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Half-Whole Dimensions in Quaternionic Quantum Mechanics

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We introduce *half-whole* dimensions for quaternionic matrices and propose a quaternionic version of the Frobenius-Schur theorem which allows us to obtain the proper quaternionic dimensionality for the representations of the Dirac and Duffin-Kemmer-Petiau (DKP) algebras.

§ 1. Introduction

We briefly recall the main properties of the quaternionic field. Such a field is characterized by three imaginary units, i, j, k, which satisfy the following multiplication rules:

$$i^2=j^2=k^2=-1\,, \tag{1a}$$

$$[i,j]=2k\,,\quad [j,k]=2i\,,\quad [k,i]=2j\,. \tag{1b}$$

In going from complex numbers to the quaternions we lose the property of commutativity. The *full*-quaternionic conjugation is denoted by † and defined by

$$1^\dagger=1\,,\quad (i,j,k)^\dagger=-(i,j,k)\,.$$

The previous definition implies

$$(\psi\phi)^\dagger=\phi^\dagger\psi^\dagger$$

for ψ, ϕ quaternionic functions.

Working in quaternionic quantum mechanics with quaternionic geometry (QQM_{qg}), there is no quaternionic self-adjoint operator with all the properties expected for a momentum operator.[1] We would like to overcome such a difficulty by using a complex scalar product[2] (or complex geometry as called by Rembieliński[3])

$$\langle\psi|\phi\rangle_c=\frac{1}{2}\left(\langle\psi|\phi\rangle-i\langle\psi|\phi\rangle i\right),$$

and defining as the appropriate momentum operator[4]

$$\boldsymbol{p}\equiv-\boldsymbol{\partial}|i\,.\quad (\boldsymbol{p}\psi\equiv-\boldsymbol{\partial}\psi i) \tag{2}$$

Note that the usual $\boldsymbol{p}\equiv-i\boldsymbol{\partial}$ still gives a self-adjoint operator with standard commutation relations with the coordinates, but such an operator does not commute with the Hamiltonian, which will be, in general, a quaternionic quantity.

In Eq. (2), a particular *barred* operator appears. We recall the barred quaternion definition (for further details, see Ref. 5)):

$$(q+p|i)r\equiv qr+pri\,.\quad [q,p,r\in\mathcal{H}]$$

We observe that the dimensionality of a complete set of states for the complex inner product $\langle\psi|\phi\rangle_c$ is *twice* that for the quaternionic inner product $\langle\psi|\phi\rangle$. Specifically, if $|\eta_m\rangle$ represent a complete set of intermediate states for the quaternionic scalar product, so that

$$\langle\psi|\phi\rangle=\sum_m\langle\psi|\eta_m\rangle\langle\eta_m|\phi\rangle\,,$$

$|\eta_m\rangle$ and $|\eta_m j\rangle$ form a complete set of states for the complex scalar product,

$$\begin{aligned}|\phi\rangle&=\sum_m(|\eta_m\rangle\langle\eta_m|\phi\rangle_c+|\eta_m j\rangle\langle\eta_m j\rangle\langle\eta_m j|\phi\rangle_c\\&=\sum_n|\chi_n\rangle\langle\chi_n|\phi\rangle_c\,,\end{aligned}$$

where χ_n represent *complex* orthogonal states. The completeness relations can be written as*)

$$\overrightarrow{\mathbf{1}}=\sum_n|\chi_n\rangle(\chi_n|\,,$$

$$\overleftarrow{\mathbf{1}}=\sum_n|\chi_n)\langle\chi_n|\,,$$

where, the standard Dirac notation is generalized by the definitions

$$(\chi_n|\phi\rangle=\langle\chi_n|\phi\rangle_c\,,$$

$$\langle\phi|\chi_n)=\langle\phi|\chi_n\rangle_c\,.$$

§ 2. Even dimensions

Within quaternionic quantum mechanics with complex geometry (QQM_{cg}), we can introduce a "new" complex-imaginary unit

$$1|i\,,\quad[(1|i)\psi\equiv\psi i]$$

$$(1|i)^2=-1\,,\quad(1|i)^\dagger=-1|i\,,$$

which commutes with i, j, k. In order to prove the antihermiticity of $1|i$, we note that with complex scalar products we have

$$\langle\psi|\phi i\rangle_c=\langle\psi|\phi\rangle_c i=i\langle\psi|\phi\rangle_c=-\langle\psi i|\phi\rangle_c\,.$$

Thanks to this "new" complex-imaginary unit, we can perform a translation between even-dimensional complex matrices and quaternionic matrices with half the dimensions.[5] Working in QQM_{cg}, a generic $2n\times2n$ complex representation M can be reduced to two n-dimensional quaternionic representations M_1 and M_2:

$$M=M_1\oplus M_2\,. \qquad (3)$$

*) For further details on these completeness relations, the reader can consult the interesting work of Horwitz and Biedenharn, cited in Ref. 2), page 455.

We give the explicit construction that establishes reducibility for the case of 2×2 complex matrices

$$M=\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}. \tag{4}$$

As a consequence of our complex geometry we have a *doubling* of states:

$$\psi=\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} j \\ 0 \end{pmatrix},\ \begin{pmatrix} 0 \\ j \end{pmatrix}. \tag{5}$$

We observe that the last two (j-complex) states cannot mix, under the action of M, with the former because of the complex nature of the 2-dimensional complex matrix M. Thus, the vector space is *reducible.* Requiring the transformation for the previous states

$$\tilde{\psi}=S\psi \tag{6}$$

with, respectively,

$$\tilde{\psi}=\begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \begin{pmatrix} j \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} 0 \\ j \end{pmatrix}, \tag{7}$$

we can quickly find the quaternionic similarity matrix $S(S^\dagger=S^{-1})$ which reduces the complex representation, M. Explicitly, we have

$$S=\begin{pmatrix} a & ja \\ -jd & d \end{pmatrix},\quad [S^\dagger=S] \tag{8}$$

with

$2a=1-i|i$ which extinguishes j-complex elements,

$2d=1+i|i$ which extinguishes complex elements.

The transformed matrix $\tilde{M}=SMS^\dagger$ is then given by

$$\tilde{M}=\begin{pmatrix} q_1+p_1|i & 0 \\ 0 & q_2+p_2|i \end{pmatrix}, \tag{9}$$

where

$$2q_1=c_1+c_4^*+j(c_3-c_2^*)\,,$$

$$2ip_1=c_1-c_4^*-j(c_3+c_2^*)\,,$$

$$2q_2=c_1^*+c_4+j(c_3^*-c_2)\,,$$

$$2ip_2=c_1^*-c_4-j(c_3^*+c_2)\,.$$

Thanks to this reduction we can obtain a set of rules for the translation. The already well-known identifications of i, j and k with $-i\boldsymbol{\sigma}/2$($\boldsymbol{\sigma}$ the Pauli matrices), and of course 1 (in $\mathcal{H}$) with the 2-dimensional unit matrix, can thus be extended to the most general 2-dimensional complex matrix:

$$M=\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \Longleftrightarrow M_1=q_1+p_1|i\,, \tag{10a}$$

$$M^*=\begin{pmatrix} c_1^* & c_2^* \\ c_3^* & c_4^* \end{pmatrix} \Longleftrightarrow M_2=q_2+p_2|i\,. \tag{10b}$$

$[c_{1,\cdots,4}\in C(1,\, i)$ and $q_{1,2},\, p_{1,2}\in\mathcal{H}]$

Obviously we can generalize the previous result for a generic $2n$-dimensional complex matrix. In particular, 4×4 complex matrices (with four *complex* states) split into 2×2 quaternionic matrices (with two *complex* + two *j-complex* states):

$$M=\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & c_7 & c_8 \\ c_9 & c_{10} & c_{11} & c_{12} \\ c_{13} & c_{14} & c_{15} & c_{16} \end{pmatrix} \Longleftrightarrow M=\begin{pmatrix} r_1+s_1|i & r_2+s_2|i \\ r_3+s_3|i & r_4+s_4|i \end{pmatrix}, \tag{11}$$

$[c_{1,\cdots,16}\in C(1,\, i)$ and $r_{1,\cdots,4},\, s_{1,\cdots,4}\in\mathcal{H}]$

$2r_1=c_1+c_6^*+j(c_5-c_2^*)\,,$

$2is_1=c_1-c_6^*-j(c_5+c_2^*)\,,$

$2r_2=c_3+c_8^*+j(c_3-c_8^*)\,,$

$2is_2=c_7-c_4^*-j(c_7+c_4^*)\,,$

$2r_3=c_9+c_{14}^*+j(c_{13}-c_{10}^*)\,,$

$2is_3=c_9-c_{14}^*-j(c_{13}+c_{10}^*)\,,$

$2r_4=c_{11}+c_{16}^*+j(c_{15}-c_{12}^*)\,,$

$2is_4=c_{11}-c_{16}^*-j(c_{15}+c_{12}^*)\,.$

§ 3. Odd dimensions

As described in a recent article[6] the above translation can be performed, using a particular trick, for odd dimensional complex representations. 3×3 complex matrices can be reduced to two overlapping 2×2 block forms (so that the (2, 2)-element is common to both blocks). We start with a generic 3×3 complex matrix

$$M=\begin{pmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{pmatrix}, \tag{12}$$

which shows the following *doubling* of base states, in the associated vector space:

$$\begin{pmatrix}1\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\1\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\1\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}j\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\j\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\j\end{pmatrix}.$$

As remarked in § 2, the vector space is *reducible*. The quaternionic similarity matrix S which transforms the previous states in

$$\begin{pmatrix}1\\0\\0\end{pmatrix},\ \begin{pmatrix}j\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\1\\0\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}0\\0\\1\end{pmatrix},\ \begin{pmatrix}0\\0\\j\end{pmatrix},\ \begin{pmatrix}0\\j\\0\end{pmatrix},$$

and performs the reduction is

$$S=\begin{pmatrix}a & ja & 0\\ 0 & 0 & 1\\ -jd & d & 0\end{pmatrix}, \tag{13a}$$

$$S^{\dagger}=\begin{pmatrix}a & 0 & ja\\ -jd & 0 & d\\ 0 & 1 & 0\end{pmatrix}. \tag{13b}$$

The transformed matrix $\tilde{M}$ is then given by

$$\tilde{M}=\begin{pmatrix}(c_1+jc_4)a+(c_5^*-jc_2^*)d & (c_3+jc_6)a & 0\\ c_7a-jc_8^*d & c_9 & jc_7^*a+c_8d\\ 0 & (-jc_3+c_6)d & (c_1^*+jc_4^*)a+(c_5-jc_2)d\end{pmatrix}. \tag{14}$$

In $\tilde{M}$, the (2, 2)-element can be written conveniently as $c_9(a+d)$, i.e., containing a sum of projection operators. Thus, we can translate a generic 3×3 complex matrix by a *particular* 2×2 quaternionic matrix

$$M=\begin{pmatrix}c_1 & c_2 & c_3\\ c_4 & c_5 & c_6\\ c_7 & c_8 & c_9\end{pmatrix}\Longleftrightarrow M_1=\begin{pmatrix}(c_1+jc_4)a+(c_5^*-jc_2^*)d & (c_3+jc_6)a\\ c_7a-jc_8^*d & c_9a\end{pmatrix}, \tag{15a}$$

$$M^*=\begin{pmatrix}c_1^* & c_2^* & c_3^*\\ c_4^* & c_5^* & c_6^*\\ c_7^* & c_8^* & c_9^*\end{pmatrix}\Longleftrightarrow M_2=\begin{pmatrix}c_9d & jc_7^*a+c_8d\\ (-jc_3+c_6)d & (c_1^*+jc_4^*)a+(c_5-jc_2)d\end{pmatrix}. \tag{15b}$$

In order to prove the last identification, note that

$$\begin{pmatrix}0 & 1\\ -j & 0\end{pmatrix}M_2\begin{pmatrix}0 & j\\ 1 & 0\end{pmatrix}=\begin{pmatrix}(c_1^*+jc_4^*)a+(c_5-jc_2)d & (c_3^*+jc_6^*)a\\ c_7^*a-jc_8d & c_9^*a\end{pmatrix}. \tag{16}$$

We conclude this section by remarking on the difference between a 2×2 quaternionic matrix which acts non-trivially only on three states (see Eq. (15a)),

$$M^{(\text{three})}=\begin{pmatrix}q+p|i & ra\\ z_1a+jz_2d & z_3a\end{pmatrix},\quad [z_{1,2,3}\in\mathcal{C}(1,i)\text{ and }q,p,r\in\mathcal{H}] \tag{17}$$

— $M^{(\text{three})}$ action —

$$M^{(\text{three})}\begin{pmatrix}0\\ jz\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}, \quad M^{(\text{three})}\begin{pmatrix}q\\ z\end{pmatrix}=\begin{pmatrix}q'\\ z'\end{pmatrix}, \quad [z, z'\in \mathcal{C}(1, i) \text{ and } q, q'\in \mathcal{H}]$$

and a generic 2×2 quaternionic matrix which acts non-trivially on four states (see Eq. (11)).

In standard theory the dimensionality of complex matrices is strictly connected to the dimensionality of the vector space, whereas working in QQM_{cg} we have a *doubling* of states, so we require the following correspondence rule between the dimensionality of quaternionic matrices (n) and the dimensionality of the vector space (n_{VS})

$$2n=n_{\text{VS}}.$$

In order to distinguish between *odd* and *even* vector spaces we introduce *half-whole* dimensions for our quaternionic matrices:

$$n_{M^{(\text{three})}}=\frac{3}{2}, \quad n_{M^{(\text{four})}}=2. \tag{18}$$

§ 4. Dirac algebra

Let us consider in abstracto, four algebraic quantities γ^μ, which satisfy the Dirac relations:

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}. \quad (\mu, \nu=0, 1, 2, 3) \tag{19}$$

We observe that in the following array (characterized by sixteen quantities)

$$\begin{array}{l} 1;\\ \gamma^0, \gamma^1, \gamma^2, \gamma^3;\\ \gamma^0\gamma^1, \gamma^0\gamma^2, \gamma^0\gamma^3, \gamma^1\gamma^2, \gamma^1\gamma^3, \gamma^2\gamma^3;\\ \gamma^0\gamma^1\gamma^2, \gamma^0\gamma^1\gamma^3, \gamma^0\gamma^2\gamma^3, \gamma^1\gamma^2\gamma^3;\\ \gamma^0\gamma^1\gamma^2\gamma^3, \end{array}$$

any product of two elements is proportional to another element of the array. We now wish to obtain appropriate matrix representations for the abstract algebraic quantities γ^μ. First of all we briefly recall the standard (complex) results. Then we will generalize our considerations, by considering, as underlying numerical fields, quaternions and complexified quaternions.

In standard (complex) theory it is very simple to prove the following theorems.*)

1. If $\gamma_A\neq 1$, one can always find a γ_B such that $\gamma_B\gamma_A\gamma_B=-\gamma_A$;
2. With the exception of the 1-element, the trace of all γ_A is zero;
3. The sixteen γ_A are linearly independent,

*) In order to simplify the following considerations we indicate the general element of the array by $\gamma_A(A=1, 2, \cdots, 16)$.

$$\sum_{A=1}^{16} a_A \gamma_A = 0\,, \quad (a_A \text{ complex numbers})$$

if and only if all sixteen coefficients a_A vanish,

4. The only hypercomplex quantity $X=\sum_{A=1}^{16} a_A \gamma_A$ which commutes with all γ_A is (a multiple of) unity.

In order to find all possible irreducible representations of the Dirac algebra, we shall need two noteworthy theorems regarding the representations of algebras.

The first is the theorem of Frobenius and Shur which may be stated as follows:

5. Let $\mathcal{A}$ be an algebra of order n possessing a unit element. Let p be number of (non-equivalent) irreducible representations of the algebra, and denote the dimensionality of these representations by $n_1, n_2, \cdots, n_p$ in turn. Then

$$n = n_1^2 + n_2^2 + \cdots + n_p^2\,. \tag{20}$$

The second theorem enables us to find the number p of the possible irreducible representations:

6. If the algebra $\mathcal{A}$ is semi-simple, then the number of possible irreducible representations is equal to the maximum number of base elements which commute with each other.

Combining these two theorems 5 and 6, we can quickly obtain the dimensionalities of the various possible irreducible representations of a semi-simple algebra with a unit element.

By virtue of the previous considerations one finds that the only *complex* irreducible representations of the Dirac algebra is *four-dimensional*.

What happens for quaternions? Obviously the theorems 1 and 6 also hold, since their demonstrations do not use the explicit form of the γ_A-matrices. In order to prove theorems 2, 3 and 4 we must introduce an appropriate definition of trace and choose *commuting* numerical coefficients a_A. Finally, the remaining (Frobenius and Shur) theorem will be modified.

§ 5. Quaternionic Dirac algebra

In a previous work, Rotelli[4] derived a *new* version of the Dirac equation by adopting quaternions as the underlying numerical field. The main difference between the quaternionic and complex Dirac equations is represented by the dimensionality of the γ^μ-matrices. In fact, working within QQM, there exists a 2×2 matrix representation for the Dirac algebra given by

$$\boldsymbol{\gamma} = \boldsymbol{Q}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad [\boldsymbol{Q} \equiv (i, j, k)] \tag{21}$$

Notwithstanding the two component structure of the quaternionic wave functions, four standard Dirac solutions are reproduced. In such an equation, the complex geometry gives a welcome doubling of states.*)

In standard (complex) quantum mechanics, multiplying the following sixteen real matrices by complex numbers

$$\begin{pmatrix}1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ \begin{pmatrix}\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{pmatrix},\ \dots,\ \begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot\end{pmatrix},\ \begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1\end{pmatrix},$$

(where dots indicate zeros) we obtain the most general 4×4 complex matrix, so such a matrix is sufficient to represent the sixteen quantities which characterize the Dirac algebra.

At first glance it seems that, within QQM, we must have a Dirac algebra on *reals* (if we need coefficients α_A which commute with our quaternionic matrices). Utilizing real numbers as multiplicative coefficients, we can understand the *reduced* dimensions of the γ^{μ}-matrices, because the four *real* matrices

$$\begin{pmatrix}1&\cdot\\\cdot&\cdot\end{pmatrix},\ \begin{pmatrix}\cdot&1\\\cdot&\cdot\end{pmatrix},\ \begin{pmatrix}\cdot&\cdot\\1&\cdot\end{pmatrix},\ \begin{pmatrix}\cdot&\cdot\\\cdot&1\end{pmatrix}$$

(if multiplied by real numbers) require the following quaternionic partners

$$\boldsymbol{Q}\begin{pmatrix}1&\cdot\\\cdot&\cdot\end{pmatrix},\ \boldsymbol{Q}\begin{pmatrix}\cdot&1\\\cdot&\cdot\end{pmatrix},\ \boldsymbol{Q}\begin{pmatrix}\cdot&\cdot\\1&\cdot\end{pmatrix},\ \boldsymbol{Q}\begin{pmatrix}\cdot&\cdot\\\cdot&1\end{pmatrix}.$$

Therefore, working with 2×2 quaternionic matrices and using real numbers as multiplicative coefficients, we can yet reproduce the *magic* number 16.

In our previous papers[5),10)] we have emphasized the possibility to use *barred* quaternions within quaternionic matrices. In this case we could multiply our matrices by *complex* numbers like

$$a+b|i\,,\quad (a,\,b\in\mathcal{R})$$

which obviously commute with any quaternionic quantities. If we allow use of such barred-complex numbers, we can generalize the standard quaternionic trace definition

$$\mathrm{tr}\begin{pmatrix}q_1&q_2\\q_3&q_4\end{pmatrix}=\mathrm{re}(q_1+q_4)$$

by

$$\mathrm{Tr}\begin{pmatrix}q_1+p_1|i&q_2+p_2|i\\q_3+p_3|i&q_4+p_4|i\end{pmatrix}=\mathrm{re}(q_1+q_4)+\mathrm{re}(p_1+p_4)|i\,. \tag{22}$$

It is straightforward to prove that the new trace definition guarantees the standard property

*) Observe that within QQM with complex geometry e^{-ipx}, je^{-ipx} represents orthogonal solutions.

$$\mathrm{Tr}(M_1M_2)=\mathrm{Tr}(M_2M_1).$$

Choosing *barred* complex coefficients a_A and generalizing the trace definition, we can easily demonstrate the theorems 2, 3 and 4, given in the previous section.

In order to complete our discussion concerning the quaternionic Dirac algebra, we must modify the Frobenius and Shur theorem as follows:

$$n=4n_1^2+4n_2^2+\cdots+4n_p^2, \tag{23}$$

in fact we must remember that for any real matrix

$$\begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

we must add three quaternionic partners

$$\boldsymbol{Q}\begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Modified Frobenius-Schur Theorem: Let $\mathcal{A}$ be an algebra of order n possessing a unit element. Let p be the number of (non-equivalent) irreducible representations of the algebra, and denote the dimensionality of these representations by $n_1, n_2, \cdots, n_p$ in turn. Then

$$n=4(n_1^2+n_2^2+\cdots+n_p^2) \tag{24a}$$

with

$$n_{1,\cdots,p}=\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots. \tag{24b}$$

§ 6. Quaternionic DKP algebra

Applying the *modified* Frobenius-Schur theorem to the Dirac algebra we find*)

$$16=4n_1^2. \tag{25}$$

Thus we have 2×2 quaternionic matrix representations for the Dirac algebra (see Eq. (21)).

In this section we briefly show an example of *half-whole dimensions* by analyzing the DKP algebra

$$\beta^\mu\beta^\nu\beta^\lambda+\beta^\lambda\beta^\nu\beta^\mu=-g^{\mu\nu}\beta^\lambda-g^{\lambda\nu}\beta^\mu. \tag{26}$$

By a similar procedure to that used in the case of the Dirac algebra, but of course with considerably more effort, one can trace the properties of the DKP algebra. We find that there are now 126 linearly independent quantities. Moreover, one finds that

*) Note that the maximum number of Dirac algebra base elements which commute with each other is one, so $n=4n_1^2$.

there are three elements which commute with all base elements ($p=3$). We now may use our *modified* theorem as given in the previous section. We have to decompose 126 into the sum of three square numbers. This is accomplished by

$$126=4\left[\left(\frac{1}{2}\right)^2+\left(\frac{5}{2}\right)^2+5^2\right]. \tag{27}$$

In summary, the DKP algebra has three quaternionic representations and these are one-half (trivial), five-halves (spin 0) and five (spin 1) dimensional representations.

We explicitly give the quaternionic representations of dimension 1/2 and 5/2 by the following 3×3 quaternionic matrices:

$$\beta^0=\begin{pmatrix} \cdot & \cdot & a \\ \cdot & \cdot & \cdot \\ -a & \cdot & \cdot \end{pmatrix}, \quad \beta^1=j\begin{pmatrix} \cdot & \cdot & a \\ \cdot & \cdot & \cdot \\ -d & \cdot & \cdot \end{pmatrix},$$

$$\beta^2=\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & a \\ \cdot & a & \cdot \end{pmatrix}, \quad \beta^3=j\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & a \\ \cdot & -d & \cdot \end{pmatrix}, \tag{28}$$

where

$$2a=\frac{1-i|i}{2}, \quad 2d=\frac{1+i|i}{2}.$$

We can immediately observe that the β^μ-matrices of Eq. (28) act trivially on the state

$$\begin{pmatrix} 0 \\ 0 \\ j \end{pmatrix}, \quad \text{[trivial case]}$$

and non-trivially on the states

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} j \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ j \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad \text{[spin 0]}$$

Such matrices represent the quaternionic counterpart of the complex matrices (spin 0 + trivial case) which appear in the standard DKP equation (a complete discussion of the quaternionic DKP equation recently appeared in the literature 7)).

§ 7. Conclusions

The renewed interest in *QQM* (Refs. 1) and 11)), suggests that we look at the quaternionic world with trust. The introduction of barred quaternions

$$q+p|i,$$

(natural objects when one works within QQM_{cg}) allow us to formulate in a consistent way the standard physical theories (like special relativity,[8] electroweak model,[9] GUT[10]). From the viewpoint of group structure, these barred quantities are very similar to complexified quaternions[12]

$$q=\mathcal{I}p$$

(the imaginary unit $\mathcal{I}$ commutes with the quaternionic imaginary units i, j, k), but in physical problems, such as eigenvalue calculations, tensor products, relativistic equations solutions, they give different results.

Barred quaternions are very useful in writing a quaternionic version of the Dirac[4] and DKP[7] equations. Nevertheless, if we wish to use quaternions as the underlying numerical field we must revise the standard assumptions. For example, due to the doubling of solutions given by the complex geometry, we must introduce *half-whole* dimensions for quaternionic matrices and modify the Frobenius-Schur theorem. Obviously, this represents only a first step towards a quaternionic world. An interesting research topic could be to generalize the group theoretical structure by our barred quaternionic operators.

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