Modern Physics Letters A, Vol. 11, No. 5 (1996) 357–366

THE QUATERNIONIC DIRAC LAGRANGIAN

STEFANO DE LEO* and PIETRO ROTELLI†

Dipartimento di Fisica, Instituto di Fisica Nucleare, Università di Lecce,
Sezione di Lecce, Lecce, 73100, Italy

Received 30 October 1995

We discuss the use of the variational principle within quaternionic quantum mechanics. This is nontrivial because of the noncommutative nature of quaternions. We derive the Dirac–Lagrangian density corresponding to the two-component Dirac equation. This Lagrangian is complex projected as anticipated in previous articles and this feature is necessary even for a classical real Lagrangian.

1. The Variational Principle

The standard derivation of the Euler–Lagrange field equations implicitly assumes that the variation in the action I of a given field $\delta\phi_i$ may be commuted to, say, the extreme right. Thus after a functional integration by parts, and neglecting the surface terms one obtains the field equations,

$$\frac{\partial \mathcal{L}(x)}{\partial \phi_i(x)} = \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial_\mu \phi_i(x)}, \tag{1}$$

where $\mathcal{L}$ is the Lagrangian density and x represents spacetime.

Somewhat surprisingly this form of the field equations survives in field theory, that is after second quantization. The reason that this is not *a priori* obvious is because when the fields become operators and noncommutating, the translation of $\delta\phi_i$ is not necessarily without consequences. Indeed, we should be obliged to define with attention the significance of the functional derivatives in Eq. (1). This fact does not in practice produce difficulties as a consequence of:

(a) For bilinear terms (e.g. mass or kinetic energy) the variations naturally lie on the right (for $\delta\phi_i$) or the left (for $\delta\phi_i^+$);

(b) In any case some authors implicitly assume that, when $\delta\phi_i$ is bosonic it commutes with everything;

(c) When ϕ_i is a fermion field $\delta\phi_i$ is assumed to commute at least with any bosonic fields and with ϕ_i itself. This last fact (by no means obvious) leads to a *natural* extraction of $\delta\phi_i$ to the right or the left.

*E-mail: deleos@le.infn.it
†E-mail: rotelli@le.infn.it

More specifically, within normal ordered products there always exists a natural direction of extract given the fact that all creation operators (bosonic or fermionic) commute (anticommute) amongst themselves.

It is however an assumption that the variation in ϕ_i, i.e. $\delta\phi_i$ has the same characteristic commutation relations of ϕ_i. These implicit assumptions are brought to the fore when one considers quaternionic fields both in classical and quantum field theory. For it is unthinkable that a quaternionic $\delta\phi_i$ created within a Lagrangian density can, in general, be commuted without consequence to the left or the right. The example of the free Dirac–Lagrangian treated in this letter will demonstrate some of the difficulties.

Before entering explicitly into the world of quaternionic fields we wish to discuss the limitations, if any, applied to $\delta\phi_i$ in standard (complex) quantum mechanics. Our objective, which will have an application in this letter, is to demonstrate that the properties of $\delta\phi_i$ may be substantially different from those of ϕ_i. For example, we shall observe first that even if the classical Lagrangian density is necessarily real, $\delta\phi_i$ or more correctly $\delta\mathcal{L}$ may be formally complex (or even quaternionic) since at the end $\delta\mathcal{L} \equiv 0$ in order to obtain the Euler–Lagrange equations. It is true that in general we may limit our attention to a subset of $\delta\phi_i$ ($\delta\mathcal{L}$) without losing the Euler–Lagrange equations, so that our observation seems academic, but it is not without consequences with quaternions as we shall see.

Consider one of the simplest of all particle Lagrangian densities, that for two classical scalar fields (ϕ_i-real $i = 1, 2$) without interactions:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 - \frac{m^2}{2}{\phi_1}^2 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{m^2}{2}{\phi_2}^2 \\ &\equiv \partial_\mu\phi^+\partial^\mu\phi - m^2\phi^+\phi\,,\end{aligned} \tag{2}$$

where

$$\phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

and

$$\phi^+ \equiv \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)\,.$$

The well-known corresponding Euler–Lagrange equations are:

$$(\partial_\mu\partial^\mu + m^2)\phi_i = 0\,, \qquad i = 1, 2\,, \tag{3}$$

or, equivalently,

$$(\partial_\mu\partial^\mu + m^2)\phi = 0\,. \tag{4}$$

Now to obtain "directly" the last equation one performs the very particular variation of ϕ (ϕ^+)

$$\begin{aligned}\phi &\to \phi\,,\\ \phi^+ &\to \phi^+ + \delta\phi^+\,,\end{aligned} \tag{5}$$

i.e. in order to obtain the corresponding Euler–Lagrangian equation one treats ϕ and ϕ^+ as *independent* fields. In second quantization these fields indeed contain independent creation and annihilation operators corresponding to positively and negatively charged particles.[a] To satisfy Eq. (5) we must necessarily have,

$$\delta\phi_1 + i\delta\phi_2 = 0 \tag{6}$$

and this means that the *variations* in the originally real ϕ_i fields are complex (if $\delta\phi_1$ is real, then $\delta\phi_2$ is pure imaginary etc.).

Of course we could obtain the equivalent result from variations separately of ϕ_1 and ϕ_2, e.g. varying ϕ_1 (with ϕ_2 constant)

$$\begin{aligned} \phi &\to \phi + \delta\phi_1\,, \\ \phi^+ &\to \phi^+ + \delta\phi_1 \end{aligned} \tag{7}$$

yielding, after a double integration by parts

$$(\partial_\mu\partial^\mu + m^2)(\phi + \phi^+) = 0 \tag{8}$$

and the corresponding result for ϕ_2. Thus while not obligatory we can in this latter approach consider only *real* $\delta\phi_i$. However, there is a subtle difference in the two approaches, which readily passes unobserved. In the case when only ϕ_1 or ϕ_2 is varied the $\delta\phi_i$ appears both to the left and to the right. Only if $\delta\phi_i$ commutes with ϕ_i are the two methods equivalent.

The point of these observations is that it is possible to impose diverse conditions on the Lagrangian density, on the fields, and on their variations. This is ignored in standard quantum mechanics but is important for what follows.

2. The Two-Component Dirac Equation

The adoption of quaternions as the base number system in quantum mechanics allows one to define a quaternionic Dirac equation[1] in which the wave function is characterized by having only two components (consequently the Schrödinger–Pauli equation applies to a one-component wave function). This follows from the fact that the Dirac algebra upon the reals (but not upon the complex) has a two-dimensional irreducible representation with quaternions. The standard 4×4 complex gamma matrices are in fact reducible. This structure is consistent if the momentum operator p^μ is defined by

$$p^\mu = \partial^\mu | i \tag{9}$$

where a *barred* operator $A|b$ acts upon a general wave function ψ by

$$(A|b)\psi \equiv A\psi b\,. \tag{10}$$

[a] Actually the "electric" charge e of the fields depends upon whether the global gauge group is to be made local or not. This fact is a choice *not* determined by the free Lagrangian.

In general A will be a matrix and b a complex $\mathcal{C}(1,i)$ number, where i is one of the imaginary (i,j,k) units of a quaternion:

$$q = a_0 + a_1 i + a_2 j + a_3 k\,, \tag{11}$$

with

$$\begin{gathered} i^2 = j^2 = k^2 = -1\,, \\ ij = -ji = k \qquad \text{(cyclic)}\,, \\ a_{0,1,2,3} \in \mathbb{R}\,. \end{gathered}$$

The quaternionic conjugation is defined by

$$q^+ = a_0 - a_1 i - a_2 j - a_3 k\,. \tag{12}$$

The Dirac equation then reads in covariant form

$$\gamma_\mu \partial^\mu \psi i - m\psi = 0 \tag{13}$$

with a possible choice of the γ_μ

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \gamma_1 &= i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \gamma_2 &= j\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \gamma_3 &= k\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{14}$$

This equation has four plane wave ($\propto e^{-ipx}$) solutions which correspond to those of the standard Dirac equation if one adopts the complex scalar product (complex geometry[2])

$$\langle\psi|\phi\rangle_{\mathcal{C}} = \frac{1}{2}\langle\psi|\phi\rangle - \frac{i}{2}\langle\psi|\phi\rangle i\,, \tag{15}$$

where $\langle\psi|\phi\rangle$ is the quaternionic scalar product,

$$\langle\psi|\phi\rangle = \int \psi^+ \phi d\tau \tag{16}$$

so that the four plane wave solutions are (complex) orthogonal. Actually the need of this scalar product is anticipated by our choice of p^μ which is not hermitian on a quaternionic geometry. The complex scalar product was first introduced in the definition of tensor products in quaternionic quantum mechanics,[3] and appears essential for the "translation" of complex quantum mechanics to a quaternionic version.[4]

Our main objective in this work is to derive the Dirac–Lagrangian (density) which yields Eq. (13), and this will be done in the next section. We conclude this section with some notes and comments upon this use of quaternions in quantum mechanics.

(a) The objective of reproducing almost all standard results in quantum mechanics is achieved. This objective is less ambitious than the development of generalized quantum dynamics by Adler and coworkers.[5]

(b) There are differences within the bosonic sector. For example there is a doubling of solutions in the Klein–Gordon and Maxwell equations. This leads to *anomalous* solutions so-called because it was at first believed that they violated four-momentum conservation. In Ref. 6 the conservation of four-momentum for these anomalous particles was demonstrated. Furthermore it has been shown that, unlike the situation in complex QM, the quaternionic version of the Duffin–Kemmer (DK) equation is *not* equivalent to the Klein–Gordon (KG) (for spin-0) and Proca (for spin-1) equations.

(c) There exist however projected equations with only the standard number of solutions. The quaternionic version of the DK equation is an example of this and can be simply identified with the complex projected ("modified") KG equation[7,8]

$$\frac{1 - i|i}{2}(\partial_\mu \partial^\mu + m^2)\phi = 0\,.$$

(d) It is possible to invent new quaternionic equations equivalent to pairs of complex equations in the same way as the Schrödinger equation can be rewritten as a pair of real equations. In nonrelativistic QM one can introduce (as Adler has done[9]) an effective potential which leads to the elimination of the quaternionic part of the wave function (*the intrinsically quaternionic part of the wave function has no running wave solutions*! — Adler[10]). Nevertheless the "effective" complex Schrödinger equation shows characteristic quaternionic effects, in particular a T-violation arising from the underlying quaternionic dynamics.[9,10] New quaternionic equations *not* translatable into corresponding complex equations also occur whenever $1|j$ and/or $1|k$ factors appear explicitly in the equations.

(e) There exists a not yet completely investigated generalization of group theory or more precisely of representation theory. In a recent work,[11] one of us (SdL) has highlighted the possibility to extend the quaternionic group theory by using *complex linear quaternions*[3]

$$q + p|i \qquad (q, p \in \mathcal{H})\,.$$

This represents only a first step towards a more complete treatment and a generalization of quaternionic group theory (for example by using *real linear quaternions* $q + p|i + r|j + s|k$ – $q, p, r, s \in \mathcal{H}$).

Finally, we are always intrigued by the fact that had Schrödinger considered quaternionic solutions to his equation, he would have found two (within a complex geometry) and probably discovered the existence of spin.

3. The Dirac–Lagrangian

Let us consider, as a first hypothesis, the traditional form for the Dirac–Lagrangian density:

$$\mathcal{L} = \frac{i}{2}[\bar{\psi}\gamma^\mu(\partial_\mu\psi) - (\partial_\mu\bar{\psi})\gamma^\mu\psi] - m\bar{\psi}\psi \tag{17}$$

as given for example by Itzykson and Zuber.[12] The position of the imaginary unit is purely conventional in (17) but with a quaternionic number field we must recognize that the ∂_μ operator is more precisely part of the first quantized momentum operator $\partial_\mu|i$ and that hence only the hermitian conjugate part of the kinetic term (second half of the bracket expression in Eq. (17)) appears with the imaginary unit in the "correct" position. Thus the correct form of the kinetic term reads:

$$\mathcal{L}_K = \frac{1}{2}[\bar{\psi}\gamma^\mu\partial_\mu\psi i - i(\partial_\mu\bar{\psi})\gamma^\mu\psi] \,. \tag{18}$$

We observe that this modification of Eq. (17) is also justified by the simple requirement that $\mathcal{L}$ be hermitian. We could of course multiply Eq. (18) by the "hermitian" $-i|i$ which inverts the order of the imaginary unit, but by integrating by parts this may be reformulated as in Eq. (18). Note that we cannot change the sign of $\mathcal{L}$ without changing the sign of the Hamiltonian $\mathcal{H}$.

The requirement of hermiticity however says nothing about the Dirac mass term in Eq. (17). It is here that appeal to the variational principle must be made. A variation $\delta\psi$ in ψ from Eq. (18) cannot be brought to the extreme right because of the imaginary unit in the first half of the expression. The only consistent procedure is to generalize the variational rule that says that ψ and $\bar{\psi}$ must be varied *independently*. We thus apply independent variations to ψ ($\delta\psi$) and ψi ($\delta(\psi i)$). Similarly for $\delta\bar{\psi}$ and $\delta(i\bar{\psi})$. Now to obtain the desired Dirac equation for ψ and its adjoint equation for $\bar{\psi}$ we are obliged to modify the mass term into

$$\mathcal{L}_m = -\frac{m}{2}[\bar{\psi}\psi - i\bar{\psi}\psi i] \,. \tag{19}$$

The final result for $\mathcal{L}$ is

$$\mathcal{L}_D = \frac{1}{2}[\bar{\psi}\gamma^\mu\partial_\mu\psi i - i(\partial_\mu\bar{\psi})\gamma^\mu\psi] - \frac{m}{2}[\bar{\psi}\psi - i\bar{\psi}\psi i] \,. \tag{20}$$

Considering this last equation we observe that it is nothing other than the complex projection of equation (17)

$$\mathcal{L}_D = \frac{1 - i|i}{2}\mathcal{L} \,. \tag{21}$$

Indeed, while $\mathcal{L}$ in Eq. (17) is quaternionic and with the modification of $\mathcal{L}_K$ in Eq. (18) hermitian, the form given in Eq. (20) is purely complex and hermitian.

This observation is even more subtle with classical fields for now $\mathcal{L}$ defined by

$$\mathcal{L} = \mathcal{L}_K - m\bar{\psi}\psi \tag{22}$$

is both hermitian and *real*. Thus it may be objected that the complex projection in Eq. (21) is superfluous. For $\mathcal{L}$ itself this is true but for *quaternionic* variations in the fields $\delta\psi$ etc. a difference exists. The variation $\delta\mathcal{L}$ of Eq. (22) is in general quaternionic while, because of Eq. (21), the variation $\delta\mathcal{L}_D$ is always *complex*. Furthermore Eq. (22) would not yield the correct field equation through the variational principle unless we limit $\delta\psi$ to complex variations notwithstanding the quaternionic nature of the fields. We consider this latter option unjustified and thus select for the formal structure of $\mathcal{L}$ that of Eq. (20).

We note that as described in Sec. 1, one should as a rule define the numerical basis of the fields and separately of the variations. For classical fields we have the case of real fields with complex variations (charged scalar fields) and in our studies the *possibility* of quaternionic fields but with only complex variations (the second option described above).

Furthermore one should specify the numerical nature of both $\mathcal{L}$ and $\delta\mathcal{L}$. For example with our choice given by Eq. (20) we have:

$\psi, \bar{\psi}\,,$	quaternionic fields;
$\delta\psi, \delta(\psi i), \delta\bar{\psi}, \delta(i\bar{\psi})\,,$	quaternionc variations;
$\mathcal{L}_D\,,$	hermitian, complex projected Lagrangian;
$\delta\mathcal{L}_D\,,$	complex but generally nonhermitian variation.

4. The Invariance Groups of $\mathcal{L}_D$

Having obtained $\mathcal{L}_D$ in the previous section we may ask which if any group (global) leaves this Lagrangian invariant. Remembering that the single particle fields are one-component quaternionic functions, the most natural transformation is,

$$\begin{aligned} \psi &\to f\psi g \\ \bar{\psi} &\to g^+\psi^+ f^+\gamma^0\,. \end{aligned} \tag{23}$$

Now the quaternionic nature of the γ^μ matrices limits f, and for arbitrary ψ (since we assume no knowledge at this stage of the field equations[b]) this leads to the conclusion that f must be real and commute with each γ^μ. On the other hand the complex projection of $\mathcal{L}_D$ permits g to be complex (it can then be "extracted" to the right and then commuted through $\mathcal{L}_D$). Furthermore for invariance,

$$gg^+ff^+ = |g|^2 f^2 = 1\,. \tag{24}$$

We find with our representation for γ^μ that f is proportional to the unit matrix, and thus being real its magnitude may be completely absorbed within g, or equivalently

[b]We recall the well-known fact that $\mathcal{L}_D$ becomes identically null, if the field equations are applied to ψ and $\bar{\psi}$.

f may set by definition to 1. Whence, the only invariance group is defined by

$$\begin{aligned} &|g|^2 = 1\,, \\ &g \in \mathcal{C}(1, i)\,. \end{aligned} \tag{25}$$

For Lie groups this implies that g is a member of $\mathrm{U}(1,c)$ the *complex* unitary group.

Remembering that the Glashow group[13] for the Weinberg–Salam theory[14] is $\mathrm{SU}(2,c) \times \mathrm{U}(1,c)$ we observe that this $\mathrm{U}(1,c)$ group may be identified with ours and that our field must necessarily be a singlet (scalar) under $\mathrm{SU}(2,c)$. Since ψ represents a *single* fermion spin-$\frac{1}{2}$ field (the γ^μ are 2×2 quaternionic matrices) it is not surprising, and indeed necessary that it represents a singlet under $\mathrm{SU}(2,c)$.

The interesting feature is what happens if we select a reducible 4×4 complex set of γ^μ matrices. Now the number of the fermionic particles is two. For example the leptons of the first family (electronic neutrin-ν_e, electron-e) can be precisely rewritten as follows[c]:

$$\nu_e + je\,,$$

where ν_e and e represent standard (complex) fields. Nevertheless, the existence of a complex $\mathrm{SU}(2,c)$ group acting on ψ from the left seems excluded. On the other hand, as we have described in detail elsewhere[15] the group $\mathrm{SU}(2,c)$ is isomorphic (at the generator algebra level) with $\mathrm{U}(1,q)$, the unitary quaternionic group with elements

$$g \sim e^{ia+jb+kc}$$

with

$$a, b, c \in \mathbb{R}\,,$$

which is the simplest of all unitary quaternionic groups.

Now it is still not obvious that this group is an invariance group of $\mathcal{L}_D^{(2)}$, where the superscript "(2)" indicates the presence of two fermionic fields (with the same mass) since the γ^μ matrices are complex, not real. However, it is readily verified that an equivalent set of γ^μ matrices (with the same commutation relations) is given if the i-factors in the 4×4 *complex* γ^μ are substituted by $1|i$. Whence the imaginary units appear to the *right* of ψ and the elements given by Eq. (25) commute with γ^μ and hence cancel within $\mathcal{L}_D$. One may also derive the similarity transformation which performs the above change of representation and thus derive the corresponding representation[d] of $\mathrm{U}(1,q)$ for our original (conventional) choice of complex γ^μ.

[c] An alternative but equivalent choice (using 4×4 *quaternionic* matrices) is given by

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}$$

with ν_e and e two-component quaternionic fields.

[d] We observe that while our $\mathrm{U}(1,q)$ group acts from the *left* on ψ and $\mathrm{U}(1,c)$ acts on the *right*, this has nothing to do with the helicity indices L and R of the Weinberg–Salam theory.

We note that our $\mathrm{U}(1,q)$ group consists of quaternionic "phase" factors multiplied by the unit 4×4 matrix.[e] We may ask if there are other 4×4 (essentially real) matrices which commute with the four-dimensional Dirac gamma matrices. However there is no other matrix that commutes with all four gamma matrices so the maximum global invariance group is indeed $\mathrm{U}(1,q)|\mathrm{U}(1,c)$ to be identified with the (complex) Glashow group $\mathrm{SU}(2) \times \mathrm{U}(1)$.

5. Conclusions

We begin our conclusions from the end results of the last section. We wish to recall that not every global invariance group is automatically gauged into the corresponding local group in gauge theory. For example, the so-called complex bosonic Lagrangian may *a priori* (before eventual gauging) represent two *neutral* equal mass particles or a complex pair of oppositely charged particle–antiparticle. Thus even in the case of a Dirac–Lagrangian with complex (4×4) Dirac matrices we do not have necessarily a doublet under $\mathrm{U}(1,q)$. So that we have not *derived* the gauge group of the Salam–Weinberg model. The pair may consist of two singlets. However it was not even obvious *a priori* that a global invariance group $\mathrm{U}(1,q)$ isomorphic at the generator level with $\mathrm{SU}(2,c)$ exists. *We have thus shown that even with quaternionic fields, it is possible to impose a Glashow group invariance and that this occurs by merely adopting reducible gamma matrices.*

Our viewpoint is that $\mathrm{SU}(2,c)$ invariance in particle physics is really indication of invariance under the simplest of all unitary quaternionic groups. We also have a complex invariance group $\mathrm{U}(1,c)$ but this is justified by the complex projection of the Lagrangian density. This complex projection was *not* imposed to obtain the $\mathrm{U}(1,c)$ factor group but in order to obtain the desired quaternionic Dirac equation. The fact that this group exists in nature as the weak-hypercharge group is an undoubted success of this model. Further, the automatic appearance of this complex unitary group is expected whatever the left acting (quaternionic) unitary group is. This strongly suggests that in the search for grand unified theories one should consider preferentially a product group of the type $G|\mathrm{U}(1,c)$ with G a *quaternionic* unitary group, conditioned to commute with the Dirac gamma matrices in the chosen representation.

Let us recall the other results of this letter. We have discussed the application of the variational principle to Lagrangians with possibly quaternionic fields. We noted *en passant* that even within standard field theory the limitation of variations in the fields to complex variations is an implicit assumption, since the variations themselves have no physical content. Thus quaternionic variations have always *existed* even if not traditionally applied. Of course this observation can be generalized to even more exotic variations, e.g. supersymmetric variations containing Grassmann terms.

[e]An additional right acting complex phase $\mathrm{U}(1,c)$ is also allowed, but this is equivalent to that already described in the previous section.

Finally, whereas in this work the need of a complex projection of the Diarc–Lagrangian density is demonstrated, for the scalar field Lagrangian it is an assumption with interesting consequences.[7] We suggest that, for reasons which still elude us, all terms in the quaternionic Lagrangians including interaction terms must be complex projected.

References

1. P. Rotelli, *Mod. Phys. Lett.* **A4**, 933 (1989).
2. J. Rembieliński, *J. Phys.* **A11**, 2323 (1978).
3. L. P. Horwitz and L. C. Biedenharn, *Ann. Phys.* **157**, 432 (1984).
4. S. De Leo and P. Rotelli, *Prog. Theor. Phys.* **92**, 917 (1994).
5. S. L. Adler and A. C. Millard, "Generalized quantum dynamics as pre-quantum mechanics", hepth/9508076 – IASSNS-HEP 95/66.
6. S. De Leo and P. Rotelli, *Phys. Rev.* **D45**, 575 (1992).
7. S. De Leo and P. Rotelli , hepth/9506179, to appear in *Int. J. Mod. Phys. A*.
8. S. De Leo, hepth/9509058, to appear in *Prog. Theor. Phys.*
9. S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields* (Oxford Univ. Press, 1995).
10. S. L. Adler, "Scattering theory in quaternionic quantum mechanics" , *From Symmetries to Strings: Forty Year of Rochester Conferences* (World Scientific, 1990).
11. S. De Leo, "Quaternions for GUTs", hepth/9506008 – IASSNS-HEP 95/45.
12. C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, 1985).
13. S. L. Glashow, *Nucl. Phys.* **22**, 579 (1961).
14. S. Weinberg, *Phys. Rev. Lett.* **19**, 1264 (1967); A. Salam, in *Proc. 8th Nobel Symposium on Weak and Electromagnetic Interaction*, ed. Svartholm (1968), p. 367.
15. S. De Leo and P. Rotelli, *Nuovo Cimento* **B110**, 33 (1995).