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Quaternionic electroweak theory

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Abstract. We explicitly develop a quaternionic version of the electroweak theory, based on the local gauge group $U(1,\ q)_L \mid U(1,\ c)_Y$. The need of a complex projection for our Lagrangian and the physical significance of the anomalous scalar solutions are also discussed.

1. Introduction

Not many mathematicians can claim to have invented a new kind of number. A rigorous definition of the reals was given by Eudoxus, after the Pythagoreans discovery that the equation $x^2 = 2$ cannot be solved for rational numbers. The Indian mathematician Brahmagupta was the first to allow zero and negative numbers to be subjected to arithmetical operations, thus permitting the translation from $\mathcal{R}^{(+)}$ to $\mathcal{R}$. Cardano, perhaps better known as a physician than as a mathematician, introduced complex numbers, probably to solve equations such as $x^2+1=0$. After Gauss had proved the fundamental theorem of algebra, there was no longer any need to introduce new numbers to solve equations [1]. In fact, it was with a different motivation in mind that quaternions were invented by WR Hamilton [2].

Hamilton was looking for numbers of the form

$$x + \mathrm{i}y + \mathrm{j}z \tag{1a}$$

with

$$\mathrm{i}^2 = \mathrm{j}^2 = -1 \tag{1b}$$

which would do for the space what complex numbers had done for the plane. Nevertheless such a number system does not represent a correct choice. Working with only two imaginary units we must express the product ij by

$$\mathrm{ij} = a + \mathrm{i}b + \mathrm{j}c \qquad (a,\ b,\ c \in \mathcal{R})\,. \tag{2}$$

Equation (2) implies

$$\mathrm{i}^2\mathrm{j} = \mathrm{i}a - b + (a + \mathrm{i}b + \mathrm{j}c)c = \cdots + \mathrm{j}c^2$$

and so the inconsistent relation

$$c^2 = -1\,.$$

In 1843 Hamilton introduced a third imaginary unit $\mathrm{k} = \mathrm{ij}$. Numbers of the form

$$q = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d \qquad (a,\ b,\ c,\ d \in \mathcal{R}) \tag{3a}$$

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were called quaternions. They were added, subtracted and multiplied according to the usual law of arithmetic, except for their non-commutative multiplication law, due to the following rules for the imaginary units i, j, k:

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1 \tag{3b}$$

$$\mathrm{ij} = -\mathrm{ji} = \mathrm{k} \qquad \mathrm{jk} = -\mathrm{kj} = \mathrm{i} \qquad \mathrm{ki} = -\mathrm{ik} = \mathrm{j}\,. \tag{3c}$$

The conjugate of (3*a*) is given by

$$q^\dagger = a - \mathrm{i}b - \mathrm{j}c - \mathrm{k}d\,. \tag{4}$$

We observe that $qq^\dagger$ and $q^\dagger q$ are both equal to the real number

$$N(q) = a^2 + b^2 + c^2 + d^2$$

which is called the norm of q. When $q \neq 0$, we can define

$$q^{-1} = q^\dagger / N(q)$$

so the quaternions form a zero-division ring. Such a non-commutative number field is denoted, in Hamilton's honour, by $\mathcal{H}$.

Our aim in this work is to show that quaternions can be used to express standard physical theories. In particular, overcoming difficulties due to their non-commutative multiplication law, we formulate a quaternionic version of the Salam–Weinberg model. In section 2 we apply the quaternionic numbers in classical and quantum physics analysing a quaternionic formulation of special relativity and giving a quaternionic version of the Dirac equation. In this section we also point out the possible predictive potential of quaternionic numbers by a qualitative study of the Schrödinger equation. A feature of our formalism, namely the need of a complex projection for quaternionic Lagrangians, is discussed in section 3. In section 4 we examine the main differences between the standard (complex) physical theory and our quaternionic version (see the doubling of solutions in the quaternionic bosonic equations). There, recalling the main steps of a previous article, we give a possible interpretation for the anomalous solutions which appear in the quaternionic Klein–Gordon equation. In section 5 we will explicitly discuss a quaternionic electroweak theory (QewT), based on the 'quaternionic Glashow group'

$$U(1,\ q)_L \mid U(1,\ c)_Y \qquad (L \leftrightarrow \text{left-handed helicity},\ Y \leftrightarrow \text{weak-hypercharge})\,.$$

Our conclusions are drawn in section 6. In the next sections we will adopt the following notation

$$\phi, \psi \quad \leftrightarrow \quad \text{complex fields}$$
$$\Phi, \Psi \quad \leftrightarrow \quad \text{quaternionic fields}$$

and use the system of natural units ($\hbar = c = 1$).

2. Quaternions in classical and quantum physics

If we represent complex numbers in a plane by

$$x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}$$

(recall that in place of i we could use any imaginary unit), we immediately observe that a rotation of α-angle around the z-axis can be given by $\mathrm{e}^{\mathrm{i}(\theta+\alpha)}$, in fact

$$\mathrm{e}^{\mathrm{i}\alpha}(x + \mathrm{i}y) = r\mathrm{e}^{\mathrm{i}(\theta+\alpha)}\,. \tag{5}$$

Using quaternions (instead of complex numbers) we can express a rotation in three-dimensional space. For example a rotation about an axis passing through the origin and parallel to a given unitary vector $\hat{\boldsymbol{u}} \equiv (u_x,\ u_y,\ u_z)$ by an angle α, can be obtained by making the transformation

$$\exp\left((\mathrm{i}u_x + \mathrm{j}u_y + \mathrm{k}u_z)\frac{\alpha}{2} \right)\,(\mathrm{i}x + \mathrm{j}y + \mathrm{k}z)\,\exp\left(-(\mathrm{i}u_x + \mathrm{j}u_y + \mathrm{k}u_z)\frac{\alpha}{2} \right) \tag{6}$$

upon the position vector $\mathcal{X} \equiv \mathrm{i}x + \mathrm{j}y + \mathrm{k}z$. This reduces, after simple manipulations, to (5) (we pose $u_x = u_y = 0$ and $u_z = 1$ in (6)), except for the appearance of k instead of i. The previous transformation leaves $\mathcal{X}^\dagger \mathcal{X} = x^2 + y^2 + z^2$ invariant.

The special theory of relativity requires the invariance of the expression

$$t^2 - x^2 - y^2 - z^2$$

under a coordinate transformation passing from a stationary frame to a moving one with constant velocity. This suggests that space and time be joined together in a quaternionic number

$$\mathcal{X} \equiv t + \mathrm{i}x + \mathrm{j}y + \mathrm{k}z \tag{7}$$

with the Lorentz invariant $\mathrm{Re}\,\mathcal{X}^2$. If complex numbers are the natural candidates to represent rotation in a plane, quaternions express concisely the Lorentz transformations. Introducing *barred-operator* $A \mid b$, which acts on quaternionic objects q as in

$$(A \mid b)q = Aqb$$

we can quickly express the generators of the Lorentz groups by the following operators [3]:

$$\begin{array}{ll}
\text{boost } (t,\ x) & \dfrac{\mathrm{k} \mid \mathrm{j} - \mathrm{j} \mid \mathrm{k}}{2} \\[2ex]
\text{boost } (t,\ y) & \dfrac{\mathrm{i} \mid \mathrm{k} - \mathrm{k} \mid \mathrm{i}}{2} \\[2ex]
\text{boost } (t,\ z) & \dfrac{\mathrm{j} \mid \mathrm{i} - \mathrm{i} \mid \mathrm{j}}{2} \\[2ex]
\text{rotation around } x & \dfrac{\mathrm{i} - 1 \mid \mathrm{i}}{2} \\[2ex]
\text{rotation around } y & \dfrac{\mathrm{j} - 1 \mid \mathrm{j}}{2} \\[2ex]
\text{rotation around } z & \dfrac{\mathrm{k} - 1 \mid \mathrm{k}}{2}\,.
\end{array}$$

The last three reproduce (6) in a new form.

An interesting application of quaternions in quantum physics is represented by the quaternionic formulation of the Dirac equation [4]. Notwithstanding the two-component structure of the wavefunction, all four standard solutions appear. This represents a stimulating example of the doubling of solutions within a quaternionic quantum mechanics with complex geometry [5]. We indicate with the terminology complex geometry the use of a complex scalar product $\langle\Psi|\Phi\rangle_{\mathrm{c}}$, defined in terms of the quaternionic counterpart $\langle\Psi|\Phi\rangle$ by

$$\langle\Psi|\Phi\rangle_{\mathrm{c}} = \frac{1 - \mathrm{i} \mid \mathrm{i}}{2}\langle\Psi|\Phi\rangle\,. \tag{8}$$

Such a scalar product was used by Horwitz and Biedenharn in order to consistently define multiparticle quaternionic states [6]. We observe that the dimensionality of a complete set of states for complex inner product $\langle \Psi | \Phi \rangle_c$ is *twice* that of the quaternionic inner product $\langle \Psi | \Phi \rangle$. Specifically, if $|\eta_l\rangle$ are a complete set of intermediate states for the quaternionic inner product, so that

$$\langle \psi | \phi \rangle = \sum_l \langle \psi | \eta_l \rangle \langle \eta_l | \phi \rangle$$

$|\eta_l\rangle$ and $|\eta_l \ \mathrm{j}\rangle$ form a complete set of states for the complex inner product

$$|\phi\rangle = \sum_l \left(\, |\eta_l\rangle \langle \eta_l | \phi \rangle_c + |\eta_l \ \mathrm{j}\rangle \langle \eta_l \ \mathrm{j} | \phi \rangle_c \right)$$

$$= \sum_m |\chi_m\rangle \langle \chi_m | \phi \rangle_c$$

where χ_m represent *complex* orthogonal states. For further details on the completeness relation, one can consult [6, p 455] and [7].

We justify the choice of a complex geometry by recalling that although there is in quaternionic quantum mechanics an anti-self-adjoint operator, $\boldsymbol{\partial}$, with all the properties of a translation operator, imposing a quaternionic geometry, there is no corresponding quaternionic self-adjoint operator with all the properties expected for a momentum operator. This hopeless situation is also highlighted in Adler's recent book [8]. Nevertheless, we can overcome such a difficulty using a complex scalar product and defining as the appropriate momentum operator

$$\boldsymbol{p} \equiv -\boldsymbol{\partial} \mid \mathrm{i} \, . \tag{9}$$

Note that the usual choice $\boldsymbol{p} \equiv -\mathrm{i}\boldsymbol{\partial}$ still gives a self-adjoint operator with standard commutation relations with the coordinates, but such an operator does not commute with the Hamiltonian, which will, in general, be a quaternionic quantity. Obviously, in order to write equations that are relativistically covariant, we must treat the space components and time in the same way, hence we are obliged to modify the standard equations by the following substitution:

$$\mathrm{i}\partial_t \rightarrow \partial_t \mid \mathrm{i} \tag{10}$$

and so the quaternionic Dirac equation becomes

$$\partial_t \Psi \mathrm{i} = (\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m)\Psi \qquad (\boldsymbol{p} \equiv -\boldsymbol{\partial} \mid \mathrm{i}) \, . \tag{11}$$

Noting that the Dirac algebra upon the *reals* (but not upon the complex) has a two-dimensional irreducible representation with quaternions. Thus the standard 4×4 complex matrices $(\boldsymbol{\alpha}, \beta)$ reduce to 2×2 quaternionic matrices. For example, a particular representation is given by

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \boldsymbol{\alpha} = \boldsymbol{Q} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (\boldsymbol{Q} \equiv (\mathrm{i}, \ \mathrm{j}, \ \mathrm{k})) \, .$$

In this representation the quaternionic plane-wave solutions are

$$\begin{aligned} E > 0: &\qquad u \ \mathrm{e}^{-\mathrm{i}px} \qquad u \ \mathrm{j}\mathrm{e}^{-\mathrm{i}px} \\ E < 0: &\qquad v \ \mathrm{e}^{-\mathrm{i}px} \qquad v \ \mathrm{j}\mathrm{e}^{-\mathrm{i}px} \end{aligned} \tag{12}$$

where

$$u = \sqrt{E+m} \begin{pmatrix} 1 \\ -\dfrac{\boldsymbol{Q} \cdot \boldsymbol{p}}{E+m} \end{pmatrix}$$

and

$$v = \sqrt{|E| + m} \begin{pmatrix} -\dfrac{\boldsymbol{Q} \cdot \boldsymbol{p}}{|E| + m} \\ 1 \end{pmatrix} .$$

Following the standard approach we can define the Hermitian spin operator

$$\boldsymbol{S} \equiv -\tfrac{1}{2} \begin{pmatrix} \boldsymbol{Q} \mid \mathrm{i} \end{pmatrix}$$

so the four complex orthogonal solutions given in (12) correspond to positive- and negative-energy solutions with $S = \frac{1}{2}$, and for $\boldsymbol{p} = (p_x,\ 0,\ 0)$, $S_x = \frac{1}{2},\ -\frac{1}{2},\ \frac{1}{2},\ -\frac{1}{2}$, respectively. Thus, although our wavefunction has a two-component structure, we find the four standard solutions to the Dirac equation. This is a *desirable* example of the so-called *doubling of solutions*.

Obviously such a doubling of solutions also occurs in the non-relativistic Schrödinger equation

$$\partial_t \Psi \mathrm{i} = -\frac{\boldsymbol{\partial}^2}{2m} \Psi \qquad \text{(quaternionic solutions: } \mathrm{e}^{-\mathrm{i}px},\ \mathrm{je}^{-\mathrm{i}px}) . \tag{13}$$

If Schrödinger had worked within a quaternionic quantum mechanics with complex geometry, finding two complex-orthogonal solutions to his equation, he would have probably discovered *spin*. Indeed the non-relativistic limit of the Dirac equation yields the Schrödinger–Pauli equation with two solutions which is formally identical with the *one-component* Schrödinger equation with quaternions. We like to call this stimulating situation within quaternionic quantum mechanics with complex geometry: *the belated theoretical discovery of spin*.

As we have already noted elsewhere [9], this doubling of solutions in the Schrödinger equation would be an impressive argument in favour of the use of quaternions within quantum mechanics, if this doubling of solutions did not also occur in bosonic equations, where it has obviously nothing to do with spin. For example, we find four complex orthogonal solutions for the Klein–Gordon equation, with the result that, in addition to the two normal solutions

$$\mathrm{e}^{-\mathrm{i}px} \qquad \text{(positive and negative energy)}$$

we discover two *anomalous* solutions

$$\mathrm{je}^{-\mathrm{i}px} \qquad \text{(positive and negative energy)} .$$

The physical significance of the anomalous solutions has been a ‘puzzle’ for the authors. Only recently, by a quaternionic study of the electroweak Higgs sector, have we been able to identify anomalous Higgs particles [10].

3. Complex projected Lagrangians

Before analysing the Higgs sector within a QewT, we must highlight a feature of quaternionic field theory, namely *the need for the use of a complex projection for our quaternionic Lagrangians*. This result has been justified in previous papers [10, 11]; here we briefly recall only some of the main steps.

The standard free Lagrangian density for two Hermitian scalar fields ϕ_1, ϕ_2 is

$$\mathcal{L} = \tfrac{1}{2} \left(\partial_\mu \phi_1 \partial^\mu \phi_1 - m^2 \phi_1^{\,2} + \partial_\mu \phi_2 \partial^\mu \phi_2 - m^2 \phi_2^{\,2} \right) \tag{14}$$

where

$$\phi_{1,\,2} = V^{-\frac{1}{2}} \sum_{\boldsymbol{k}} (2\omega_{\boldsymbol{k}})^{-\frac{1}{2}} [a_{1,\,2}(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}kx} + a^{\dagger}_{1,\,2}(\boldsymbol{k})\mathrm{e}^{+\mathrm{i}kx}] \,.$$

The Lagrangian (14) can be concisely rewritten, by complex scalar fields ϕ, $\phi^{\dagger}$, as follows:

$$\mathcal{L} = \partial_{\mu}\phi^{\dagger}\partial^{\mu}\phi - m^2\phi^{\dagger}\phi \qquad \left(\phi \equiv \frac{\phi_1 + \mathrm{i}\phi_2}{\sqrt{2}}\;,\; \phi^{\dagger} \equiv \frac{\phi_1 - \mathrm{i}\phi_2}{\sqrt{2}}\right). \tag{15}$$

Note that the cross term $\phi_1\mathrm{i}\phi_2 - \mathrm{i}\phi_2\phi_1$ is trivially null ($[\phi_1,\ \phi_2] = 0$ and i commutes with $\phi_{1,\,2}$). We now wish to extend the previous considerations to four Hermitian scalar fields ϕ_1, ϕ_2, ϕ_3, ϕ_4. In order to rewrite

$$\mathcal{L} = \tfrac{1}{2}\,(\partial_{\mu}\phi_1\partial^{\mu}\phi_1 - m^2{\phi_1}^2 + \partial_{\mu}\phi_2\partial^{\mu}\phi_2 - m^2{\phi_2}^2 + \partial_{\mu}\phi_3\partial^{\mu}\phi_3 - m^2{\phi_3}^2 + \partial_{\mu}\phi_4\partial^{\mu}\phi_4 - m^2{\phi_4}^2) \tag{16}$$

by quaternionic scalar fields Φ, $\Phi^{\dagger}$

$$\Phi \equiv \frac{\phi_1 + \mathrm{i}\phi_2 + \mathrm{j}\phi_3 + \mathrm{k}\phi_4}{\sqrt{2}} \qquad \left(\Phi^{\dagger} \equiv \frac{\phi_1 - \phi_2\mathrm{i} - \phi_3\mathrm{j} - \phi_4\mathrm{k}}{\sqrt{2}}\right) \tag{17}$$

we must require a complex projection of our Lagrangian. Such a complex projection kills the 'pure' quaternionic cross terms

$$\phi_1\mathrm{j}\phi_3 - \phi_3\mathrm{j}\phi_1 \qquad \phi_1\mathrm{k}\phi_4 - \phi_4\mathrm{k}\phi_1 \qquad -\phi_2\mathrm{ij}\phi_3 - \phi_3\mathrm{ji}\phi_2 \qquad -\phi_2\mathrm{ik}\phi_4 - \phi_4\mathrm{ki}\phi_2\,.$$

So the quaternionic Klein–Gordon Lagrangian, with four Hermitian scalar fields, reads†

$$\mathcal{L}_{\mathrm{c}} = \frac{1 - \mathrm{i}\mid\mathrm{i}}{2}\,(\partial_{\mu}\Phi^{\dagger}\partial^{\mu}\Phi - m^2\Phi^{\dagger}\Phi)\,. \tag{18}$$

The complex projection for scalar Lagrangian can be avoided by requiring 'double-barred' operators within quaternionic scalar fields‡. However, *this* solution fails for fermionic fields.

Let us consider the standard (complex) Dirac Lagrangian

$$\mathcal{L}^{\mathrm{D}} = \bar{\psi}\gamma^{\mu}\partial_{\mu}\psi\mathrm{i} - m\bar{\psi}\psi\,. \tag{19}$$

As well known, it represents an Hermitian operator, in fact

$$(\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi\mathrm{i})^{\dagger} = -\mathrm{i}(\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi$$

and after integration by parts, gives (using the fact that here the fields are complex)

$$\mathrm{i}\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi \equiv \bar{\psi}\gamma^{\mu}\partial_{\mu}\psi\mathrm{i}\,.$$

In our quaternionic formalism the different position of the imaginary unit suggests modification of the kinetic term in the Dirac Lagrangian. In order to obtain an Hermitian operator we consider

$$\tfrac{1}{2}\,[\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi\mathrm{i} - \mathrm{i}(\partial_{\mu}\bar{\Psi})\gamma^{\mu}\Psi] \tag{20}$$

† In order to maintain the canonical commutation relations for the creation–annihilation operators, we must assume either commutation or anticommutation relations with j:

$$[a(\boldsymbol{k}),\ a^{\dagger}(\boldsymbol{k}')] = \delta_{\boldsymbol{kk}'} \Rightarrow -\mathrm{j}[a(\boldsymbol{k}),\ a^{\dagger}(\boldsymbol{k}')]\mathrm{j} = -\mathrm{j}\delta_{\boldsymbol{kk}'}\mathrm{j}$$
$$\Rightarrow [-\mathrm{j}a(\boldsymbol{k})\mathrm{j},\ -\mathrm{j}a^{\dagger}(\boldsymbol{k}')\mathrm{j}] = \delta_{\boldsymbol{kk}'}$$
$$\Rightarrow -\mathrm{j}a(\boldsymbol{k})\mathrm{j} = \pm a(\boldsymbol{k})\,.$$

‡ For *double-barred* operators, $a \parallel \mathrm{e}^{\pm\mathrm{i}kx}$, the exponential acts from the right on the vacuum state (or any state vector), whereas for *barred* operators, $A \mid b$, b acts from the right on the *fields*. If j commutes with $a_m(\boldsymbol{k})$ the cross terms are automatically killed.

which, after integration by parts, reduces to

$$\frac{1-\mathrm{i}\mid\mathrm{i}}{2}\,(\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi\mathrm{i})\,.$$

So a first modification of the standard Dirac Lagrangian is justified by the simple requirement that $\mathcal{L}^{\mathrm{D}}$ be Hermitian. Nevertheless, this requirement says nothing about the Dirac mass term. It is here that we must invoke the 'quaternionic' variational principle which generalizes the variational rule that says that Ψ and $\bar{\Psi}$ must be varied independently.

If we consider the kinetic term (20) we immediately note that a variation $\delta\Psi$ gives

$$\tfrac{1}{2}\,[\bar{\Psi}\gamma^{\mu}\partial_{\mu}\delta\Psi\mathrm{i}-\mathrm{i}(\partial_{\mu}\bar{\Psi})\gamma^{\mu}\delta\Psi]$$

and since within a quaternionic field theory $[\delta\Psi,\ \mathrm{i}]\neq 0$, we cannot mechanically extract the field equation from the Lagrangian

$$\mathcal{L}^{\mathrm{D}}=(\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi\mathrm{i})_{\mathrm{c}}-m\bar{\Psi}\Psi\,. \tag{21}$$

In order to obtain the desired Dirac equation for Ψ and $\bar{\Psi}$ we are obliged to treat Ψ and $\Psi\mathrm{i}$ (similarly $\bar{\Psi}$ and $\mathrm{i}\bar{\Psi}$) as independent fields and modify the mass term in the Dirac Lagrangian to

$$-\tfrac{1}{2}m\,(\bar{\Psi}\Psi-\mathrm{i}\bar{\Psi}\Psi\mathrm{i})\,.$$

The final result is the need of a 'full' complex projection

$$\mathcal{L}^{\mathrm{D}}_{\mathrm{c}}=\frac{1-\mathrm{i}\mid\mathrm{i}}{2}\,(\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi\mathrm{i}-m\bar{\Psi}\Psi)\,. \tag{22}$$

Any complex projection, under extreme right or left multiplication by a complex number, behaves as follows:

$$(z\mathcal{L}\tilde{z})_{\mathrm{c}}=z\mathcal{L}_{\mathrm{c}}\tilde{z}=z\tilde{z}\mathcal{L}_{\mathrm{c}}\,.$$

Thus if $z\tilde{z}=1$ we have invariance. When the transformation is attributed to the fields Φ and Ψ, this implies that $\tilde{z}=z^{*}$ and hence

$$z\in U(1,\ c)\,.$$

The automatic appearance of this complex unitary group is expected whatever the left-acting (quaternionic) unitary group is [10].

If we analyse the scalar field Lagrangian (18) we immediately note that the 'full' quaternionic gauge group is

$$U(1,\ q)\mid U(1,\ c) \tag{23}$$

which is isomorphic at the Lie algebra level with the (complex) Glashow group $SU(2,\ c)\times U(1,\ c)$. This invariance group and the four quaternionic Klein–Gordon solutions, equal to the Higgs particle number before spontaneous symmetry breaking, suggests that the Salam–Weinberg theory contains an interpretation of the anomalous particles.

4. Anomalous solutions

Since the only fundamental scalar could be the Higgs boson, in order to interpret the anomalous scalars we believe it to be natural to concentrate our attention on the Higgs sector of the electroweak theory. Moreover, as we pointed out in section 3, the number of Higgs particles, before spontaneous symmetry breaking, is four:

$$h^{0}\qquad h^{-}\qquad \bar{h}^{0}\qquad h^{+}$$

and this agrees with the number of quaternionic solutions to the Klein–Gordon equation.

Remembering that the standard (complex) term $\Phi^\dagger\Phi$ splits into $(\Phi^\dagger\Phi)_c$ when Φ becomes a quaternionic scalar field, we write the quaternionic Higgs Lagrangian as follows:

$$\mathcal{L}^{\mathrm{H}} = (\partial_\mu \Phi^\dagger \partial^\mu \Phi)_c - \mu^2 (\Phi^\dagger\Phi)_c - |\lambda|(\Phi^\dagger\Phi)_c^2 \qquad (24)$$

with

$$\Phi \equiv h^0 + \mathrm{j}h^+ \qquad \text{where } h^0 \text{ and } h^+ \text{ are complex scalar fields}.$$

The Lagrangian (24) is obviously invariant under the global group

$$U(1,\ q) \mid U(1,\ c) \qquad (\Phi \ \rightarrow\ \exp(-g\boldsymbol{Q}\cdot\boldsymbol{\alpha}/2)\ \Phi\ \exp(\mathrm{i}\tilde{g}Y_\Phi\beta/2)).$$

If we wish to impose a local gauge invariance we must compensate the derivative terms which appear in the Lagrangian by introducing a quaternionic covariant derivative

$$\partial^\mu\Phi \rightarrow \mathcal{D}^\mu\Phi \equiv \partial^\mu\Phi - \tfrac{1}{2}g\ (\mathrm{i}\Phi W_1^\mu + \mathrm{j}\Phi W_2^\mu + \mathrm{k}\Phi W_3^\mu) + \tfrac{1}{2}\tilde{g}\ Y_\Phi \Phi B^\mu \mathrm{i}. \qquad (25)$$

The Hermitian gauge fields $\boldsymbol{W}^\mu$ and B^μ have the well known gauge transformation properties

$$\boldsymbol{W}^\mu \rightarrow \boldsymbol{W}^\mu - \partial^\mu\boldsymbol{\alpha} - g\boldsymbol{\alpha}\wedge\boldsymbol{W}^\mu$$

$$B^\mu \rightarrow B^\mu - \partial^\mu\beta.$$

Thus the Higgs Lagrangian, invariant under the *local* group $U(1,\ q) \mid U(1,\ c)$, reads

$$\mathcal{L}^{\mathrm{H}} = [(\mathcal{D}_\mu\Phi)^\dagger \mathcal{D}^\mu\Phi]_c - \mu^2(\Phi^\dagger\Phi)_c - |\lambda|(\Phi^\dagger\Phi)_c^2. \qquad (26)$$

Let us consider $\mu^2 < 0$ and examine the consequences of spontaneous symmetry breaking. As in the standard theory, we choose a *real* minimum value for the Higgs potential

$$\Phi_0 = \frac{v}{\sqrt{2}} \qquad \left(v = \sqrt{-\mu^2/|\lambda|}\right)$$

which breaks both $U(1,\ q)$ and $U(1,\ c)$ symmetries, but preserves an invariance under the symmetry generated by a *mixed* generator

$$\begin{array}{ll} U(1,\ q) \text{ generators}: & \begin{array}{l} -\mathrm{i}\Phi_0 = -\mathrm{i}v/\sqrt{2} \\ -\mathrm{j}\Phi_0 = -\mathrm{j}v/\sqrt{2} \\ -\mathrm{k}\Phi_0 = -\mathrm{k}v/\sqrt{2} \end{array} \\ U(1,\ c) \text{ generator } (Y_\Phi = +1): & +\Phi_0\mathrm{i} = +\mathrm{i}v/\sqrt{2} \\ (-\mathrm{i} + 1 \mid \mathrm{i})\ \Phi_0 = 0. & \end{array} \qquad (27)$$

We can identify this residual ‘complex’ group as the electromagnetic gauge group. Rewriting W_1^μ and B^μ as a linear combination of the physical fields A^μ and Z^μ

$$W_1^\mu = \sin\theta_{\mathrm{w}} A^\mu + \cos\theta_{\mathrm{w}} Z^\mu \qquad (28a)$$

$$B^\mu = \cos\theta_{\mathrm{w}} A^\mu - \sin\theta_{\mathrm{w}} Z^\mu \qquad (28b)$$

where θ_{w} is the Weinberg angle, and

$$g\sin\theta_{\mathrm{w}} = \tilde{g}\cos\theta_{\mathrm{w}} = e$$

we can quickly obtain the minimal coupling in terms of the electromagnetic field A^μ:

$$\partial^\mu \rightarrow \partial^\mu - \tfrac{1}{2}eA^\mu(\mathrm{i} - 1 \mid \mathrm{i}) + \cdots \qquad \left(p^\mu \rightarrow p^\mu - \tfrac{1}{2}eA^\mu(1 + \mathrm{i} \mid \mathrm{i}) + \cdots\right). \qquad (29)$$

The electric charge operator

$$\tfrac{1}{2}e(1 + \mathrm{i} \mid \mathrm{i}) \qquad (30)$$

allows us to connect the complex scalar fields h^0 with the neutral Higgs bosons and the (anomalous) *pure* quaternionic scalar fields $\mathrm{j}h^+$ with the charged Higgs bosons.

5. QewT

In this section, we summarize the structure of the quaternionic electroweak Lagrangian. Having introduced in the previous sections the gauge group $U(1,\ q) \mid U(1,\ c)$, which represents the quaternionic counterpart of the 'complex' Glashow group $SU(2,\ c)\times U(1,\ c)$, we wish to construct a fermionic Lagrangian invariant under such a group. If we consider a single-particle (two-component) field Ψ, we have no hope of achieving this. In fact the most general transformation

$$\Psi \to f\Psi g \qquad \text{(where } f, g \text{ are quaternionic numbers)}$$

is right-limited from the complex projection of our Lagrangian and left-limited from the presence of quaternionic (two-dimensional) γ^μ matrices. So we can only write a Lagrangian invariant under a right-acting complex $U(1,\ c)$ group. The situation changes drastically if we use a 'left-real' (four-component) Dirac equation, in this case we could commute the quaternionic phase and restore the invariance under the left-acting quaternionic unitary group†. Indeed in this case Ψ represents two fermions.

Recalling the standard representation for the γ^μ matrices [12]

$$\tilde{\gamma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}$$

we can immediately write the desired 'left-real' Dirac equation

$$(\tilde{\gamma}^\mu \partial_\mu \mid \mathrm{i} - m)\Psi = 0$$

where

$$\tilde{\gamma}^\mu \equiv \gamma^\mu\text{-matrices with i-factors substituted by } 1 \mid \mathrm{i}\,.$$

The massless fermionic Lagrangian in our QewT reads‡

$$\mathcal{L}_{\mathrm{c}}^{\mathrm{F}} = (\bar{\Psi}_l \tilde{\gamma}^\mu \partial_\mu \Psi_l \mathrm{i} + \bar{\Psi}_q \tilde{\gamma}^\mu \partial_\mu \Psi_q \mathrm{i})_{\mathrm{c}} \tag{31}$$

with

$$\Psi_l = e + \mathrm{j}\nu \qquad \Psi_q = d + \mathrm{j}u \qquad \text{(where } e, \nu, d, u \text{ are complex fermionic fields)}\,.$$

This Lagrangian is globally invariant under the following transformations:
left-handed fermions:

$$e_L + \mathrm{j}\nu_L \to \mathrm{e}^{-\frac{1}{2}g\boldsymbol{Q}\cdot\boldsymbol{\alpha}}\ (e_L + \mathrm{j}\nu_L)\ \exp\left(\tfrac{1}{2}\tilde{g}\mathrm{i}Y_l^{(L)}\beta\right)$$

$$d_L + \mathrm{j}u_L \to \mathrm{e}^{-\frac{1}{2}g\boldsymbol{Q}\cdot\boldsymbol{\alpha}}\ (d_L + \mathrm{j}u_L)\ \exp\left(\tfrac{1}{2}\tilde{g}\mathrm{i}Y_q^{(L)}\beta\right)$$

right-handed fermions:

$$e_R \to e_R\ \exp\left(\tfrac{1}{2}\tilde{g}\mathrm{i}Y_e^{(R)}\beta\right)$$

$$d_R + \mathrm{j}u_R \to d_R\ \exp\left(\tfrac{1}{2}\tilde{g}\mathrm{i}Y_d^{(R)}\beta\right) + \mathrm{j}u_R\ \exp\left(\tfrac{1}{2}\tilde{g}\mathrm{i}Y_u^{(R)}\beta\right).$$

† The left (right) action of the quaternionic (complex) unitary group has nothing to do with the helicity indices L and R of the Salam–Weinberg theory

‡ Note that once more the complex projection kills the *undesired* cross terms. Recall that complex projection is also justified by the requirements of Hermiticity.

Requiring that the electric charge operator be represented (in units of e) by

$$\mathcal{Q} = \frac{Y + \mathrm{i} \mid \mathrm{i}}{2}$$

leads to the *weak-hypercharge* assignments

$$Y_l^{(L)} = -1 \qquad Y_q^{(L)} = +\tfrac{1}{3} \qquad Y_e^{(R)} = -2 \qquad Y_d^{(R)} = -\tfrac{2}{3} \qquad Y_u^{(R)} = +\tfrac{4}{3}\,.$$

In order to construct a local-invariant theory, we must introduce the gauge bosons

$$\boldsymbol{W}^\mu \qquad \text{for } U(1,\ q)_L$$

$$B^\mu \qquad \text{for } U(1,\ c)_Y$$

by the covariant derivatives

$$\mathcal{D}^\mu(e_L + \mathrm{j}\nu_L) \equiv \left[\partial^\mu - \tfrac{1}{2}g\ (\mathrm{i} \mid W_1^\mu + \mathrm{j} \mid W_2^\mu + \mathrm{k} \mid W_3^\mu) - \tfrac{1}{2}\tilde{g} \mid B^\mu \mathrm{i}\right](e_L + \mathrm{j}\nu_L)$$

$$\mathcal{D}^\mu(d_L + \mathrm{j}u_L) \equiv \left[\partial^\mu - \tfrac{1}{2}g\ (\mathrm{i} \mid W_1^\mu + \mathrm{j} \mid W_2^\mu + \mathrm{k} \mid W_3^\mu) + \tfrac{1}{6}\tilde{g} \mid B^\mu \mathrm{i}\right](d_L + \mathrm{j}u_L)$$

$$\mathcal{D}^\mu u_R \equiv (\partial^\mu + \tfrac{2}{3}\tilde{g} \mid B^\mu \mathrm{i})u_R$$

$$\mathcal{D}^\mu d_R \equiv (\partial^\mu - \tfrac{1}{3}\tilde{g} \mid B^\mu \mathrm{i})d_R$$

$$\mathcal{D}^\mu e_R \equiv (\partial^\mu - \tilde{g} \mid B^\mu \mathrm{i})e_R$$

the substitution $\partial^\mu \to \mathcal{D}^\mu$ in (31) making our Lagrangian locally invariant.

Working with quaternions we can concisely express the four gauge fields $\boldsymbol{W}^\mu$ and B^μ by only one quaternionic gauge field. Actually, in analogy with the quaternionic Higgs scalar $\Phi = h^0 + \mathrm{j}h^+$ (where h^0, h^+ are complex scalar fields), we introduce the following quaternionic gauge field:

$$W_\mu = W_\mu^0 + \mathrm{j}W_\mu^+ \qquad \left(W_\mu^0 = (B_\mu^0 + \mathrm{i}W_\mu^1)/\sqrt{2}\,,\ W_\mu^+ = (W_\mu^2 - \mathrm{i}W_\mu^3)/\sqrt{2}\right) \tag{32}$$

and so the gauge-kinetic term is represented by

$$\mathcal{L}_\mathrm{c}^G = -\tfrac{1}{2}(F_{\mu\nu}^\dagger F^{\mu\nu})_\mathrm{c} \tag{33}$$

with

$$F^{\mu\nu} = \partial^\mu W^\nu - \partial^\nu W^\mu - g\boldsymbol{Q} \cdot \boldsymbol{W}^\mu \wedge \boldsymbol{W}^\nu\,.$$

Now we can add an interaction term which involves Yukawa couplings of the scalars to the fermions:

$$\mathcal{L}_\mathrm{c}^\mathrm{Y} = -\{G_\mathrm{e}\bar{e}_R[\Phi^\dagger(e_L + \mathrm{j}\nu_L)]_\mathrm{c} + G_\mathrm{d}\bar{d}_R[\Phi^\dagger(d_L + \mathrm{j}u_L)]_\mathrm{c} + G_\mathrm{u}\bar{u}_R[\tilde{\Phi}^\dagger(d_L + \mathrm{j}u_L)]_\mathrm{c}\} + \mathrm{HC}$$

$$(\tilde{\Phi} \equiv \Phi\mathrm{j})\,. \tag{34}$$

Equation (34) transforms under local $U(1,\ q)_L \mid U(1,\ c)_Y$ as

$$\bar{e}_R[\Phi^\dagger(e_L + \mathrm{j}\nu_L)]_\mathrm{c}$$

$$\to\ \exp\left(-\tfrac{1}{2}\tilde{g}\mathrm{i}Y_e^{(R)}\beta\right)\bar{e}_R\ \exp\left(-\tfrac{1}{2}\tilde{g}\mathrm{i}Y_\Phi\beta\right)[\Phi^\dagger(e_L + \mathrm{j}\nu_L)]_\mathrm{c}\exp\left(\tfrac{1}{2}\tilde{g}\mathrm{i}Y_l^{(L)}\beta\right)$$

$$\bar{d}_R[\Phi^\dagger(d_L + \mathrm{j}u_L)]_\mathrm{c}$$

$$\to\ \exp\left(-\tfrac{1}{2}\tilde{g}\mathrm{i}Y_d^{(R)}\beta\right)\bar{d}_R\ \exp\left(-\tfrac{1}{2}\tilde{g}\mathrm{i}Y_\Phi\beta\right)[\Phi^\dagger(d_L + \mathrm{j}u_L)]_\mathrm{c}\exp\left(\tfrac{1}{2}\tilde{g}\mathrm{i}Y_q^{(L)}\beta\right)$$

$\bar{u}_R[\tilde{\Phi}^\dagger(d_L + \mathrm{j}u_L)]_\mathrm{c}$

$$\to \exp\left(-\tfrac{1}{2}\tilde{g}\mathrm{i}Y_u^{(R)}\beta\right)\bar{u}_R\ \exp\left(+\tfrac{1}{2}\tilde{g}\mathrm{i}Y_\Phi\beta\right)[\tilde{\Phi}^\dagger(d_L + \mathrm{j}u_L)]_\mathrm{c}\exp\left(\tfrac{1}{2}\tilde{g}\mathrm{i}Y_q^{(L)}\beta\right).$$

Because the Ψ_R fields are complex all the *complex* phase factors can be brought together. Thus invariance follows if

$$Y_e^{(R)} + Y_\Phi - Y_l^{(L)} = 0$$

etc. If we expand the Lagrangian about the minimum of the Higgs potential by writing

$$\Phi = \exp\left(-\frac{\boldsymbol{Q}\cdot\boldsymbol{u}}{2v}\right)(v + H^0)/\sqrt{2} \qquad \text{(where } H^0 \text{ is a Hermitian scalar field)} \tag{35}$$

and transforming at once to a U-gauge:

$$\Phi \to \Phi' = \exp\left(-\frac{\boldsymbol{Q}\cdot\boldsymbol{u}}{2v}\right)\phi = (v + H^0)/\sqrt{2}$$

$$\Psi_{l,\,q} \to \Psi'_{l,\,q} = \exp\left(-\frac{\boldsymbol{Q}\cdot\boldsymbol{u}}{2v}\right)\Psi_{l,\,q}$$

$$\boldsymbol{Q}\cdot\boldsymbol{W}_\mu \to \boldsymbol{Q}\cdot\boldsymbol{W}'_\mu$$

we can re-express our Lagrangian in terms of the U-gauge fields. The Yukawa term becomes

$$\mathcal{L}_\mathrm{c}^\mathrm{Y} = -\frac{1}{\sqrt{2}}\{G_\mathrm{e}\bar{e}_R[(v + H^0)(e_L + \mathrm{j}\nu_L)]_\mathrm{c} + G_\mathrm{d}\bar{d}_R[(v + H^0)(d_L + \mathrm{j}u_L)]_\mathrm{c}$$
$$+ G_\mathrm{u}\bar{u}_R[-\mathrm{j}(v + H^0)(d_L + \mathrm{j}u_L)]_\mathrm{c}\} + \mathrm{HC}$$
$$= -\frac{v}{\sqrt{2}}(G_\mathrm{e}\bar{e}e + G_\mathrm{d}\bar{d}d + G_\mathrm{u}\bar{u}u) + \text{ coupling between } H^0 \text{ and fermions}$$

so the electron and the quarks d, u acquire a mass

$$m_{\mathrm{e,\,d,\,u}} = G_{\mathrm{e,\,d,\,u}}\frac{v}{\sqrt{2}}.$$

From the scalar term in the Lagrangian we recognize a physical Higgs boson with mass

$$M_H = \sqrt{-2\mu^2}$$

and the intermediate boson masses

$$M_{W^\pm} = g\frac{v}{2} \qquad M_Z = M_W\sqrt{1 + \frac{\tilde{g}^2}{g^2}}.$$

The interactions between the gauge bosons and fermions may be read off from

$$\mathcal{L}_\mathrm{c}^\mathrm{I} = -\tfrac{1}{2}g\,[(\bar{e}_L - \bar{\nu}_L\mathrm{j})\tilde{\gamma}_\mu(\mathrm{i}\mid W_1^\mu\mathrm{i} + \mathrm{j}\mid W_2^\mu\mathrm{i} + \mathrm{k}\mid W_3^\mu\mathrm{i})(e_L + \mathrm{j}\nu_L)]_\mathrm{c}$$
$$+\tfrac{1}{2}\tilde{g}\,[(\bar{e}_L - \bar{\nu}_L\mathrm{j})\tilde{\gamma}_\mu(e_L + \mathrm{j}\nu_L)B_\mu]_\mathrm{c}$$
$$-\tfrac{1}{2}g\,[(\bar{d}_L - \bar{u}_L\mathrm{j})\tilde{\gamma}_\mu(\mathrm{i}\mid W_1^\mu\mathrm{i} + \mathrm{j}\mid W_2^\mu\mathrm{i} + \mathrm{k}\mid W_3^\mu\mathrm{i})(d_L + \mathrm{j}u_L)]_\mathrm{c}$$
$$-\tfrac{1}{6}\tilde{g}\,[(\bar{d}_L - \bar{u}_L\mathrm{j})\tilde{\gamma}_\mu(d_L + \mathrm{j}u_L)B_\mu]_\mathrm{c}$$
$$-\tfrac{2}{3}\tilde{g}\,\bar{u}_R\tilde{\gamma}^\mu u_R B_\mu + \tfrac{1}{3}\tilde{g}\,\bar{d}_R\tilde{\gamma}^\mu d_R B_\mu + \tilde{g}\,\bar{e}_R\tilde{\gamma}^\mu e_R B_\mu\,.$$

For the charged gauge bosons we find

$$\mathcal{L}_{\mathrm{c}}^{W-l,q} = -\tfrac{1}{2} g\ [(\bar{e}_L - \bar{\nu}_L \mathrm{j})\tilde{\gamma}_\mu (\mathrm{j} \mid W_2^\mu \mathrm{i} + \mathrm{k} \mid W_3^\mu \mathrm{i})(e_L + \mathrm{j}\nu_L)]_{\mathrm{c}}$$

$$- \tfrac{1}{2} g\ [(\bar{d}_L - \bar{u}_L \mathrm{j})\tilde{\gamma}_\mu (\mathrm{j} \mid W_2^\mu \mathrm{i} + \mathrm{k} \mid W_3^\mu \mathrm{i})(d_L + \mathrm{j}u_L)]_{\mathrm{c}}$$

$$= -\frac{g}{\sqrt{2}}\ (\bar{\nu}_L \tilde{\gamma}^\mu e_L W_\mu^+ \mathrm{i} + \bar{u}_L \tilde{\gamma}^\mu d_L W_\mu^+ \mathrm{i}) + \mathrm{HC}\,.$$

Similarly, rewriting W_1^μ and B^μ as a linear combination of the physical fields A^μ and Z^μ (see equations (28*a*), (28*b*)), we can reproduce the standard results for the neutral gauge boson couplings to fermions.

6. Conclusions

We have formulated in the previous sections a quaternionic version of the electroweak theory which reproduces the standard results. Notwithstanding the quaternionic nature of the fields, our Lagrangians are complex projected and this represents a *desirable* feature of our formalism. We have identified the quaternionic counterpart of the complex Glashow group $SU(2,\ c) \times U(1,\ c)$ with $U(1,\ q) \mid U(1,\ c)$ and argued that the right-acting $U(1,\ c)$ group (at first sight unnatural in the context of quaternionic groups) is a direct consequence of the complex projection of our Lagrangians. Such a complex projection opens the door to all possible right-acting complex groups, for example if we consider the following fermionic fields:

$$\Psi_q = \begin{pmatrix} d_r + \mathrm{j}u_r \\ d_g + \mathrm{j}u_g \\ d_b + \mathrm{j}u_b \end{pmatrix} \qquad \Psi_l = e + \mathrm{j}\nu_e \qquad ((r,\ g,\ b) \leftrightarrow (\text{red, green, blue}))$$

we can quickly write a Lagrangian density

$$\mathcal{L}_{\mathrm{c}}^{\mathrm{F}} = (\bar{\Psi}_l \tilde{\gamma}^\mu \partial_\mu \Psi_l \mathrm{i} + \bar{\Psi}_q \tilde{\gamma}^\mu \partial_\mu \Psi_q \mathrm{i})_{\mathrm{c}}$$

invariant under the global gauge group

$$SU(3,\ \tilde{c}) \times U(1,\ q) \times U(1,\ \tilde{c}) \qquad (\tilde{c} = a + b \mid \mathrm{i} \text{ with } a, b \in \mathcal{R})\,.$$

So the complex projection of our Lagrangian, required in order to obtain the proper field equations, represents a fundamental ingredient in reformulating the quaternionic electroweak theory and the standard model. The complex projection of $\mathcal{L}$ allows us to confront our quaternionic Lagrangian densities with those of the standard theory by means of the *rules of translation* [13], obtained for complex scalar products. We have not, however, been able to *derive* a complex geometry from the assumption of the complex projection of $\mathcal{L}$ (such a connection is not yet clear to us and is currently under investigation).

It is also important to recall that the possibility of rewriting standard particle physics theories in quaternionic form is a non-trivial objective; in fact the non-commutative nature of quaternions alters the conventional approach (as in tensor products, variational calculus, bosonic equations).

We observe that our long-standing perplexity as to the physical significance of the anomalous solutions is overcome. We had already observed that if the anomalous photon existed the field had to be non-Hermitian and hence treated in analogy with the weak charged currents [9]. Now our quaternionic version of the Salam–Weinberg model shows that the anomalous photon can be identified with one of the charged W particles, and not with Z^0 as in our original hypothesis [9]. Of course, without spontaneous symmetry breaking this identification would appear embarrassing, since we would have expected the anomalous photon to have zero mass. In a similar manner the anomalous solutions of the Klein–Gordon equation for the Higgs fields have been identified in this work with the charged scalar fields before spontaneous symmetry breaking.

Now let us discuss the potential generalizations which the use of quaternions suggests. We have noted that $U(1,\ q)$ is the most natural quaternionic invariance group for particle physics and this coincides nicely with the practical importance of $SU(2,\ c)$ (spin, isospin, etc). This type of argument (based on groups) for an underlying quaternionic number system is not as ephemeral as the above example seems to be. For example, while we have already noted that no complex group can be excluded *a priori*, the existence of quite simple invariance quaternionic groups such as $U(n,\ q)$, isomorphic to the unitary symplectic complex groups $USp(2n,\ c)$ [14], would surely not go unnoticed. In short, certain unusual groups could become 'natural' with quaternions. A possible application of these considerations is to grand unification theories.

The relevance of unitary quaternionic groups and of the, as yet unexplored, mathematics of groups consisting of generalized quaternionic elements

$$q_{\mathrm{c}} = p + q \mid \mathrm{i} \qquad \text{and} \qquad q_r = p + q \mid \mathrm{i} + r \mid \mathrm{j} + s \mid \mathrm{k} \qquad (p, q, r, s \in \mathcal{H})$$

is still under investigation.

This work presents an explicit quaternionic translation of the standard (complex) theory based on the even group $SU(2,\ c) \times U(1,\ c)$. We wish to underline that while all even-dimensional complex group can be translated into our quaternionic formalism with half the dimensions [13] (an undoubted practical advantage), see $U(1,\ q) \mid U(1,\ c)$, work is still necessary for the translation of odd-dimensional complex groups.

A different approach with quaternions to the Salam–Weinberg model is taken by Morita. He uses complexified quaternions [15] in which an additional complex unit $\mathcal{I}$ which *commutes* with the quaternions appears. It is this unit which is identified with the imaginary unit of complex quantum mechanics (e.g. in the plane-wave exponential) and applies this formalism to the generalization of the Glashow group to $SU(2)_L \times SU(2)_R \times U(1)$, identified with the complex-quaternion group [16] $U(1,\ q)_L \times U(1,\ q)_R \times U(1,\ ;\mathcal{C})$, where $\mathcal{C} = (1,\ \mathcal{I})$.

Since quaternions appear in many related works (even if not specificatly applied to the Salam–Weinberg model) we cannot mention them all. We wish, however, to recall the works of Nash and Joshi and in particular that on composite systems [17] in which there is a fundamental difference between configuration space and momentum space representations. This must be considered an alternative to the complex geometry advocated by Horwitz and Biedenharn [6] and that represented in purely quaternionic language by the present authors.

Another related field originates from the study of quaternion Kähler manifolds [18] and their use for Yang–Mills fields in general [19]. This approach appears also in certain string models [20].

Having shown in this work the possibility of rewriting standard theory in quaternionic form, the question that comes to mind is: do there exist interesting quaternionic equations corresponding to new physics? The analogy is always with the Schrödinger equation. This equation can always be written as a *pair* of real equations but the existence of complex numbers, for example in the wavefunction, would be evidenced by the 'rule' for expressing probability amplitudes in terms of the two real wavefunctions. In this context we recall the work of Adler [21] who assumes quaternionic probability amplitudes formulating a quite revolutionary quantum theory.

We conclude by hoping that the work presented in this paper will be considered an encouragement for the use of a quaternionic quantum mechanics with complex geometry.

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