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A ONE-COMPONENT DIRAC EQUATION

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We develop a relativistic free wave equation on the complexified quaternions, linear in the derivatives. Even if the wave functions are only one-component, we show that four independent solutions, corresponding to those of the Dirac equation, exist. A partial set of translations between complex and complexified quaternionic quantum mechanics may be defined.

1. Introduction

In physics, particularly quantum mechanics, we are accustomed to distinguishing between "states" and "operators." Even when the operators are represented by numerical matrices, the squared form of operators distinguishes them from the column structure of the spinor states. Only for one-component fields and operators is there potential confusion.

In extending quantum mechanics defined over the complex field to quaternions[1–4] or even complexified quaternions,[5] it has almost always been assumed that matrix operators contain elements which are "numbers," indistinguishable from those of the state vectors. This is an unjustified limitation.

As the title of this paper indicates, we shall display a one-component Dirac equation which has only the standard solutions and thus avoids the doubling of solutions found by other authors who used complexified quaternions. This is achieved by the introduction of *$\mathcal{H}$-real linear complexified quaternions* within operators.

After defining quaternions, complexified quaternions and their generalizations we will give the simplest example of their use (namely the one-component Dirac equation) and introduce another fundamental ingredient, the scalar product, in order to formulate quantum mechanics.

In 1843, Hamilton,[6] after more than a decade of attempts to generalize complex numbers (seen as a representation of rotations in two dimensions) in order to describe rotations in three dimensions, discovered quaternions. Instead of an entity which he expected to be characterized by three real numbers, the Irish physicist

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found that four real numbers were required. He wrote a quaternion such as[a]

$$q = a + \mathcal{I}b + \mathcal{J}c + \mathcal{K}d\,, \qquad a,\ b,\ c,\quad d \in \mathcal{R}\,,$$

with operations of multiplication defined according to the following rules for the imaginary units:

$$\begin{aligned}\mathcal{I}^2 = \mathcal{J}^2 = \mathcal{K}^2 &= -1\,,\\ \mathcal{J}\mathcal{K} = -\mathcal{K}\mathcal{J} &= \mathcal{I}\,,\\ \mathcal{K}\mathcal{I} = -\mathcal{I}\mathcal{K} &= \mathcal{J}\,,\\ \mathcal{I}\mathcal{J} = -\mathcal{J}\mathcal{I} &= \mathcal{K}\,.\end{aligned}$$

We can immediately extend the previous algebra by introducing a quantity i whose square is -1 and commutes with $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$:

$$q_{\mathcal{C}} = \alpha + \mathcal{I}\beta + \mathcal{J}\gamma + \mathcal{K}\delta\,, \qquad \alpha,\ \beta,\ \gamma,\ \delta \in \mathcal{C}(1,i)\,. \tag{1}$$

The complexified quaternions (1) were introduced in physics to rederive, more elegantly, all the expressions of special relativity. A review by Synge[7] covers developments up to the 1960's; modern presentations have been given by Sachs,[8] Gough[9] and Abonyi.[10] Quantum mechanics on the complexified quaternions has been formulated by Morita;[5] in his papers we can find an interesting quaternionic version of the electroweak theory.

Remembering the noncommutativity of the quaternionic multiplication, we must specify whether the complexified quaternionic Hilbert space $V_{\mathcal{H}_{\mathcal{C}}}$ ($\mathcal{H}_{\mathcal{C}}$ indicates our complexified quaternionic field) is to be formed by right or by left multiplication of vectors by scalars. Besides, we must specify if our scalars are quaternionic, complex or real numbers. We will follow the usual choice[5] to work with a linear vector space under multiplication by complex $[\mathcal{C}(1,i)]$ scalars, and so the two different conventions, represented by right or by left multiplication, give isomorphic versions of the theory.

In the next section we briefly recall the complexified quaternionic algebra and introduce an appropriate scalar product (or geometry, as called by Rembieliński[11]). In the third section we explicitly give a one-component formulation of the Dirac equation; even if the wave function is only one-component, we show that four i-complex independent solutions appear. In the following section we identify a set of rules which permit translations between complex and complexified quaternionic quantum mechanics. Our conclusions are drawn in the final section.

2. Complexified Quaternionic Algebra and $\mathcal{H}$-Real Geometry

Working with complexified quaternionic numbers we have different opportunities to define conjugation operations on the imaginary units:

[a]In this paper we will use the symbols $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ instead of the common i, j, k, since we believe it is useful to distinguish the *quaternionic* imaginary units from the *complex* imaginary unit i which appears in standard quantum mechanics.

Number of conjugations	$\mathcal{I}$	$\mathcal{J}$	$\mathcal{K}$	i
	$-$	$+$	$+$	
1	$+$	$-$	$+$	$+$
	$+$	$+$	$-$	
	$+$	$-$	$-$	
2	$-$	$+$	$-$	$+$
	$-$	$-$	$+$	
3	$-$	$-$	$-$	$+$

Nevertheless the conjugations expressed in the previous chart are not independent. In fact we can prove that the conjugation operations which change only one imaginary unit are connected between themselves and with the conjugation of all the three imaginary units through similarity transformations, as in

$$-\mathcal{I}\beta + \mathcal{J}\gamma + \mathcal{K}\delta = -\mathcal{K}(\mathcal{I}\beta - \mathcal{J}\gamma + \mathcal{K}\delta)\mathcal{K}\,,$$

$$-\mathcal{I}\beta + \mathcal{J}\gamma + \mathcal{K}\delta = -\mathcal{I}(-\mathcal{I}\beta - \mathcal{J}\gamma - \mathcal{K}\delta)\mathcal{I}\,.$$

An analogous observation can be formulated for the conjugation of two imaginary units:

$$-\mathcal{I}\beta - \mathcal{J}\gamma + \mathcal{K}\delta = -\mathcal{I}(-\mathcal{I}\beta + \mathcal{J}\gamma - \mathcal{K}\delta)\mathcal{I}\,,$$

$$-\mathcal{I}\beta - \mathcal{J}\gamma + \mathcal{K}\delta = -\mathcal{K}(\mathcal{I}\beta + \mathcal{J}\gamma + \mathcal{K}\delta)\mathcal{K}\,.$$

If we now conjugate the imaginary unit i, three possible conjugations appear:

Symbol	$\mathcal{I}$	$\mathcal{J}$	$\mathcal{K}$	i
$q_c{}^\star$	$-$	$-$	$-$	$+$
$q_c{}^*$	$+$	$+$	$+$	$-$
$q_c{}^+$	$-$	$-$	$-$	$-$

It is straightforward to prove that

$$(q_c p_c)^\star = p_c{}^\star q_c{}^\star\,,$$

$$(q_c p_c)^* = q_c{}^* p_c{}^*\,,$$

and so we immediately have

$$(q_c p_c)^+ = (q_c p_c)^{*\star} = (q_c{}^* p_c{}^*)^\star = p_c{}^{*\star} q_c{}^{*\star} = p_c{}^+ q_c{}^+\,.$$

In quantum mechanics, probability amplitudes, rather than probabilities, superimpose, so we must determine what kinds of number system can be used for the probability amplitudes $\mathcal{A}$. We need a real modulus function $N(\mathcal{A})$ such that

$$\text{Probability} = [N(\mathcal{A})]^2 .$$

The first four assumptions on the modulus function are basically technical in nature:

$$\begin{gathered} N(0) = 0 , \\ N(\mathcal{A}) > 0 \quad \text{if} \quad \mathcal{A} \neq 0 , \\ N(r\mathcal{A}) = |r| N(\mathcal{A}) , \quad r \text{ real} , \\ N(\mathcal{A}_1 + \mathcal{A}_2) \leq N(\mathcal{A}_1) + N(\mathcal{A}_2) . \end{gathered}$$

A final assumption about $N(\mathcal{A})$ is physically motived by imposing the *correspondence principle* in the following form: We require that in the absence of quantum interferences effects, probability amplitude superimposition should reduce to probability superimposition. So we have an additional condition on $N(\mathcal{A})$:

$$N(\mathcal{A}_1 \mathcal{A}_2) = N(\mathcal{A}_1) N(\mathcal{A}_2) .$$

A remarkable theorem of Albert[12] shows that the only algebras over the reals, admitting a modulus functions with the previous properties, are the reals $\mathcal{R}$, the complex $\mathcal{C}$, the real quaternions $\mathcal{H}_{\mathcal{R}}$ and the octonions $\mathcal{O}$.

The previous properties of the modulus function seem to constrain us to work with *division algebras* (which are finite-dimensional algebras for which $a \neq 0$, $b \neq 0$ imply that $ab \neq 0$); in fact

$$\mathcal{A}_1 \neq 0 , \qquad \mathcal{A}_2 \neq 0$$

implies that

$$N(\mathcal{A}_1 \mathcal{A}_2) = N(\mathcal{A}_1) N(\mathcal{A}_2) \neq 0 ,$$

which gives

$$\mathcal{A}_1 \mathcal{A}_2 \neq 0 .$$

A simple example of a nondivision algebra is provided by the algebra of complexified quaternions since

$$(1 + i\mathcal{I})(1 - i\mathcal{I}) = 0$$

guarantees that there are nonzero divisors of zero.

So if the probability amplitudes are assumed to be complexified quaternions we cannot give a satisfactory probability interpretation. Nevertheless we know that probability amplitudes are connected with inner products, and so we can overcome the above difficulty by defining an appropriate scalar product. In order to obtain the right properties of the modulus functions, a possibility is represented by choosing an $\mathcal{H}$-*real* (real with respect to $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$) projection of the complexified quaternionic scalar product:[b]

[b]The barred operators $\mathcal{O}|q$ act on quaternionic objects ϕ as follows:

$$(\mathcal{O}|q)\phi \equiv \mathcal{O}\phi q .$$

$$\mathcal{A} = \langle\psi|\phi\rangle_{\mathcal{C}(1,i)} = \frac{1 - \mathcal{I}|\mathcal{I} - \mathcal{J}|\mathcal{J} - \mathcal{K}|\mathcal{K}}{4}\langle\psi|\phi\rangle\,. \tag{2}$$

We observe that the dimensionality of a complete set of states for the complex inner product $\langle\psi|\phi\rangle_{\mathcal{C}(1,i)}$ is *four times* that for the complexified quaternionic inner product $\langle\psi|\phi\rangle$. Specifically, if $|\eta_l\rangle$ are a complete set of intermediate states for the complexified quaternionic inner product, so that

$$\langle\psi|\phi\rangle = \sum_l \langle\psi|\eta_l\rangle\langle\eta_l|\phi\rangle\,,$$

$|\eta_l\rangle$, $|\eta_l\mathcal{I}\rangle$, $|\eta_l\mathcal{J}\rangle$, $|\eta_l\mathcal{K}\rangle$ form a complete set of states for the complex inner product:

$$\begin{aligned}|\phi\rangle &= \sum_l (|\eta_l\rangle\langle\eta_l|\phi\rangle_{\mathcal{C}(1,i)} + |\eta_l\mathcal{I}\rangle\langle\eta_l\mathcal{I}|\phi\rangle_{\mathcal{C}(1,i)} \\ &\quad + |\eta_l\mathcal{J}\rangle\langle\eta_l\mathcal{J}|\phi\rangle_{\mathcal{C}(1,i)} + |\eta_l\mathcal{K}\rangle\langle\eta_l\mathcal{K}|\phi\rangle_{\mathcal{C}(1,i)}) \\ &= \sum_m |\chi_m)(\chi_m|\phi\rangle_{\mathcal{C}(1,i)}\,,\end{aligned}$$

where χ_m represents *complex* orthogonal states. Thus the completeness relation can be written as[c]

$$\overrightarrow{\mathbf{1}} = \sum_m |\chi_m)(\chi_m|\,,$$

$$\overleftarrow{\mathbf{1}} = \sum_m |\chi_m)(\chi_m|\,,$$

and so in our formalism we generalize the Dirac notation by the definitions

$$(\chi_m|\phi\rangle = \langle\chi_m|\phi\rangle_{\mathcal{C}(1,i)}\,,$$

$$\langle\phi|\chi_m) = \langle\phi|\chi_m\rangle_{\mathcal{C}(1,i)}\,.$$

The probability amplitude $\mathcal{A}$ (now i-complex-valued) satisfies the required properties, including the superposition property. In this paper we will treat a complexified quaternionic quantum mechanics with $\mathcal{H}$-real (i-complex) geometry.

In the expression (2) we recognize a particular *$\mathcal{H}$-real linear complexified quaternion.* These objects represent the most general transformation on complexified quaternions. Explicitly, such transformations are expressed by[d]

$$\mathcal{O} = q_{\mathcal{C}} + p_{\mathcal{C}}|\mathcal{I} + r_{\mathcal{C}}|\mathcal{J} + s_{\mathcal{C}}|\mathcal{K}\,, \qquad q_{\mathcal{C}},\ p_{\mathcal{C}},\ r_{\mathcal{C}},\quad s_{\mathcal{C}} \in \mathcal{H}_{\mathcal{C}}\,. \tag{3}$$

[c] For further details on the completeness relation, one can consult an interesting work of Horwitz and Biedenharn; see Ref. 13, p. 455.

[d] Note that a sum of terms such as in (3) cannot be written in the factorized form $q_{\mathcal{C}}|p_{\mathcal{C}}$. In fact operators like $q_{\mathcal{C}}|p_{\mathcal{C}}$ are characterized by only 7 complex parameters, whereas the transformation (3) contains 16 complex parameters and represents the most general transformation that we can make on complexified quaternions.

An operator like (3) satisfies the relation

$$\mathcal{O}(\psi\alpha) = (\mathcal{O}\psi)\alpha\,,$$

for an arbitrary $\alpha \in \mathcal{C}(1,i)$; from here we get the term "$\mathcal{H}$-real" (real with respect to $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$) or "i-complex" linear operator.

Thanks to *$\mathcal{H}$-real linear complexified quaternions* we will be able to write a one-component Dirac equation (characterized by four orthogonal solutions) and will overcome previous problems.

3. One-Component Dirac Equation

Various formulations of the Dirac relativistic equation on the complexified quaternionic field have been considered since the 1930's. A pioneer in this field was certainly Conway;[14] more recent presentations can be found in the papers of Edmonds[15] and Gough.[9,16] When written in this manner, a doubling of solutions from four to eight occurs.

We briefly recall the "standard" complexified quaternionic Dirac equation, as formulated by Edmonds and Gough. If we represent the energy–momentum by the complexified quaternion

$$\mathcal{P} = E + i(\mathcal{I}p_x + \mathcal{J}p_y + \mathcal{K}p_z)\,, \tag{4}$$

a relativistic free wave equation is quickly obtained by the substitution

$$\mathcal{P} \to i\mathcal{D} = i\partial_t + \mathcal{I}\partial_x + \mathcal{J}\partial_y + \mathcal{K}\partial_z \tag{5}$$

in

$$\mathcal{P}\psi_a = m\psi_b\,, \qquad \mathcal{P}^\star\psi_b = m\psi_a\,. \tag{6}$$

The spinors ψ_a and ψ_b satisfy the Klein–Gordon equation; in fact we have

$$\begin{aligned}\mathcal{P}^\star\mathcal{P}\psi_a &= m\mathcal{P}^\star\psi_b = m^2\psi_a\,,\\ \mathcal{P}\mathcal{P}^\star\psi_b &= m\mathcal{P}\psi_a = m^2\psi_b\,,\\ \mathcal{P}\mathcal{P}^\star &= \mathcal{P}^\star\mathcal{P} = -{\partial_t}^2 + \vec{\partial}^2\,.\end{aligned}$$

The covariance is obtained if, under Lorentz transformations

$$\mathcal{P}' = \Lambda\mathcal{P}\Lambda^+\,, \qquad \Lambda^\star\Lambda = 1\,, \qquad \Lambda \in \mathcal{H}_C\,,$$

the spinors ψ_a and ψ_b transform as follows:

$$\psi_a{}' = \Lambda^*\psi_a\,, \qquad \psi_b{}' = \Lambda\psi_b\,. \tag{7}$$

Equations (6) can be rewritten in matrix form:

$$\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} i\partial_t + \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\vec{Q}\cdot\vec{\partial}\right]\begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix} = m\begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix}\,, \qquad \vec{Q} \equiv (\mathcal{I},\mathcal{J},\mathcal{K})\,. \tag{8}$$

We recognize, in the complexified quaternionic Dirac equation, eight orthogonal solutions, whereas the standard Dirac equation involves only four. For $\vec{p} = 0$, we find that

$$E = +m\colon \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\mathcal{I},\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\mathcal{J},\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\mathcal{K};$$

$$E = -m\colon \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix},\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\mathcal{I},\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\mathcal{J},\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\mathcal{K}.$$

Remember that with $\mathcal{H}$-real geometry, ψ, $\psi\mathcal{I}$, $\psi\mathcal{J}$, $\psi\mathcal{K}$ represent orthogonal states. The possible physical significance of these additional solutions has been a matter of speculation.[17]

We will show (see Sec. 4) that this doubling of solutions is strictly connected with the use of reducible matrices

$$\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}, \qquad \begin{pmatrix}0 & -\vec{Q}\\ \vec{Q} & 0\end{pmatrix}, \tag{9}$$

and so there is no "new physics" in the complexified quaternionic Dirac equation. Finally, we note that when one works with complexified quaternions, matrix operators like (9) represent an unjustified limitation.

We now derive, following the standard Dirac approach, a *one-component* equation with only *four* solutions.

In order to obtain a positive-definite probability density ρ, we require an equation linear in ∂_t; then, for relativistic covariance, the equation must also be linear in $\vec{\partial}$. The simplest equation that we can write down is

$$i\partial_t\psi = \vec{Q}\cdot\vec{\mathcal{P}}\psi\mathcal{I} + mi\psi\mathcal{J}, \tag{10}$$

with

$$\vec{\mathcal{P}} \equiv -i\vec{\partial}.$$

Our Hamiltonian, given by

$$\vec{Q}\cdot\vec{\mathcal{P}}|\mathcal{I} + mi|\mathcal{J},$$

represents a particular *$\mathcal{H}$-real linear complexified quaternion.* It is straightforward to prove that ψ satisfies the Klein–Gordon equation; in fact the operators

$$\vec{\alpha} \equiv \vec{Q}|\mathcal{I}, \qquad \beta \equiv i|\mathcal{J} \tag{11}$$

verify the usual relations[e]

$$\{\vec{\alpha},\beta\} = \{\alpha_x,\alpha_y\} = \{\alpha_x,\alpha_z\} = \{\alpha_y,\alpha_z\} = 0,$$
$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1.$$

[e]Note that

$$(q_{\mathcal{C}}|q)(p_{\mathcal{C}}|p) = q_{\mathcal{C}}p_{\mathcal{C}}|pq, \qquad q_{\mathcal{C}},\ p_{\mathcal{C}} \in \mathcal{H}_{\mathcal{C}},\ q,\ p \in \mathcal{H}_{\mathcal{R}}.$$

In order to find a positive-definite probability density, we must consider an appropriate conjugation. Noting that

$$\begin{aligned}
\psi^* \psi &= \psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2 \,, \\
\psi^* \psi &= |\psi_1|^2 - |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2 + (\mathcal{I}, \mathcal{J}, \mathcal{K}) \text{ terms} \,, \\
\psi^+ \psi &= |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 + (\mathcal{I}, \mathcal{J}, \mathcal{K}) \text{ terms} \\
[\psi &= \psi_1 + \mathcal{I}\psi_2 + \mathcal{J}\psi_3 + \mathcal{K}\psi_4 \,; \quad \psi_1,\ \psi_2,\ \psi_3,\ \psi_4 \in \mathcal{C}(1,i)] \,,
\end{aligned}$$

we choose

$$\rho = (\psi^+ \psi)_{\mathcal{C}(1,i)} \,. \tag{12}$$

We need an $\mathcal{H}$-real (i-complex) geometry and a $^+$ conjugation. Using such geometry, given an operator like (3), we can immediately write its Hermitian conjugate as

$$\mathcal{O}^+ = q_{\mathcal{C}}^+ - p_{\mathcal{C}}^+ \,|\, \mathcal{I} - r_{\mathcal{C}}^+ \,|\, \mathcal{J} - s_{\mathcal{C}}^+ \,|\, \mathcal{K} \,. \tag{13}$$

We now consider a new equivalent representation for the operators $\vec{\alpha}$ and β, which allows us to simplify the steps

$$\vec{\alpha} \equiv (-\mathcal{J} \,|\, \mathcal{I}, i \,|\, \mathcal{K} \,, \quad \mathcal{K} \,|\, \mathcal{I}) \,, \quad \beta \equiv -\mathcal{I} \,|\, \mathcal{I} \,. \tag{14}$$

We seek plane wave solutions of the quaternionic Dirac equation (10), i.e. solutions of the forms

$$\begin{aligned}
\psi^+(x) &= u(p) e^{-i(Et - \vec{p} \cdot \vec{x})} \quad \text{positive energy} \,, \\
\psi^-(x) &= v(p) e^{+i(Et + \vec{p} \cdot \vec{x})} \quad \text{negative energy} \,,
\end{aligned}$$

with the condition that E is positive.

Following the standard procedure, we obtain[f]

$$\begin{aligned}
u^{(1)}(p) &= [2m(m+E)]^{-\frac{1}{2}} (E + m + \mathcal{K}(p_x + ip_y) + \mathcal{J} p_z) \,, \\
u^{(2)}(p) &= [2m(m+E)]^{-\frac{1}{2}} (E + m + \mathcal{K}(p_x - ip_y) + \mathcal{J} p_z) \mathcal{I} \,, \\
v^{(1)}(p) &= [2m(m+E)]^{-\frac{1}{2}} (E + m + \mathcal{K}(p_x + ip_y) + \mathcal{J} p_z) \mathcal{J} \,, \\
v^{(2)}(p) &= [2m(m+E)]^{-\frac{1}{2}} (E + m + \mathcal{K}(p_x - ip_y) + \mathcal{J} p_z) \mathcal{K} \,.
\end{aligned}$$

If we rewrite the complexified quaternionic Dirac equation in covariant form,

$$i\gamma^\mu \partial_\mu \psi = m\psi \,, \tag{15}$$

[f]The normalization factors have been chosen in order that

$$u^{(1)+} u^{(1)} = u^{(2)+} u^{(2)} = v^{(1)+} v^{(1)} = v^{(2)+} v^{(2)} = \frac{E}{m} \,.$$

where

$$\gamma^\mu \equiv (\beta, \beta\,\vec{\alpha})\,,$$

we can quickly extract the standard results, since our γ^μ matrices, now $\mathcal{H}$-real linear complexified quaternionic numbers, satisfy the i-complex Dirac algebra. For example, in order to obtain the relativistic covariance we must assume that, under Lorentz transformations ($x' = \Lambda x$), there is a linear relation between the wave function ψ in the first frame and the wave function ψ' in the transformed frame, namely

$$\psi' = \mathcal{T}(\Lambda)\psi\,,$$

with

$$\mathcal{T}(\Lambda) = e^{\frac{1}{8}[\gamma^\mu,\gamma^\nu]\omega_{\mu\nu}}\,, \qquad \omega_{\mu\nu} \text{ antisymmetric in } \mu\nu\,.$$

Considering an infinitesimal rotation around z and finding the corresponding transformation of the wave function ψ, we obtain the spin operator

$$\mathcal{S}_z = -\frac{1}{2}\mathcal{J}\,|\,\mathcal{J}\,, \tag{16}$$

and so our four solutions $u^{(1,2)}$, $v^{(1,2)}$ correspond to positive and negative energy solutions with $\mathcal{S} = \frac{1}{2}$ and for $\vec{p} = (0, 0, p_z)$, to $\mathcal{S}_z = \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$, respectively.

We can introduce in the Dirac equation a potential A^μ by the minimal coupling:

$$i\partial^\mu \to i\partial^\mu - eA^\mu\,, \qquad \text{where } e = -|e| \text{ for the electron}\,.$$

We will seek a transformation $\psi \to \psi_c$ reversing the charge, such that

$$(i\partial^\mu - eA^\mu)\gamma_\mu\psi = m\psi\,, \tag{17}$$

$$(i\partial^\mu + eA^\mu)\gamma_\mu\psi_c = m\psi_c\,. \tag{18}$$

To construct ψ_c we conjugate (* operation) the first equation and multiply it for $i|j$:

$$(\mathcal{I}|\mathcal{J})(-i\partial^\mu - eA^\mu)\gamma_\mu^*\psi^* = m\mathcal{I}\psi^*\mathcal{J}\,.$$

In our representation, it is straightforward to prove that

$$(\mathcal{I}|\mathcal{J})\gamma_\mu^*\psi^* = -\gamma_\mu\mathcal{I}\psi^*\mathcal{J}$$

and thus

$$\psi_c = \mathcal{I}\psi^*\mathcal{J}\,, \tag{19}$$

which represents the quaternionic version of the charge conjugation operation.

We conclude this section with the following consideration: Comparing Eq. (15) with the standard Dirac equation

$$i\gamma^\mu\partial_\mu\psi = m\psi\,,$$

where

$$\gamma^0 = \begin{pmatrix} 1 & \cdot \\ \cdot & -1 \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} \cdot & \vec{\sigma} \\ -\vec{\sigma} & \cdot \end{pmatrix},$$

we can obtain an interesting relation between 4×4 complex matrices and $\mathcal{H}$-real linear complexified quaternions

$$i\gamma^0 \leftrightarrow -i\mathcal{I}|\mathcal{I}, \qquad i\vec{\gamma} \leftrightarrow (-i\mathcal{K}, \mathcal{I}|\mathcal{J}, -i\mathcal{J}). \tag{20}$$

With this as encouragement we will derive a complete translation in the following section.

4. A Possible Translation

In this section we give explicitly a set of rules for passing back and forth between standard (complex) quantum mechanics and our complexified quaternionic version. Nevertheless we must note that this translation will not be possible in all situations, and so it is only partial. We will be able to pass from $4n$-dimensional complex matrices to n-dimensional $\mathcal{H}$-real linear complexified quaternionic matrices.

We know that 16 complex numbers are necessary in order to define the most general 4×4 complex matrix but only four are needed to define the most general complexified quaternion:

$$\psi = \psi_1 + \mathcal{I}\psi_2 + \mathcal{J}\psi_3 + \mathcal{K}\psi_4 \quad [\psi_1,\ \psi_2,\ \psi_3,\ \psi_4 \in \mathcal{C}(1,i)].$$

Therefore, in order to achieve a translation, we need 12 new complex $\mathcal{C}(1,i)$ numbers. If we work with $\mathcal{H}$-real linear complexified quaternions, these new degrees of freedom are represented by

$$\phi|\mathcal{I} + \eta|\mathcal{J} + \xi|\mathcal{K} \quad (\phi,\ \eta,\ \xi \in \mathcal{H}_{\mathcal{C}}).$$

The rules for translating between complex and complexified quaternionic quantum mechanics are

$$i \leftrightarrow \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix},$$

$$\mathcal{I} \leftrightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad 1|\mathcal{I} \leftrightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\mathcal{J} \leftrightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad 1|\mathcal{J} \leftrightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

We can obviously obtain the remaining rules from the previous ones; for example,

$$\mathcal{K} = \mathcal{I}\mathcal{J} \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$1|\mathcal{K} = (1|\mathcal{J})(1|\mathcal{I}) \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

With these rules we can translate any $4n$-dimensional complex matrix into an equivalent n-dimensional $\mathcal{H}$-real linear complexified matrix, and *vice versa*:

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{pmatrix} \leftrightarrow q_{\mathcal{C}} + p_{\mathcal{C}}|\mathcal{I} + r_{\mathcal{C}}|\mathcal{J} + s_{\mathcal{C}}|\mathcal{K}, \tag{21}$$

with

$$\begin{aligned} 4q_{\mathcal{C}} &= a_1 + b_2 + c_3 + d_4 + \mathcal{I}(a_2 - b_1 + c_4 - d_3) \\ &\quad + \mathcal{J}(a_3 - b_4 - c_1 + d_2) + \mathcal{K}(a_4 + b_3 - c_2 - d_1), \\ 4p_{\mathcal{C}} &= a_2 - b_1 - c_4 + d_3 - \mathcal{I}(a_1 + b_2 - c_3 - d_4) \\ &\quad - \mathcal{J}(a_4 + b_3 + c_2 + d_1) + \mathcal{K}(a_3 - b_4 + c_1 - d_2), \\ 4r_{\mathcal{C}} &= a_3 + b_4 - c_1 - d_2 + \mathcal{I}(a_4 - b_3 - c_2 + d_1) \\ &\quad - \mathcal{J}(a_1 - b_2 + c_3 - d_4) - \mathcal{K}(a_2 + b_1 + c_4 + d_3), \\ 4s_{\mathcal{C}} &= a_4 - b_3 + c_2 - d_1 - \mathcal{I}(a_3 + b_4 + c_1 + d_2) \\ &\quad + \mathcal{J}(a_2 + b_1 - c_4 - d_3) - \mathcal{K}(a_1 - b_2 - c_3 + d_4). \end{aligned}$$

We can obviously invert the previous development and obtain, given $q_{\mathcal{C}}$, $p_{\mathcal{C}}$, $r_{\mathcal{C}}$, $s_{\mathcal{C}}$, the complex coefficients a, b, c, d, i.e. as in

$$\begin{aligned} a_1 &= (q_{\mathcal{C}} + \mathcal{I}p_{\mathcal{C}} + \mathcal{J}r_{\mathcal{C}} + \mathcal{K}s_{\mathcal{C}})_{\mathcal{C}(1,i)}, \\ c_2 &= (s_{\mathcal{C}} + \mathcal{I}r_{\mathcal{C}} + \mathcal{J}p_{\mathcal{C}} + \mathcal{K}q_{\mathcal{C}})_{\mathcal{C}(1,i)}. \end{aligned}$$

We now give an example of this translation. Consider the spin operator which appears in the standard Dirac equation (see, for example, Ref. 18):

$$S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The only nonzero coefficients are $a_1 = c_3 = -b_2 = -d_4 = 1$, and so the complexified quaternionic version of the spin operator is

$$S_z = -\frac{1}{2}\mathcal{J}|\mathcal{J}.$$

In a similar way we can obtain all the results given in the third section. We must only translate from the standard (complex) Dirac equation, so we cannot have a doubling of the solutions.

We conclude this section by giving the matrix $\mathcal{S}$ which reduces the complexified quaternionic matrices used by Edmonds[15] and Gough.[9] In this matter we have elegantly explained the doubling of solutions in their papers.[g]

The matrix $\mathcal{S}$, which satisfies

$$\mathcal{S}i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathcal{S}^{-1} = (1|\mathcal{J})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathcal{S}\vec{Q}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathcal{S}^{-1} = (\vec{Q}|\mathcal{I})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

is

$$\mathcal{S} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 + i|\mathcal{I} - 1|\mathcal{J} - i|\mathcal{K} & -i + 1|\mathcal{I} - i|\mathcal{J} + 1|\mathcal{K} \\ i + 1|\mathcal{I} + i|\mathcal{J} + 1|\mathcal{K} & -1 + i|\mathcal{I} + 1|\mathcal{J} - i|\mathcal{K} \end{pmatrix}.$$

5. Conclusions

In this paper we have defined a set of rules for translating from standard (complex) quantum mechanics to complexified quaternionic quantum mechanics; with our rules we can obtain a rapid counterpart of the standard quantum-mechanical results and overcome previous difficulties (like doubling of the solutions in the Dirac equation).

We note that there is no "new physics" in the complexified quaternionic Dirac equation, in contrast with the standard folklore. We also emphasize that our translation is only a *partial* translation. In fact the isomorphism between complex and complexified quaternionic quantum mechanics, which we have presented in this work, requires a 4:1 reduction, and so only works between $4n$-dimensional complex space and n-dimensional complexified quaternionic space; other dimensions *do not* admit this isomorphism.

[g] In Ref. 19 we read the following comment about the complexified quaternionic version of the Dirac equation: "When written in this manner a doubling of components of the wave function from 4 to 8 occurs." This difficulty is now overcome.

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