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QUATERNION HIGGS AND THE ELECTROWEAK GAUGE GROUP

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We show that, in quaternion quantum mechanics with a complex geometry, the minimal four Higgs of the unbroken electroweak theory naturally determine the quaternion invariance group which corresponds to the Glashow group. Consequently, we are able to identify the physical significance of the anomalous Higgs scalar solutions. In addition, we introduce and discuss the complex projection of the Lagrangian density.

1. Introduction

One of the primary objectives of the present authors in recent years has been to demonstrate the possibility (if not necessity) of using quaternions in the description of elementary particles, in both first and second quantization. An essential ingredient in the version of quaternion quantum mechanics used by the authors is what Rembeliński[1] called long ago the adoption of a complex geometry (complex scalar product). This choice is certainly less ambitious than that of Adler,[2,3] who advocates the use of a quaternion geometry and seeks a completely new quantum mechanics. However, we recall that up to a decade ago the use of quaternions in QM seemed doomed to failure. The noncommutative nature of quaternions (and hence quaternion wave functions) made the definition of tensor products ambiguous and self-destructive; for example, in general an algebraic product of fermionic wave functions no longer satisfies the single particle wave equations.

A complex geometry thus seems necessary, if not sufficient, for reproducing standard QM. In fact we have recently shown[4] that with the use of generalized quaternions (see Sec. 2) a translation exists between *even-dimensional* quantum mechanics and our quaternion version. This by no means concludes the study of this subject. Apart from the eventual extension beyond standard QM to, for example, the study of intrinsically quaternion field equations (in the sense in which the

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Schrödinger equation is intrinsically complex because of the explicit appearance of the imaginary unit), we have to admit a difference in the bosonic sector (odd-dimensional) in which additional *anomalous* solutions appear. There is also a somewhat surprising difference in the physical content of Lie group representations again associated with the odd-dimensional (bosonic) sector, notwithstanding the isomorphism of the corresponding Lie algebras.[5]

The authors have long been puzzled by the significance of the anomalous solutions. Although the initial fear of nonconservation of momentum has been overcome,[6] *we have not been able to identify an anomalous particle before this work.* We had considered the possibility that with quaternions one might be able to distinguish between particles and pseudoparticles. This would be very attractive since in QM the distinction is by definition. Furthermore, where anomalous solutions do not occur, such as in the quaternion Dirac equation,[7,8] we see that, as a physical justification, both parities appear (for particle and antiparticle). However, to date such an identification has not been possible and the results of this paper lead elsewhere. Indeed, we shall argue that to reproduce the Weinberg[9]–Salam[10] model, or more precisely the Higgs sector, we require anomalous Higgs solutions of the Klein–Gordon equation, and that these are the charged Higgs that eventually lead to massive $W^{\pm}$ gauge bosons.

Another possible justification for the use of quaternions would be if certain (correct) choices became *natural* with them. Now, while we are well aware that naturalness has no rigorous definition and is often synonymous with habit or some form of analogy, we will argue in just these terms for the gauge group of the electroweak model. We shall show that the invariance group of the Klein–Gordon equation (for a given four-momentum) is $\mathrm{U}(1,q)|\mathrm{U}(1,c)$. The *bar* separates the left-acting unitary quaternion group in one dimension from the right-acting complex group $\mathrm{U}(1,c)$. Here left and right have nothing to do with helicity. This group substitutes the Glashow[11] group $\mathrm{SU}(2)\times\mathrm{U}(1)$. We recall that the Lie algebra $\mathrm{u}(1,q)$ is isomorphic to that of $\mathrm{su}(2,c)$ as long as one uses anti-Hermitian generators.[5] We shall then *assume* that this global group is an invariance group of the Lagrangian density, and this will imply the need for a complex projection of the dynamic Higgs term. For quite different reasons a complex projection is needed in the fermionic sector, but this will be explained elsewhere. Analogy with the standard theory then tells us how we must proceed for the potential terms.

In the next section we recall some previous results about the quaternion Dirac equation. We then discuss the quaternion Kemmer equation and show that anomalous scalar (and vector) solutions can be avoided if necessary. We also recall in this section our rules for translation from complex to quaternion QM and vice-versa mentioned above. In Sec. 3 we discuss the Higgs particles and derive the above-quoted results. Furthermore we shall obtain a particularly elegant form of $\mathrm{U}(1)_{\mathrm{em}}$ and hence the corresponding rule for minimal coupling, which is *a priori* ambiguous with quaternions. In Sec. 4 we shall describe the introduction of the gauge fields by gauging the above group. Our conclusions are drawn in Sec. 5.

2. The Dirac and the Kemmer Equation

We use standard nomenclature for quaternions q,

$$q = r_0 + ir_1 + jr_2 + kr_3$$
$$(r_m \in \mathcal{R}(\text{reals}),\ m = 0, \ldots, 3)\,, \tag{1}$$

with i, j, k the quaternion imaginary units $i^2 = j^2 = k^2 = -1$, which satisfy

$$ij = -ji = k \quad (\text{and cyclic})\,. \tag{2}$$

An alternative (symplectic) decomposition of q is

$$q = z_1 + jz_2\,, \qquad z_m \in \mathcal{C}(1, i)\,, \quad m = 1, 2\,. \tag{3}$$

This form implies a choice of one of the imaginary units (i in this case) which occurs quite naturally in complex wave or field equations such as the Schrödinger or Dirac ones. For us, the unit i will always correspond to the imaginary unit in standard (complex) QM.

We shall now justify the choice of a complex geometry by recalling a particular derivation of the (irreducible) quaternion Dirac equation. Noncommutativity implies an *a priori* ambiguity in the form of the Dirac equation with quaternions. One possibility is

$$i\partial_t\psi = H\psi = (\alpha \cdot \wp + \beta m)\psi\,, \tag{4}$$

where (for covariance arguments) we must define the momentum operator $\wp$ in the standard way (quaternion Hermitian):

$$\wp \equiv -i\partial\,. \tag{5}$$

Unfortunately this choice leads to nonconservation of the norm N,

$$N = \int \psi^+\psi\, d\mathbf{x}\,, \tag{6}$$

if H is not complex (i.e. if α and β are not the standard matrices). Furthermore, if H is assumed to be quaternionic then $\wp$ is not even a conserved quantity, and so forth. This choice therefore obliges one to adopt the standard four-dimensional complex Dirac matrices. Thus only the wave function ψ would be quaternion. This use of the standard γ^μ matrices even with quaternions goes back many decades.

An alternative choice which *automatically* conserves the norm is[7]

$$\partial_t\psi i = H\psi = (\alpha \cdot \wp + \beta m)\psi\,. \tag{7}$$

This requires for consistency (and covariance)

$$\wp \equiv -\partial|i\,, \tag{8}$$

where the bar separates left- and right-acting elements. We are thus led to define as *generalized quaternions* Q [4] the numerical operators

$$Q = q_1 + q_2|i\,, \qquad q_m \in \mathcal{H}\,, \qquad m = 1, 2\,, \tag{9}$$

such that, applied to a state vector ψ,

$$Q\psi = q_1\psi + q_2\psi i \quad (\text{note that } q_1 \equiv q_1|1)\,. \tag{10}$$

The choice (8) for the momentum operator (in first quantization) would not be Hermitian unless one introduces the complex scalar product indicated by the subscript $\mathcal{C}$:

$$(\psi, \phi)_{\mathcal{C}} \equiv \frac{1 - i|i}{2}\,\langle \psi, \phi \rangle\,, \tag{11}$$

where

$$\langle \psi, \phi \rangle \equiv \int \psi^{+} \phi \, d\mathbf{x} \tag{12}$$

is the quaternion scalar product. H may now be quaternion and we are hence allowed to use the irreducible two-dimensional quaternion Dirac matrices, which in turn define a *real* Dirac algebra. As an aside, we note that algebraic theorems which demonstrate that the minimum number of dimensions of the Dirac matrices is four, assume the existence of a *complex* Dirac algebra.

Our choice for $\wp^{\mu}$ ($\wp^{0} \equiv \partial_t|i$) displays the fact that, in order to retain the standard QM commutation relations, a special imaginary unit must be selected from the quaternions. All the momentum eigenstates will then be characterized by standard plane wave functions on the *right* of any spinor or polarization vector (on the left for the adjoint wave functions).

The natural appearance of generalized quaternions has another useful by-product. While it has long been known that any quaternion can be represented by a subset of 2×2 complex matrices, we are now able to identify any two-dimensional complex matrix with a generalized quaternion and vice versa. A particular choice is given by

$$\begin{aligned} 1 &\leftrightarrow \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}, & i &\leftrightarrow \begin{pmatrix} i & \cdot \\ \cdot & -i \end{pmatrix}, \\ j &\leftrightarrow \begin{pmatrix} \cdot & -1 \\ 1 & \cdot \end{pmatrix}, & k &\leftrightarrow \begin{pmatrix} \cdot & -i \\ -i & \cdot \end{pmatrix}, \end{aligned} \tag{13}$$

$$1|i \leftrightarrow \begin{pmatrix} i & \cdot \\ \cdot & i \end{pmatrix}, \qquad i|i \leftrightarrow \begin{pmatrix} -1 & \cdot \\ \cdot & 1 \end{pmatrix}, \tag{14}$$

This translation is valid for all operators while for *states* we use the symplectic representation

$$q = z_1 + j z_2 \leftrightarrow \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} . \tag{15}$$

This still leaves out all odd-dimensional (complex) operators in QM characteristic of the standard bosonic equations for particles with mass, such as the Klein–Gordon equation. There is an exception to this last comment that we wish to investigate further, namely the Kemmer equation[12]

$$\beta^\mu \partial_\mu \phi = m\phi \tag{16}$$

(here the $1|i$ of the momentum operator has been absorbed into β^μ, whose elements must anyway be assumed to be generalized quaternions). β^μ satisfy the Duffin–Kemmer–Petiau condition[13–15]

$$\beta^\mu \beta^\nu \beta^\lambda + \beta^\lambda \beta^\nu \beta^\mu = -g^{\mu\nu} \beta^\lambda - g^{\lambda\nu} \beta^\mu . \tag{17}$$

This implies that β^μ are not invertible so that this equation *cannot* be written in the Dirac form (4). Equation (17), however, guarantees that each element of ψ satisfies the Klein–Gordon equation. The Kemmer equation has spin content 0 and 1 and the representations for the scalar particle are five-dimensional. There also exists, however, a trivial one-dimensional solution ($\beta^\mu \equiv 0$) which if added to the spin 0 representation yields a six-dimensional representation which can be translated into 3×3 generalized quaternions:

$$\begin{aligned} \beta^0 &= \begin{pmatrix} \cdot & \cdot & a \\ \cdot & \cdot & \cdot \\ -a & \cdot & \cdot \end{pmatrix} , \qquad \beta^1 = j \begin{pmatrix} \cdot & \cdot & a \\ \cdot & \cdot & \cdot \\ -d & \cdot & \cdot \end{pmatrix} , \\ \beta^2 &= \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & a \\ \cdot & a & \cdot \end{pmatrix} , \qquad \beta^3 = j \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & a \\ \cdot & -d & \cdot \end{pmatrix} , \end{aligned} \tag{18}$$

with

$$a = \frac{1 - i|i}{2} , \qquad d = \frac{1 + i|i}{2} .$$

Now, before proceeding we must digress to describe the so-called anomalous solutions, in particular those of the Klein–Gordon equation. In QM this equation,

$$(\partial^\mu \partial_\mu + m^2)\phi = 0 , \tag{19}$$

has two solutions (positive and negative energy),

$$\phi = e^{-ipx} , \qquad p_0 = \pm\sqrt{\mathbf{p}^2 + m^2} . \tag{20}$$

With quaternions and with a complex geometry the number of solutions doubles; in addition to Eq. (20) we have the complex orthogonal solutions

$$\phi = je^{-ipx}\,, \qquad p_0 = \pm\sqrt{\mathbf{p}^2 + m^2}\,. \tag{21}$$

These are the *anomalous* or pure quaternion solutions.

This doubling of solutions does *not* occur for our Dirac equation. The reason is that the doubling of solutions is there compensated for by the reduced number of spinor components. The question is: What happens in the Kemmer equation? Direct analogy with the Dirac equation is not possible because the number of solutions no longer corresponds to the number of components of ψ. However, we can begin with our Kemmer equation, find the explicit (nontrivial) solutions and simply count them or express them in derivative terms (possible for Kemmer but not for Dirac) so as to obtain the second order equivalent equation.

In fact the solutions to our Kemmer equation are only two in number:

$$\psi = \begin{pmatrix} -\frac{ip_0 + kp_x}{m} \\ \frac{ip_y - kp_z}{m} \\ 1 \end{pmatrix} e^{-ipx}\,, \qquad p_0 = \pm\sqrt{\mathbf{p}^2 + m^2}\,. \tag{22}$$

This can be rewritten in terms of the ϕ in Eq. (20):

$$\psi = \begin{pmatrix} \frac{(\partial_t + j\partial_x)\phi}{m} \\ \frac{(\partial_y + j\partial_z)\phi}{m} \\ \phi \end{pmatrix}, \tag{23}$$

from which we derive the necessary and sufficient equation for the scalar field ϕ:

$$\frac{1 - i|i}{2}(\partial^\mu \partial_\mu + m^2)\phi = 0\,. \tag{24}$$

This is what we shall call the *modified Klein–Gordon equation.* It does not have anomalous solutions because the projection operator $\frac{1-i|i}{2}$ kills all j, k terms.

There also exists the alternative modified Klein–Gordon equation

$$\frac{1 + i|i}{2}(\partial^\mu \partial_\mu + m^2)\phi = 0\,, \tag{25}$$

which kills the complex solutions. Note that Eqs. (24) and (25) are related by a "quaternion similarity" transformation:

$$\frac{1 - i|i}{2} \to -j\left(\frac{1 - i|i}{2}\right)j = \frac{1 + i|i}{2}$$

and

$$\phi \to -j\phi\,. \tag{26}$$

All of this tells us that we may readily eliminate the anomalous solutions by invoking the *modified bosonic equations.* The correct equation and the corresponding Lagrangian are thus in practice determined only when the number of particles in the theory is fixed. For the Higgs of the next section we shall use the standard Klein–Gordon equation which contains four particles (parity apart).

3. The Higgs Sector

We know that before spontaneous symmetry breaking the minimal number of Higgs is four: $\mathcal{H}^0$, $\mathcal{H}^+$, $\bar{\mathcal{H}}^0$, $\mathcal{H}^-$. We therefore adopt as a consequence of the count of states of the previous section a free Higgs Lagrangian which yields the Klein–Gordon equation

$$\mathcal{L}_{\text{free}} = \partial_\mu \phi^+ \partial^\mu \phi\,, \tag{27}$$

where ϕ is a massless quaternion field. The field equation

$$\partial_\mu \partial^\mu \phi = 0 \tag{28}$$

is obviously invariant under the global group $\mathrm{U}(1,q)|\mathrm{U}(1,c)$:

$$\phi \to e^{i\alpha + j\beta + k\gamma} \phi e^{-i\delta}\,, \qquad \alpha,\ \beta,\ \gamma,\ \delta \text{ real parameters}\,. \tag{29}$$

The limitation of the right-acting group to $\mathrm{U}(1,c)$ instead of $\mathrm{U}(1,q)$ follows from the implicit additional requirement that the complex plane wave structure be conserved. In first quantization this would correspond to maintaining the given momentum; in second quantization, to the desire of not assigning a creation or annihilation operator with the incorrect plane wave structure, which would then violate the corresponding Heisenberg equation[6] (or yield negative energies).

In order to impose this maximal group invariance of the field equation upon the free Lagrangian, we must *assume* that the Lagrangian is defined as a complex projection (in addition to the Hermitian nature of $\mathcal{L}$, which, however, involves the creation and annihilation operators).

Thus in fact we can define the Lagrangian density $\mathcal{L}$ as

$$\mathcal{L}_{\text{free}} = \frac{1 - i|i}{2} (\partial_\mu \phi^+ \partial^\mu \phi) \equiv (\partial_\mu \phi^+ \partial^\mu \phi)_C\,. \tag{30}$$

This complex projection is automatic for spin $\frac{1}{2}$ fields in order to reproduce the standard form of the Dirac equation from the variational principle. In fact ψ and ψi must be varied independently in analogy with ψ^+ and ψ, but we shall not go into detail here.

Any complex projection under extreme right or left multiplication by a complex number behaves as follows:

$$(z\mathcal{L}z')_C = z(\mathcal{L})_C z' = zz'(\mathcal{L})_C\,. \tag{31}$$

Thus if $zz' = 1$ we have invariance. When the transformation is attributed to the ϕ field in Eq. (30), this implies that $z' = z^*$ and hence

$$z \in U(1, c) . \tag{32}$$

It is obvious that if this complex projection is generalized to all terms in $\mathcal{L}$ the standard Higgs Lagrangian is an invariant under $\mathrm{U}(1,q)|\mathrm{U}(1,c)$, since $(\phi^+\phi)_{\mathcal{C}}$ is.[a] Hence at this level our global invariance group is isomorphic with the Glashow group $SU(2,c) \times \mathrm{U}(1,c)$. We must, however, remember that the group representations are not totally isomorphic.

In the classical field treatment of spontaneous symmetry breaking we want the field $\mathcal{H}^0$ to develop a constant *real* vacuum expectation value. This fixes the neutral Higgs to be purely complex fields (the anomalous fields have no real part). This in turn fixes the $\mathrm{U}(1)_{\mathrm{em}}$ gauge, which will survive spontaneous symmetry breaking. Indeed, the requirement of invariance of the neutral Higgs under $\mathrm{U}(1)_{\mathrm{em}}$ can only be achieved if $\mathcal{H}^0$ is a complex field in accordance with the above argument, and

$$\mathrm{U}(1)_{\mathrm{em}} = e^{ig\alpha}|e^{-ig'\delta} , \tag{33}$$

with

$$g\alpha = g'\delta , \tag{34}$$

so that the phases cancel after commuting with the Higgs field. The different signs of the arguments in Eq. (33) are nothing other than a convention of the authors. Here we have explicitly used the (real) coupling strengths g, g' characteristic of the Glashow group. We observe the analogy of the above result with the standard theory, where, however, one must assume the weak isospin, weak hypercharge and electric charge relationship. Here we appear to have no freedom of choice.

We have a certain number of observations to make:

(1) The above result fixes the mode of minimal coupling (see below).
(2) Since under $\mathrm{U}(1)_{\mathrm{em}}$ the complex Higgs is neutral, the anomalous Higgs (pure j, k) are necessarily charged.
(3) Had we imposed by fiat the condition that the $\mathrm{U}(1)_{\mathrm{em}}$ be either the (weak hypercharge) right-acting $\mathrm{U}(1,c)$ or the left-acting $\mathrm{U}(1,c)$ subgroup of $\mathrm{U}(1,q)$, we would have modified the sense of minimal coupling and imposed a common electric charge on all the Higgs fields. If $g\alpha \neq g'\delta$ we would also have had four charged Higgs fields but with different charges.

We return for some further comments upon the complex projection of the Lagrangian density. We already noted that this condition is obligatory in the Dirac sector, and therefore it is natural to assume it to be a property of the full Lagrangian. One may object that in classical field theory the Lagrangian is real anyway, so that a complex projection is irrelevant. However, for quantum fields the reality of $\mathcal{L}$ is

[a]The quartic term of the Higgs Lagrangian will also be assumed in the more restrictive form of $|\lambda|{(\phi^+\phi)_{\mathcal{C}}}^2$, so that the plane wave factors of all ϕ, ϕ^+ fields may be factorized as in normal QM.

substituted by the hermiticity of $\mathcal{L}$, so that $\mathcal{L}$ is *not* in general real. Furthermore, even for classical field theory the reality of $\mathcal{L}$ does not in general extend to the variations $\delta\mathcal{L}$, which may be complex (and for us even quaternion). Thus it is for these variations that the complex projection plays a nontrivial role.

We conclude this section by explicitly writing the Higgs part of our electroweak Lagrangian density:

$$\mathcal{L}^{\mathcal{H}} = (\partial_\mu \phi^+ \partial^\mu \phi)_{\mathcal{C}} - \mu^2 (\phi^+ \phi)_{\mathcal{C}} - |\lambda| (\phi^+ \phi)_{\mathcal{C}}{}^2 . \tag{35}$$

Note that the quartic potential term is a product of complex projections and not merely the complex projection of a product (see footnote a).

4. Local Group Invariance and Minimal Coupling

The content of this section follows faithfully the standard procedure, so that we sketch only the various steps. We wish to impose a local gauge invariance [parameters $\boldsymbol{\theta} \equiv (\alpha, \beta, \gamma)$ and δ with x^μ dependence]. In order to compensate for the derivative terms that then appear in the Lagrangian, we introduce four Hermitian vector fields by the following substitution:

$$\partial^\mu \to \partial^\mu + \frac{g}{2} \mathbf{Q} \cdot \mathbf{W}^\mu - \frac{g'}{2} B^\mu | i , \tag{36}$$

where $\mathbf{Q} \equiv (i, j, k)$ are the quaternion imaginary units. The gauge fields have the well-known but peculiar gauge transformation properties. To find them we impose the condition that

$$(\mathcal{D}_\mu \phi)' = U (\mathcal{D}_\mu \phi) V , \tag{37}$$

where U and V characterize the transformation of the scalar field ϕ,

$$\phi(x) \to \exp\left[\frac{g}{2} \mathbf{Q} \cdot \boldsymbol{\theta}(x)\right] \phi(x) \exp\left[- i \frac{g'}{2} \delta(x)\right] = U \phi V ,$$

and $\mathcal{D}_\mu$ represents the covariant derivative

$$\partial_\mu \to \mathcal{D}_\mu = \partial_\mu + \tilde{W}_\mu + \tilde{B}_\mu ,$$

with

$$\tilde{W}_\mu = \frac{g}{2} \mathbf{Q} \cdot \mathbf{W}^\mu ,$$

$$\tilde{B}_\mu = - \frac{g'}{2} B^\mu | i .$$

Therefore we have

$$(\mathcal{D}_\mu \phi)' = (\partial_\mu U) U^{-1} \phi' + U (\partial_\mu \phi) V + \phi' V^{-1} (\partial_\mu V) + \tilde{W}'_\mu \phi' + \tilde{B}'_\mu \phi' , \tag{38}$$

$$U (\mathcal{D}_\mu \phi) V = U (\partial_\mu \phi) V + U \tilde{W}_\mu U^{-1} \phi' + U \tilde{B}_\mu U^{-1} \phi' . \tag{39}$$

By confronting Eqs. (38) and (39) and noting that U commutes with $\tilde{B}_\mu$, we find that

$$\tilde{W}'_\mu = U\tilde{W}_\mu U^{-1} - (\partial_\mu U)U^{-1}, \tag{40}$$

$$\tilde{B}'_\mu = \tilde{B}_\mu - 1|V^{-1}(\partial_\mu V). \tag{41}$$

The infinitesimal transformations for the gauge fields are

$$\begin{gathered}\mathbf{Q}\cdot\mathbf{W}^\mu \to \mathbf{Q}\cdot\mathbf{W}^\mu - \mathbf{Q}\cdot\theta + \frac{g}{2}[\mathbf{Q}\cdot\partial^\mu\theta, \mathbf{Q}\cdot\mathbf{W}^\mu]\\ (\mathbf{W}^\mu \to \mathbf{W}^\mu - \partial^\mu\theta + g\theta\wedge\mathbf{W}^\mu),\end{gathered} \tag{42}$$

$$B^\mu \to B^\mu - \partial^\mu\delta. \tag{43}$$

Since we have already identified the electromagnetic gauge group, we can already anticipate the residual gauge invariance in terms of the electromagnetic field A^μ. By remembering that we can write[b] W_1^μ and B^μ as a linear combination of A^μ and Z^μ,

$$\begin{aligned}B^\mu &= \cos\theta_W A^\mu - \sin\theta_W Z^\mu,\\ W_1^\mu &= \sin\theta_W A^\mu + \cos\theta_W Z^\mu,\end{aligned} \tag{44}$$

we have

$$\partial^\mu \to \partial^\mu + \frac{e}{2}A^\mu(i - 1|i) \quad (e: \text{ electric charge}), \tag{45}$$

which can be written in terms of the quaternion projection operator $(1 + i|i)$ as

$$\begin{gathered}\partial^\mu|i \to \partial^\mu|i + \frac{e}{2}A^\mu(1 + i|i),\\ \text{with } e = \frac{gg'}{\sqrt{g'^2 + g^2}} = g\sin\theta_W = g'\cos\theta_W,\end{gathered} \tag{46}$$

but with the understanding that the projection operator acts upon the scalar field ϕ, and guarantees the charge neutrality (invariance) of the pure complex fields. For simplicity it is convenient to think of the gauge fields as classical real fields. Thus their position within the Lagrangian density is irrelevant. In second quantization the situation is somewhat more complicated. The gauge fields are Hermitian operators that act upon the kets (essentially the vacuum) and are *bared* operators with the plane wave structures as right-acting factors. In this way standard energy–momentum conservation occurs without the complication of noncommutativity. Henceforth, unless stated otherwise, we therefore treat the gauge fields as real classical fields. It is nonetheless interesting to note that if the $\mathbf{Q}$ factors are absorbed within the definition of the gauge fields then W_1^μ and B^μ become complex

[b] With our convention, W_1^μ (and not W_3^μ as in the standard model) is the neutral member of the *weak isospin* triplet.

(indeed pure imaginary in this classical limit) while W_2^μ and W_3^μ are both pure quaternion (j, k) or anomalous in our terminology.

We also recall for completeness here the form of the gauge kinetic terms

$$\mathcal{L}^B = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W^+_{\mu\nu} \cdot W^{\mu\nu}, \tag{47}$$

where

$$B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu, \tag{48}$$

$$W^{\mu\nu} = \mathbf{Q} \cdot (\partial^\mu \mathbf{W}^\nu - \partial^\nu \mathbf{W}^\mu) + \frac{g}{2}[\mathbf{Q} \cdot \mathbf{W}^\mu, \mathbf{Q} \cdot \mathbf{W}^\nu]. \tag{49}$$

5. Conclusions

In this work we have studied the Higgs sector of the electroweak theory from the point of view of quaternion quantum mechanics (QQM) with a complex geometry. The Higgs fields are assumed to be four in number and this coincides with the number of solutions (counting both positive and negative energies — particles and antiparticles) of the standard Klein–Gordon equation *within* QQM. We have also shown that the quaternion global invariance group of the one-component Klein–Gordon equation is $\mathrm{U}(1,q)|\mathrm{U}(1,c)$-isomorphic at the Lie algebra level with the Glashow group. The right-acting phase transformation is limited to the complex numbers because it must not modify the four-momentum of the specific solution considered.

The hypothesis that this group is the invariance group for the Lagrangian density then imposes an overall complex projection of the Lagrangian density. This result is *consistent* with the need for a complex projection for the Dirac Lagrangian density in order to obtain the Dirac field equation. We have pointed out that a complex projection is nontrivial because it automatically kills the j–k quaternion variations in $\delta\mathcal{L}$ which naturally occur when fields and their adjoint are varied independently.

As an aside we have shown that there exist *modified Klein–Gordon equations* (the same applies to Maxwell equations, etc.) with only half of the solutions of the standard equations. Thus anomalous solutions can always be eliminated if so desired. Although this result seems obvious *a posteriori*, it was derived from a study of the quaternion Kemmer equation and we have sketched the essential steps in Sec. 2. Thus the use of the standard Klein–Gordon equation is an assumption in QQM with physical consequence (e.g. the invariance group) and not obligatory for scalars as in complex quantum mechanics. It is, however, to be emphasized that once the desired field equation has been chosen the invariance group is fixed, unlike the normal situation, in which there is no relationship between the Klein–Gordon equation and the multiplicity structure of Higgs fields under $\mathrm{SU}(2) \times \mathrm{U}(1)$.

Spontaneous symmetry breaking of the neutral Higgs field then determines the resultant form of the residual $\mathrm{U}(1)_{\mathrm{em}}$. Specifically it is the complex subgroup of $\mathrm{U}(1,q)|\mathrm{U}(1,c)$ with rotation angles equal in magnitude but opposite in signs. As a consequence the two other Higgs fields are anomalous and charged. The significance of anomalous Higgs fields is thus connected with their electric charge. *This is the first time that a physical property has been associated with pure quaternion fields.*

Our justification for a complex-projected Lagrangian density in the case of the Higgs sector becomes a derivation within the fermion sector, which we have only outlined in this work. The assumption that all Lagrangian densities be complex-projected then implies that all symmetry groups will necessarily have a $G|\mathrm{U}(1,c)$ structure. This is particularly significant for grand-unified theories.[16] We note that recently much attention has been paid to the complex group $\mathrm{SU}(5)\times\mathrm{U}(1)$.[17] Within QQM we suggest that $\mathrm{SU}(3,Q)|\mathrm{U}(1,c) \sim \mathrm{SU}(6)\times\mathrm{U}(1)$ be a natural candidate for grand unification.

We conclude by recalling the main point of this work. The need for four Higgs fields *suggests* the use of the standard Klein–Gordon equation. This equation is invariant under the group $\mathrm{U}(1,q)|\mathrm{U}(1,c)$. The alternative choice of two modified Klein–Gordon equations would be invariant only under $\mathrm{U}(1,c)|\mathrm{U}(1,c)$, which, beyond being purely complex in contradiction with the spirit of the use of quaternions and quaternion groups, would not give rise to the charged intermediate vector bosons. We therefore claim that the natural group of any quaternion Lagrangian is of the form $G|\mathrm{U}(1,c)$, the simplest (lowest-dimensional) unitary group being $G=\mathrm{U}(1,q)\sim\mathrm{SU}(2,c)$. In this sense the Glashow gauge group appears *naturally* as the choice of the minimal quaternion unitary group for G.

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