However, the **nonassociative** algebras include the octonions $\mathcal{O}$ (an interesting discussion concerning nonassociative algebras is presented in Ref. 10)). In this paper we deal with octonions and their generalizations.

We summarize our notation for the octonionic algebra and introduce the concept of barred operators. There is a number of equivalent ways to represent the octonions multiplication table. Fortunately, it is always possible to choose an orthonormal basis $(e_0, \cdots, e_7)$ such that

$$\mathcal{O}=r_0+\sum_{m=1}^{7} r_m e_m\,, \quad (r_{0,\cdots,7} \text{ reals}) \tag{1}$$

where e_m are elements obeying the noncommutative and nonassociative algebra

$$e_m e_n=-\delta_{mn}+\epsilon_{mnp} e_p \quad (m,\, n,\, p=1, \cdots, 7) \tag{2}$$

with ϵ_{mnp} totally antisymmetric and equal to unity for the seven combinations

123, 145, 176, 246, 257, 347 and 365

(each cycle represents a quaternionic subalgebra). The octonionic conjugate $\mathcal{O}^\dagger$ is given by

$$\mathcal{O}^\dagger=r_0-\sum_{m=1}^{7} r_m e_m\,. \tag{3}$$

Let us now introduce the concept of **barred octonions**. Working with noncommutative algebras, we must distinguish between left and right multiplication:

$$e_m \mathcal{O} \neq \mathcal{O} e_m\,.$$

It is appropriate to indicate the left-action of the octonionic imaginary units by

$$e_m\,, \tag{4}$$

and the right-action by

$$1|e_m \quad [(1|e_m)\mathcal{O} \equiv \mathcal{O} e_m]\,. \tag{5}$$

In recent papers[11),12)] the successful applications of barred quaternions in quantum mechanics, field theory and gauge theories suggest that we investigate possible potentialities of octonionic barred numbers.

§ 3. Octonionic momentum operator

We begin this section by presenting an apparently hopeless problem related to the nonassociativity of the octonionic field. Working in quantum mechanics we require that an antihermitian operator satisfies the following relation:

$$\int dx \psi^{\dagger}(A\phi) = -\int dx (A\psi)^{\dagger}\phi . \qquad (6)$$

In octonionic quantum mechanics (OQM) we can immediately verify that ∂ represents an antihermitian operator with all the properties of a translation operator. Nevertheless, while in complex (CQM) and quaternionic (QQM) quantum mechanics we can define a corresponding hermitian operator by multiplying the operator ∂ by an imaginary unit, one encounters in OQM the following problem:

no imaginary unit, e_m, represents an antihermitian operator.

In fact, the nonassociativity of the octonionic algebra implies, in general (for arbitrary ψ and ϕ)

$$\int dx \psi^{\dagger}(e_m\phi) \neq -\int dx (e_m\psi)^{\dagger}\phi = \int dx (\psi^{\dagger} e_m)\phi . \quad (m=1, \cdots, 7) \qquad (7)$$

This contrasts with the situation within complex and quaternionic quantum mechanics. Such a difficulty is overcome by using complex projection of the scalar product (complex geometry), with respect to one of our imaginary units. We break the symmetry between the seven imaginary units $e_1, \cdots, e_7$ and choose as the projection plane that one characterized by $(1, e_1)$. The new scalar product is quickly obtained, performing, in the standard definition, the following substitution:

$$\int dx \rightarrow \int_c dx \equiv \frac{1-e_1|e_1}{2} \int dx .$$

Working in OQM with **complex geometry**, e_1 represents an antihermitian operator. In order to simplify the proof we write the octonionic functions ψ and ϕ as follows:

$$\psi = \psi_1 + e_2\psi_2 + e_4\psi_3 + e_6\psi_4 ,$$

$$\phi = \phi_1 + e_2\phi_2 + e_4\phi_3 + e_6\phi_4 ,$$

$$[\psi_{1,\cdots 4} \text{ and } \phi_{1,\cdots,4} \in \mathcal{C}(1, e_1)] .$$

The antihermiticity of e_1 is demonstrated if

$$\int_c dx \psi^{\dagger}(e_1\phi) = -\int_c dx (e_1\psi)^{\dagger}\phi . \qquad (8)$$

In the previous equation the only nonvanishing terms are represented by **diagonal** terms ($\sim \psi_1^{\dagger}\phi_1, \psi_2^{\dagger}\phi_2, \psi_3^{\dagger}\phi_3, \psi_4^{\dagger}\phi_4$). In fact, **off-diagonal** terms, like $\psi_2^{\dagger}\phi_3$, $\psi_3^{\dagger}\phi_4$, are killed by the complex projection

$$(\phi_2^\dagger e_2)[e_1(e_4\phi_3)] \sim (\alpha_2 e_2 + \alpha_3 e_3)(\alpha_4 e_4 + \alpha_5 e_5) \sim \alpha_6 e_6 + \alpha_7 e_7 ,$$

$$[(\phi_3^\dagger e_4)e_1](e_6\phi_4) \sim (\beta_4 e_4 + \beta_5 e_5)(\beta_6 e_6 + \beta_7 e_7) \sim \beta_2 e_2 + \beta_3 e_3 ,$$

$$[\alpha_{2,\cdots 7} \text{ and } \beta_{2,\cdots,7} \in \mathfrak{R}] .$$

The diagonal terms give

$$\int_c d\boldsymbol{x}\phi^\dagger(e_1\phi) = \phi_1^\dagger(e_1\phi_1) - (\phi_2^\dagger e_2)[e_1(e_2\phi_2)] - (\phi_3^\dagger e_4)[e_1(e_4\phi_3)] - (\phi_4^\dagger e_6)[e_1(e_6\phi_4)] , \tag{9a}$$

$$-\int_c d\boldsymbol{x}(e_1\phi)^\dagger \phi = (\phi_1^\dagger e_1)\phi_1 - [(\phi_2^\dagger e_2)e_1](e_2\phi_2) - [(\phi_3^\dagger e_4)e_1](e_4\phi_3) - [(\phi_4^\dagger e_6)e_1](e_6\phi_4) . \tag{9b}$$

The parenthesis in (9a, b) are not of relevance since

$\phi_1^\dagger e_1 \phi_1$	$(1, e_1)$	is a complex number ,
$\phi_2^\dagger e_2 e_1 e_2 \phi_2$	(subalgebra 123) ,	
$\phi_3^\dagger e_4 e_1 e_4 \phi_3$	(subalgebra 145) ,	
$\phi_4^\dagger e_6 e_1 e_6 \phi_4$	(subalgebra 176)	are quaternionic numbers .

The above-mentioned demonstration does not work for the imaginary units $e_2, \cdots, e_7$ (breaking the symmetry between the seven octonionic imaginary units).

Now, we can define an hermitian operator multiplying by e_1 the operator $\boldsymbol{\partial}$. However, such an operator is not expected to commute with the Hamiltonian, which will be, in general, an octonionic quantity. The final step towards an appropriate definition of the momentum operator is represented by choosing as the imaginary unit the barred operator $1|e_1$ (the antihermiticity proof is very similar to the previous one). In OQM with complex geometry **the appropriate** momentum operator is then given by

$$\boldsymbol{p} \equiv -\boldsymbol{\partial}|e_1 . \tag{10}$$

Obviously, in order to write equations relativistically covariant, we must treat the space components and time in the same way. Hence we are obliged to modify the standard QM operator, $i\partial_t$, by the following substitution:

$$i\partial_t \rightarrow \partial_t|e_1 ,$$

and so the octonionic Dirac equation becomes

$$\partial_t\psi e_1 = \boldsymbol{\alpha}\cdot(\boldsymbol{p}\psi) + m\beta\psi . \quad (\boldsymbol{p} \equiv -\boldsymbol{\partial}|e_1) \tag{11}$$

The possibility to write a consistent momentum operator represents for us an impressive argument in favor of the use of a complex geometry in formulating OQM. In addition, such a complex geometry gives us a welcome **quadrupling** of solutions. In fact,

$$\psi\,,\quad e_2\psi\,,\quad e_4\psi\,,\quad e_6\psi\qquad \psi\in \boldsymbol{C}(1,e_1)$$

now represent complex-orthogonal solutions. Therefore, we have the possibility to write a one-component octonionic Dirac equation in which all four standard Dirac free-particle solutions appear.[13]

§ 4. Octonionic Hilbert space

In the early seventies, Gürsey and Günaydin[6] made an attempt to extend the underlying number field of quantum mechanics from complex numbers to octonions. The main results are the following: One can realize the representations of the Poincaré group over an octonionic Hilbert space with complex scalar product which has an automorphism group $SU(3)$. Such octonionic Hilbert space can be divided into an observable subspace corresponding to the usual complex Hilbert space (CHS) of quantum mechanics and an unobservable subspace corresponding to the nonassociative components of the underlying octonionic algebra.

Rembieliński[8] established the isomorphism between octonionic Hilbert space (OHS) with complex geometry and the standard complex Hilbert space carrying the self-representation of the group $U(4)$. The octonionic structure is appropriately defined allowing a consistent tensor product of OHS.

In summary, life is not easy in the generic OHS; we cannot define a tensor product in a concise way, and there is no notion of hermiticity. But using complex geometry, all these problems soon vanish, and our OHS behaves in a suitable way.

We now give two examples of places where the nonassociativity of octonionic numbers is overcome by adopting complex scalar products. A first objection to the nonextendability to an octonionic formulation for the Hilbert space, concerns the completeness relation (Ref. 14), page 50). In writing the completeness formula

$$\begin{aligned}\langle\psi|\phi\rangle&=\langle\psi|\sum_l(|\eta_l\rangle\langle\eta_l|)|\phi\rangle\\&=\sum_l\langle\psi|\eta_l\rangle\langle\eta_l|\phi\rangle\,,\end{aligned}\tag{12}$$

we have assumed, in moving parentheses, that a product of three factors is independent of the order of multiplication. If we take a two-dimensional Hilbert space with the following orthonormal states,

$$|\psi\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}e_4\\e_6\end{pmatrix}\quad\text{and}\quad|\phi\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}-e_1\\e_3\end{pmatrix},\tag{13}$$

$$\langle\psi|\phi\rangle=0$$

and for the complete set $|\eta_l\rangle$ we consider

$$|\eta_1\rangle=\begin{pmatrix}e_3\\0\end{pmatrix}\quad\text{and}\quad|\eta_2\rangle=\begin{pmatrix}0\\e_5\end{pmatrix},\tag{14}$$

we find

$$|\eta_1\rangle\langle\eta_1|+|\eta_2\rangle\langle\eta_2|=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\mathbf{1}_{2\times 2}\,, \tag{15}$$

and so

$$\langle\psi|(\sum_l|\eta_l\rangle\langle\eta_l|)|\phi\rangle=0\,. \tag{16}$$

Now, by direct calculations and using the octonionic algebra, we have

$$\sum_l\langle\psi|\eta_l\rangle\langle\eta_l|\phi\rangle=e_5\neq 0\,. \tag{17}$$

Nevertheless by requiring a complex projection for our scalar products

$$\langle\ |\ \rangle_c\equiv\frac{1-e_1|e_1}{2}\langle\ |\ \rangle, \tag{18}$$

Eq. (16) is obviously not changed, whereas Eq. (17) vanishes. Thus, by imposing complex geometry, we satisfy the completeness relation.

A second objection in writing an octonionic quantum mechanics is the violation of the norm conservation (Ref. 14), page 51).

$$\begin{aligned}\partial_t\langle\psi(t)|\phi(t)\rangle&=(\partial_t\langle\psi(t)|)|\phi(t)\rangle+\langle\psi(t)|(\partial_t|\phi(t)\rangle)\\&=(-\langle\psi(t)|\tilde{H}^{\dagger}(t))|\phi(t)\rangle+\langle\psi(t)|(-\tilde{H}(t)|\phi(t)\rangle)\\&=(\langle\psi(t)|\tilde{H}(t))|\phi(t)\rangle+\langle\psi(t)|(-\tilde{H}(t)|\phi(t)\rangle)\,,\end{aligned} \tag{19}$$

where $\tilde{H}$ represents the anti-self-adjoint Hamiltonian. Again, working in a two-dimensional Hilbert space, let $|\psi(0)\rangle$ and $|\phi(0)\rangle$ be orthonormal states

$$|\psi(0)\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}e_4\\ e_6\end{pmatrix},\quad |\phi(0)\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}-e_1\\ e_3\end{pmatrix}, \tag{20}$$

$$\langle\psi(0)|\phi(0)\rangle=0$$

and consider

$$\tilde{H}=\begin{pmatrix}e_3 & 0\\ 0 & e_5\end{pmatrix}. \tag{21}$$

The time evolution operator will have the following form:

$$U(t,0)=\begin{pmatrix}e^{-e_3t} & 0\\ 0 & e^{-e_5t}\end{pmatrix}=\begin{pmatrix}\cos t-e_3\sin t & 0\\ 0 & \cos t-e_5\sin t\end{pmatrix}. \tag{22}$$

Then we find

$$|\psi(t)\rangle=U(t,0)|\psi(0)\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}e_4\cos t-e_7\sin t\\ e_6\cos t+e_3\sin t\end{pmatrix},$$

$$|\phi(t)\rangle = U(t,0)|\phi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -e_1\cos t + e_2\sin t \\ e_3\cos t - e_6\sin t \end{pmatrix}, \tag{23}$$

leading to

$$\langle\psi(t)|\phi(t)\rangle = e_6\sin t\cos t + e_5\sin^2 t \neq 0\,. \tag{24}$$

It is clear that by imposing complex scalar products, we have

$$\langle\psi(t)|\phi(t)\rangle_c = \langle\psi(0)|\phi(0)\rangle_c = 0\,, \tag{25}$$

which implies norm conservation. Everything is correct as it should be, as required by Rembieliński's isomorphism (scalar product projects the OHS into the physical CHS).

We observe that the dimensionality of a complete set of states for a complex inner product $\langle\psi|\phi\rangle_c$ is *four times* that for the octonionic inner product $\langle\psi|\phi\rangle$. Specifically if $|\eta_l\rangle$ are a complete set of intermediate states for the octonionic inner product, so that

$$\langle\psi|\phi\rangle = \sum_l \langle\psi|\eta_l\rangle\langle\eta_l|\phi\rangle\,,$$

$|\eta_l\rangle$, $|\eta_l e_2\rangle$, $|\eta_l e_4\rangle$, $|\eta_l e_6\rangle$ form a complete set of states for the complex inner product,

$$\begin{aligned}|\phi\rangle &= \sum_l (|\eta_l\rangle\langle\eta_l|\phi\rangle_c + |\eta_l e_2\rangle\langle\eta_l e_2|\phi\rangle_c \\ &\qquad + |\eta_l e_4\rangle\langle\eta_l e_4|\phi\rangle_c + |\eta_l e_6\rangle\langle\eta_l e_6|\phi\rangle_c) \\ &= \sum_m |\chi_m\rangle\langle\chi_m|\phi\rangle_c\,,\end{aligned}$$

where χ_m represent *complex* orthogonal states. Thus the completeness relation can be written as

$$\overrightarrow{1} = \sum_m |\chi_m\rangle\langle\!\langle\chi_m|\,,$$

$$\overleftarrow{1} = \sum_m |\chi_m\rangle\!\rangle\langle\chi_m|$$

(for further details on the completeness relation, one can consult an interesting work of Horwitz and Biedenharn, (Ref. 7), page 455). Thus in our formalism we generalize Dirac's notation by the definitions

$$\langle\!\langle\chi_m|\phi\rangle = \langle\chi_m|\phi\rangle_c\,,$$

$$\langle\phi|\chi_m\rangle\!\rangle = \langle\phi|\chi_m\rangle_c\,.$$

§ 5. Conclusions

This paper is aimed at giving a clear exposition of the potentiality of barred octonions in quantum mechanics. We know that quantum mechanics is the basic tool for different physical applications. Many physicists believe that imaginary numbers are related to the deep secret of quantization. Penrose[15] considers quantization to be completely based on complex numbers. Attempting to overcome the problem of quantum gravity, he proposed to complexify the Minkowskian space-time. This represents the main assumption behind the twistor program. Adler[14] believes that quantization processes should not be limited to complex numbers but should be extended to another member of the division algebras rank, the quaternionic field. He postulates that a successful unification of the fundamental forces will require a generalization beyond complex quantum mechanics. Adler envisages a two-level correspondence principle:

	classical physics and fields,
distance scale ↓	complex quantum mechanics and fields,
	quaternionic quantum field dynamics (preonic level[16])

with quaternionic quantum dynamics interfacing with complex quantum theory, and then with complex quantum theory interfacing in the familiar manner with classical physics (Ref. 14), page 498).

Following this approach, we are tempted to postulate that octonionic quantum field theory may play an essential role in an even deeper fundamental level of physical structure.

Quaternionic quantum mechanics, using complex geometry[11),12)] or quaternionic geometry,[14),16),17)] seems to be consistent from a mathematical point of view. Due to the octonions nonassociativity property, octonionic quantum mechanics seems to be a puzzle. In this paper we have presented an alternative way to look at the octonionic world.

The first motivation, in using octonions numbers in physics, can be concisely stated as follows: We hope to obtain a better understanding of standard theories if we have more than one concrete realization. In this way we can recoginize the fundamental postulates which hold for any generic numerical field.

We emphasize that a characteristic of our formalism is the **absolute need for a complex scalar product**, whereas in quaternionic quantum mechanics, the use of a complex geometry is not obligatory, and thus a question of choice.[14)] Using complex geometry we overcame the hermiticity problem and gave the appropriate and unique definition of the momentum operator. We also realized Rembieliński isomorphism and were able to write down a correct completeness relation and norm conservation.

Obviously to check the consistence of our octonionic formulation of quantum mechanics we need an explicit application. Thus in a forthcoming paper we will develop a relativistic wave equation based on the octonionic field.[13)]

We hope that the work presented in this paper demonstrates that octonionic quantum mechanics may constitute a coherent and well-defined branch of theoretical physics. We are convinced that octonionic quantum mechanics represents largely uncharted and potentially very interesting terrain in theoretical physics.

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