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Representations of $U(1, q)$ and Constructive Quaternion Tensor Products.

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Summary. — The representation theory of the group $U(1, q)$ is discussed in detail because of its possible application in a quaternion version of the Salam-Weinberg theory. As a consequence, from purely group-theoretical arguments we demonstrate that the eigenvalues must be right eigenvalues and that the only consistent scalar products are the complex ones. We also define an explicit quaternion tensor product which leads to a set of additional group representations for integer «spin».

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1. – Introduction.

Quaternions have been somewhat an enigma for physicists since their discovery by Hamilton [1] in 1843. Notwithstanding Hamilton's conviction that quaternions would soon play a role comparable to, if not greater than, that of complex numbers, the use of quaternions in physics is very limited. Amongst the contributions to quaternion quantum mechanics we draw attention to the fundamental works of Finkelstein *et al.* [2-5] (on foundations of quaternion quantum mechanics, on quaternionic representations of compact groups, etc.), of Horwitz and Biedenharn [6] (on quaternion quantum mechanics, second quantization and gauge fields) and to the many stimulating papers of Adler [7-9] (on quaternion potentials and *CP* violation, on quaternion field theory, etc.).

Complex numbers in physics have played a dual role, first as a technical tool in resolving differential equations (*e.g.*, in classical optics) or via the theory of analytic functions for performing real integrations, summing series etc.; secondly, in a more essential way in the development of quantum mechanics (and later field theory) characterized by complex wave functions and for fermions by complex wave equations. With quaternions, for the first type of application, *i.e.* as a means to simplify calculations, we can quote the original work of Hamilton, but this only because of the late development of vector algebra. Even Maxwell [10] used quaternions as a tool in his calculations. The more exciting possibility that quaternion equations will eventually play a significant role is synonymous, for some physicists (but not for the present authors), with the advent of a revolution in physics comparable to that of quantum mechanics.

Our own particular point of view is that even if quaternions do not simplify calculations, it would be very strange if standard quantum mechanics did not permit a quaternion description other than in the trivial sense that complex numbers are contained within the quaternions. In other words, given the validity of quantum mechanics at the elementary-particle level, we predict the existence, even at this level, of quaternion versions of all standard theories. One of the authors [11, 12] has indeed succeeded in this with a quaternion version of the Dirac equation. This equation, thanks to the use of the complex scalar product, reproduces the standard results notwithstanding the two-component nature of the wave functions due to the existence of two-dimensional quaternion gamma matrices. The same doubling of solutions implies that the Schrödinger equation has automatically two plane-wave solutions corresponding to spin up and spin down. This doubling of solutions continues even for bosonic equations. As a result, two photonic solutions exist (one called anomalous), two scalar solutions of the Klein-Gordon equation exist and so forth. It has also been demonstrated [13] that these new anomalous solutions can be associated with corresponding anomalous fields. We observe that the existence of these new solutions implies that the use of quaternions is not without predictive power, at least with the formalism described above.

Coherent with our point of view, it appears to us desirable to develop a quaternion version of the Salam-Weinberg theory of electroweak interactions. This theory may also provide a proving ground for the anomalous particles. For it is possible that the anomalous photon be identified with one of the massless neutral intermediate vector fields, prior to the creation of mass via spontaneous symmetry breaking. In other words, it has been suggested [13] that the anomalous photon could be identified with the Z^0.

As a preliminary to this non-trivial objective, one must decide upon the appropriate quaternion version of the Glashow group $SU(2, c) \times U(1, c)$. The c here specifies *complex group* and implies only complex matrix elements, *i.e.* standard group theory. A q within a group name will imply a quaternion group with, in general, quaternion matrix elements, even if this does not exclude the appearance of purely complex or even real group representations. Surprisingly, the complex group $U(1, c)$ remains as such even for a quaternion version of Salam-Weinberg. This is not difficult to justify, but we leave this explanation to a subsequent article. The group $SU(2, c)$ is particularly interesting, first because this Lie group is not only the weak isospin group of Salam-Weinberg, but is very common in particle physics (spin, isospin, etc.) and second because we do indeed have an alternative choice in the quaternion unitary group $U(1, q)$, also referred to as the symplectic group $Sp(1, q)$. It is well known that these groups are isomorphic to $SU(2, c)$ [14]. However, as we shall demonstrate in this paper, this does not guarantee identical physical content. For example, with the complex group $SU(2, c)$ all representations are obtainable from the spinor representation with the aid of tensor products. This will not be the case for $U(1, q)$, and indeed the definition of a suitable quaternion tensor product is still of primary interest [15-17].

In the next section we shall develop the representation theory of $U(1, q)$ in analogy with that of $SU(2, c)$ (we shall henceforth use the term «spin» to identify the physical observable associated with these groups). In particular, we obtain a group-theoretical justification both for the use of the right eigenvalue equations and for the adoption of the complex scalar product. We also derive a set of representations for each «spin» value. In sect. 3 we define an explicit right complex linear tensor product in terms of *quaternion* column matrices, suggested by the work of Horwitz and Biedenharn. As a consequence, we discover additional non-equivalent (see appendix B) matrix representations for integer spins, characterized by the absence of anomalous eigenstates. The physical significance of these results is discussed in our conclusions in the final section.

2. – Comparison of $U(1, q)$ and $SU(2, c)$ representations.

The representation theory of $SU(2, c)$ is well known. We recall, in particular, the importance of the Pauli matrices which represent twice the generators of «spin» 1/2. The unitary quaternion group $U(1, q)$ may be usefully confronted with both $U(1, c)$ and $SU(2, c)$. $U(1, q)$ is in fact the natural generalization of $U(1, c)$ to the non-commutative quaternion numbers, defined by

$$q = q_0 + q_1 i + q_2 j + q_3 k \quad (q_m \in \boldsymbol{R},\ m = 0, \ldots, 3)\,, \tag{1}$$

where there are three imaginary elements i, j, k ($i^2 = j^2 = k^2 = -1$) and

$$ijk = -1\,. \tag{2}$$

The complex numbers $C(1, i)$, with bases 1 and i, are a subset of the quaternions. More precisely, there are infinite, *a priori*, equivalent complex planes in the four-dimensional quaternion space. The above plane will, however, be identified with the standard complex numbers.

The Abelian group $U(1, c)$ is therefore represented by the one-dimensional general group elements

$$g \sim \exp[-i\alpha] \qquad \alpha \in \boldsymbol{R}\,. \tag{3}$$

The non-Abelian group $U(1, q)$ is represented at the lowest non-trivial level by

$$g \sim \exp[-i\alpha - j\beta - k\gamma]\,, \qquad \alpha, \beta, \gamma \in \boldsymbol{R}\,. \tag{4}$$

If we identify the numbers $i/2, j/2, k/2$ as the generators of the group, these define a Lie algebra with *anti-Hermitian* generators A_m ($m = 1, 2, 3$) satisfying

$$[A_m,\ A_n] = \varepsilon_{mnp}\, A_p\,. \tag{5}$$

This equation is readily identified with that of the Lie algebra $su(2, c)$ if one considers as generators of the corresponding group $-i\sigma_m/2$, where σ_m for $m = 1, 2, 3$ are the Pauli matrices. This is the connection between $U(1, q)$ and $SU(2, c)$ which, as a consequence, are isomorphic groups [14]. We shall demonstrate in this work that the isomorphism of the abstract groups is not automatically reflected in the corresponding matrix representations. Indeed, we anticipate that we shall find for $U(1, q)$ two inequivalent matrix representations for the same integer spin values. We shall also encounter generators which are formally irreducible into smaller block structures, but which operate on a vector space which is fully reducible.

We begin our comparison between $U(1, q)$ and $SU(2, c)$ by choosing a *polarization direction*, say that corresponding to the $i/2$ generator. In $SU(2, c)$ the choice of the polarization direction leads naturally to the choice of the corresponding eigenvectors as a basis in spin space and the automatic diagonalization of the generator. This is of course not the case for $U(1, q)$ since all three generators, being one-dimensional, are already «diagonal». We can, however, still seek the eigenvectors for $i/2$. However, before doing so, we need to recall some facts involving quaternions.

First note that for any quaternion operator A with eigenvector ψ and eigenvalue a, we have, *a priori*, two options for the eigenvalue equation, either the left eigenvalue equation

$$A\psi = a\psi \,, \tag{6}$$

or the right eigenvalue equation

$$A\psi = \psi a \,. \tag{7}$$

We also observe at this point that the most general quaternion operator A is of the «bared» type $A = B|b$ defined by its action upon any state vector ψ,

$$A\psi \equiv B\psi b \,. \tag{8}$$

This complication is due to the non-commutative nature of quaternions. Obviously, the modulus of b may always be set to unity without loss of generality. We further recall that with the adjoint (†) defined as

$$(q_0 + q_1 i + q_2 j + q_3 k)^\dagger = q_0 - q_1 i - q_2 j - q_3 k \tag{9}$$

and

$$(q_1 q_2)^\dagger = q_2^\dagger \, q_1^\dagger \tag{10}$$

the norm of each quaternion is given by

$$|q|^2 = q^\dagger q > 0 \tag{11}$$

and thus each non-null quaternion has an inverse

$$q^{-1} = \frac{q^\dagger}{|q|^2} \,. \tag{12}$$

Now if, as in our case, $A^\dagger = -A$ then, depending upon the use of the left or right eigenvalue equation, we will have, either

$$\psi^\dagger A\psi = \psi^\dagger a\psi \tag{13}$$

or

$$\psi^\dagger A\psi = \psi^\dagger \psi a \,. \tag{14}$$

In either case by taking adjoints we readily demonstrate that a does not have a real part,

$$a^\dagger = -a \tag{15}$$

(notice that in the latter case we need the fact that $\psi^\dagger\psi$ is real and hence commutes with any a).

At this point we run the risk of having infinite eigenvectors for (spin) $s = 1/2$.

To impose only a finite number of solutions to our eigenvalue equations, we select a preferential complex plane, that based upon the units 1 and i. We then require that the eigenvalues be proportional to the i unit. This accords well with the eigenvalues of $-i\sigma_m/2$, the $SU(2, c)$ counterpart. Now our two choices of eigenvalue equation may be written explicitly as

$$\frac{i}{2}\psi_{\mathrm{L}} = (-i\lambda)\psi_{\mathrm{L}} \tag{16}$$

or

$$\frac{i}{2}\psi_{\mathrm{R}} = \psi_{\mathrm{R}}(-i\lambda) \tag{17}$$

with $\lambda \in \boldsymbol{R}$. It is easy to solve these equations. The left eigenvalue equation is right quaternion linear (if ψ is a solution, so is ψq with q any quaternion) but it has only one eigenvalue compared to the two ($\lambda = \pm 1/2$) of $SU(2, c)$. Only the right eigenvalue equation (eq. (17)) yields the two desired eigenvalues, corresponding, respectively, to the eigenvectors $\psi = jz$ and $\psi = z'$, where z and z' are arbitrary $C(1, i)$ complex numbers. These solutions are characterized by being only right complex linear. Ignoring for the moment the right complex phases (z, z'), we can summarize our results by saying that to conform with standard $SU(2, c)$ results for spin 1/2 we are obliged to choose the right eigenvalue equation and, as a consequence, obtain the solutions

$$\psi_+ = j \Leftrightarrow \lambda = \frac{1}{2}, \tag{18}$$

$$\psi_- = 1 \Leftrightarrow \lambda = -\frac{1}{2}. \tag{19}$$

The choice of the generator is, as expected, not of any significance in all this. If we had «diagonalized» the $k/2$ generator, we would not have found any left eigenvectors at all, but exactly two right eigenvectors $(1 \pm j)z$ with the same λ eigenvalues $\pm 1/2$ found previously.

We have not yet finished with spin 1/2, in $SU(2, c)$ the two eigenvectors are automatically orthogonal. This is not the case for $U(1, q)$. Indeed in order to impose orthogonality we are obliged to adopt what is known as the «complex scalar product». The quaternion scalar product is defined as a straightforward generalization of the standard complex counterpart. Thus for *simple numbers* this is defined by

$$\langle f | g \rangle \equiv f^{\dagger} g. \tag{20}$$

The complex scalar product with quaternions, is simply the projection of the above upon the privileged complex plane, *i.e.* it is defined as follows:

$$(f | g) \equiv \frac{1}{2}\left[\langle f | g \rangle - i \langle f | g \rangle i\right]. \tag{21}$$

In this way 1 and j are complex orthogonal, as it occurs for the corresponding eigenvectors in $SU(2, c)$.

It is interesting to recall that the complex scalar product was first proposed by Rembieliński in 1980 [18-20] and later by Horwitz and Biedenharn in 1984 [6] for quaternion quantum mechanics as a means of performing quaternion tensor products and defining a suitable Fock space. Subsequently, it was rederived from a quaternion Dirac equation by the natural requirement that the (non-conventional) momentum operator be Hermitian. The present derivation has the merit of being a consequence of group-theoretical arguments, independent of any physical dynamical equation of motion. It must however be admitted that Adler has in recent years become a fervent advocate of the use of a quaternion scalar product. It would not then be possible to reproduce with $U(1, q)$ the properties of $SU(2, c)$. Indeed we suspect that only the *complex groups* could be employed with this purer quaternion hypothesis, if one wishes to preserve standard group theory in physics. Paradoxically, this contradicts the *a priori* rejection of a preferential complex plane.

We also observe that with the introduction of the complex scalar product it is possible to pass from the intrinsic anti-Hermitian nature of the generators of $U(1, q)$ to an equivalent (complex) Hermitian set J_m defined by

$$\frac{i}{2},\ \frac{j}{2},\ \frac{k}{2} \quad\Leftrightarrow\quad \frac{i}{2}\,|\,i,\ \frac{j}{2}\,|\,i,\ \frac{k}{2}\,|\,i \tag{22}$$

which satisfies the Lie algebra for the *generalized angular momentum* if this algebra is redefined as

$$[J_m,\ J_n] = \varepsilon_{mnp}\, J_p\,|\,i\,. \tag{23}$$

From this algebra, following the standard method of derivation (with some care in the position of the i factors) it is possible to derive the well-known angular momentum spectrum. We emphasize that this can only be done after the adoption of the complex scalar product.

As an aid in determining the quaternion generators for higher spin values, we observe that the $-i\sigma_m/2$ of $SU(2, c)$ are reducible with quaternions (appendix A) to a diagonal form with our generators A_m as elements (this is not true for $\pm\sigma_m/2$ *without* the i factor). Thus we expect that all semi-integer spin value representations can be derived by reduction from the corresponding $SU(2, c)$ generators, yielding matrices with exactly half the dimensions.

For integer spin values the complex representations derivable from those of $SU(2, c)$ are not reducible even with quaternions, but we shall justify this claim only in appendix B. As a consequence, for integer spins, we encounter «anomalous» eigenvector solutions obtained from any standard complex eigenvector by simply multiplying from the right by j. For example, for spin 1 the complete set of eigenvectors are

$$\begin{pmatrix}1\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\1\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\1\end{pmatrix};\ \begin{pmatrix}1\\0\\0\end{pmatrix}j,\ \begin{pmatrix}0\\1\\0\end{pmatrix}j,\ \begin{pmatrix}0\\0\\1\end{pmatrix}j$$

and, furthermore, the two sets of states divided by the semi-colon above are manifestly invariant subspaces (recall that the generators and hence group elements are complex for this case). Thus we encounter the situation, anticipated in the introduction, of non-reducible matrix representations for the generators o $U(1, q)$ notwithstanding a completely reducible eigenvector space.

Let us list the first few representations found,

$$s=0: \quad A_1=A_2=A_3=0,$$

$$s=\frac{1}{2}: \quad A_1=\frac{i}{2},\ A_2=\frac{j}{2},\ A_3=\frac{k}{2},$$

$s=1:$ standard complex representation,

$$s=\frac{3}{2}: A_1=\begin{bmatrix}\frac{3i}{2} & 0\\ 0 & \frac{i}{2}\end{bmatrix},\quad A_2=\begin{bmatrix}0 & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & j\end{bmatrix},\quad A_3=\begin{bmatrix}0 & \frac{\sqrt{3}i}{2}\\ \frac{\sqrt{3}i}{2} & k\end{bmatrix}$$

We can tabulate the general half-integer spins representation as follows

$$(24)\quad \left\{\begin{aligned}
A_1&=\begin{bmatrix}si & 0 & 0 & \cdot & 0 & 0\\ 0 & (s-1)\,i & 0 & \cdot & 0 & 0\\ 0 & 0 & (s-2)\,i & \cdot & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{3}{2}i & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\end{bmatrix},\\
A_2&=\begin{bmatrix}0 & a & 0 & \cdot & 0 & 0\\ -a & 0 & b & \cdot & 0 & 0\\ 0 & -b & 0 & \cdot & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & v & 0\\ 0 & 0 & 0 & -v & 0 & z\\ 0 & 0 & 0 & 0 & -z & aj\end{bmatrix},\\
A_3&=\begin{bmatrix}0 & ai & 0 & \cdot & 0 & 0\\ ai & 0 & bi & \cdot & 0 & 0\\ 0 & bi & 0 & \cdot & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & vi & 0\\ 0 & 0 & 0 & vi & 0 & zi\\ 0 & 0 & 0 & 0 & zi & ak\end{bmatrix}.
\end{aligned}\right.$$

The conditions

$$(25)\qquad \begin{cases} A^2 \equiv A_1^2 + A_2^2 + A_3^2 = -s(s+1), \\ \varepsilon_{mnp} A_p = [A_m, A_n], \\ A_m^\dagger = -A_m \end{cases}$$

then determine the corresponding matrix elements

$$(26)\qquad \begin{cases} a = \sqrt{\dfrac{s}{2}}, & d = \sqrt{\dfrac{4s-6}{2}}, \\ b = \sqrt{\dfrac{2s-1}{2}}, & e = \sqrt{\dfrac{5s-10}{2}}, \\ c = \sqrt{\dfrac{3s-3}{2}}, & \text{etc.} \end{cases}$$

For the first few half-integer spins these results are collected in table I.

TABLE I. – *Half-integer spin matrix element coefficients.*

spin	a	b	c	d	e	f	α
$\frac{1}{2}$							$\frac{1}{2}$
$\frac{3}{2}$	$\sqrt{\frac{s}{2}}$						1
$\frac{5}{2}$	$\sqrt{\frac{s}{2}}$	$\sqrt{\frac{2s-1}{2}}$					$\frac{3}{2}$
$\frac{7}{2}$	$\sqrt{\frac{s}{2}}$	$\sqrt{\frac{2s-1}{2}}$	$\sqrt{\frac{3s-3}{2}}$				2
$\frac{9}{2}$	$\sqrt{\frac{s}{2}}$	$\sqrt{\frac{2s-1}{2}}$	$\sqrt{\frac{3s-3}{2}}$	$\sqrt{\frac{4s-6}{2}}$			$\frac{5}{2}$
$\frac{11}{2}$	$\sqrt{\frac{s}{2}}$	$\sqrt{\frac{2s-1}{2}}$	$\sqrt{\frac{3s-3}{2}}$	$\sqrt{\frac{4s-6}{2}}$	$\sqrt{\frac{5s-10}{2}}$		3
$\frac{13}{2}$	$\sqrt{\frac{s}{2}}$	$\sqrt{\frac{2s-1}{2}}$	$\sqrt{\frac{3s-3}{2}}$	$\sqrt{\frac{4s-6}{2}}$	$\sqrt{\frac{5s-10}{2}}$	$\sqrt{\frac{6s-15}{2}}$	$\frac{7}{2}$

In conclusion we have found a spin spectrum for $U(1, q)$ in conformity with those of $SU(2, c)$, except that the dimension of the semi-integer representations are halved and the integer spin complex representations exhibit anomalous solutions. As we shall see in the next section, not all is satisfactory and indeed the above set of representations is not complete.

3. – Constructive quaternion tensor product and new integer representations.

There is a difficulty with the representations found in the previous section. The multiplicity of states for a given integer spin are double those of $SU(2, c)$. Apart from the question of the physical interpretation of the anomalous states, we have a problem with the multiplicities of the yet undefined tensor product. For example, consider the spin (tensor) product and decomposition

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0\,, \tag{27}$$

which in $SU(2, c)$ corresponds to the multiplicity count $2 \times 2 = 3 + 1$. For $U(1, q)$ the multiplicity count fails because the number of states with spin one is six and those with spin zero is two. Apart from the drastic choice of abandoning the tensor product, the solution to this incongruence is potentially very interesting. Perhaps we must reinterpret the significance of the tensor products, or admit that the anomalous solutions are spurious and can be avoided in some way (other than abandoning the complex scalar product). In any case, another difference between the two groups must be admitted. The solution we present below is based upon the definition of an explicit (constructive) tensor product which, even if not the final word upon quaternion tensor products, has the undoubted merit of bringing to light a new set of integer spin representations. We shall see that the above failure in the multiplicity counts is due to the fact that *none* of the states of integer spin so far listed «created» in the tensor product. The new integer spin states will have the *correct* multiplicities.

The difficulty in defining quaternion tensor products lies in the non-commutative property of quaternions. For one-component functions in quantum mechanics we simply multiply the functions to obtain the tensor product. This is possible because the product of two complex functions of independent variables continue to satisfy their individual dynamical equations of motion. With matrices the standard rules for tensor products maintain the same feature thanks to the commutativity of complex numbers. All this is lost with quaternion functions. For our particular case we would also be faced with the additional problem that the algebraic product of two quaternion numbers (spin-(1/2) generators) being itself a quaternion number does not have the correct dimensions to represent any higher spin state.

As we have anticipated in the introduction, it was Horwitz and Biedenharn [6] who first proposed the complex scalar product together with a corresponding quaternion tensor product which satisfies the property of being only right complex linear (and not quaternion right linear). These authors write each quaternion q in *symplectic* form

$$q = z + j\,z' \tag{28}$$

and thus represent q as an ordered pair of complex numbers, which we write as a column matrix rather than in row form as originally done,

$$q \sim \begin{pmatrix} z \\ z' \end{pmatrix}. \tag{29}$$

The tensor product $q_1 \otimes q_2$ of Horwitz and Biedenharn is then defined to be

$$q_1 \otimes q_2 \sim \begin{pmatrix} z_1 \\ z'_1 \end{pmatrix} \otimes \begin{pmatrix} z_2 \\ z'_2 \end{pmatrix} = \begin{bmatrix} z_1 z_2 \\ z_1 z'_2 \\ z'_1 z_2 \\ z'_1 z'_2 \end{bmatrix}. \tag{30}$$

This formalism together with the standard rules for complex matrix scalar products is such that

$$|q|^2 = |z|^2 + |z'|^2 , \tag{31}$$

$$(q_1 \otimes q_2 \,|\, q_3 \otimes q_4) = (q_1 \,|\, q_3)(q_2 \,|\, q_4) , \tag{32}$$

where we recall that $(\,|\,)$ is the *complex* scalar product between quaternions.

This formalism has a defect in that beyond the first level (single particle) one loses sight of any possible quaternion structure in the tensor products. The symplectic formalism confirms, however, our previous results in that

$$1 \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad j \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{33}$$

so that the analogy between $U(1, q)$ and $SU(2, c)$ at the spin-$(1/2)$ level is manifest. We now suggest the following modification in the above formalism. We define a two-component quaternion column matrix as the result of a tensor product between two quaternion numbers (or functions),

$$q_1 \otimes q_2 = \begin{pmatrix} q_1 z_2 \\ q_1 z'_2 \end{pmatrix} \tag{34}$$

where z_2 and z'_2 are as before the symplectic parts of q_2. This definition is still right complex linear both for q_1 and q_2. The complex scalar products between tensor products continue to satisfy the same decomposition property as above, as the following equations demonstrate

$$(35)\qquad \begin{cases} (q_1 \otimes q_2 \,|\, q_3 \otimes q_4) \equiv \left((z_2^* q_1^\dagger \;\; z_2'^* \; q_1^\dagger) \begin{pmatrix} q_3 z_4 \\ q_3 z'_4 \end{pmatrix} \right)_C , \\ (q_1 \otimes q_2 \,|\, q_3 \otimes q_4) = (z_2^* \; q_1^\dagger q_3 z_4 + z_2'^* \; q_1^\dagger q_3 z'_4)_C , \\ (q_1 \otimes q_2 \,|\, q_3 \otimes q_4) = z_2^* \; (q_1^\dagger q_3)_C z_4 + z_2'^* (q_1^\dagger q_3)_C z'_4 , \\ (q_1 \otimes q_2 \,|\, q_3 \otimes q_4) = (q_1^\dagger q_3)_C (z_2^* \; z_4 + z_2'^* z'_4), \\ (q_1 \otimes q_2 \,|\, q_3 \otimes q_4) = (q_1^\dagger q_3)_C (q_2^\dagger q_4)_C , \\ (q_1 \otimes q_2 \,|\, q_3 \otimes q_4) = (q_1 \,|\, q_3)(q_2 \,|\, q_4) \end{cases}$$

as before. In the above derivation $(q)_C$ stands for the complex $C(1, i)$ projection of the quaternion q. Our formalism maintains a quaternion structure for the tensor products and is readily generalizable, *e.g.*,

$$(36)\qquad \begin{cases} q_1 \otimes q_2 \otimes q_3 = \begin{pmatrix} q_1 z_2 \\ q_1 z'_2 \end{pmatrix} \otimes q_3 , \\ q_1 \otimes q_2 \otimes q_3 \equiv q_1 \otimes \begin{pmatrix} q_2 z_3 \\ q_2 z'_3 \end{pmatrix}, \\ q_1 \otimes q_2 \otimes q_3 = \begin{bmatrix} q_1 z_2 z_3 \\ q_1 z_2 z'_3 \\ q_1 z'_2 z_3 \\ q_1 z'_2 z'_3 \end{bmatrix} . \end{cases}$$

The main advantage for us in this formalism will be the extraction of a set of generators for $s = 1 \oplus 0$ obtainable directly from those of spin $1/2$ and automatically consistent with the multiplicity rules. It is immediately obvious from the matrix dimensions of the tensor product $q_1 \otimes q_2$ that the generators in question will necessarily be of dimension two (if not reducible) in contrast with those of spin-1 $\oplus$ spin-0 discussed in the previous section, which have dimension four $(3 + 1)$. Indeed, after a straightforward calculation we find the (unseparated) generators of spin $1 \oplus 0$ to be

$$(37)\qquad \frac{1}{2}\begin{pmatrix} i + 1\,|\,i & 0 \\ 0 & i - 1\,|\,i \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} j & 1\,|\,i \\ 1\,|\,i & j \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} k & -1 \\ 1 & k \end{pmatrix}.$$

Note that $[a\,|\,b] \times [c\,|\,d] = ac\,|\,bd$ and that all simple quaternion numbers q are formally $q|1$. The eigenvectors of the first generator above are

(38)
$$\begin{cases} s=1: \begin{pmatrix} 0 \\ j \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} j \\ i \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ s=0: \frac{1}{\sqrt{2}}\begin{pmatrix} -j \\ i \end{pmatrix}. \end{cases}$$

It is to be noted that one does not obtain a second set of complex orthogonal eigenvectors to those listed above by multiplying from the right by j, as occurs with the anomalous solutions. This is due to the somewhat surprising fact that here A^2 is neither diagonal nor proportional to the identity matrix. Indeed the expression for A^2 is

(39)
$$A^2 = -\frac{1}{2}\begin{pmatrix} 3-i|i & k-j|i \\ -k-j|i & 3+i|i \end{pmatrix}$$

and the barred nature of its matrix elements means that right multiplication by j will not leave the eigenvalue of A^2 unaltered. The four solutions are simply mixed under right multiplication by j. We also observe that the same generators are valid both for spin 1 and spin 0 and they cannot be further reduced.

If this analysis is extended to $\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2}$, we obtain the generators for $\frac{3}{2}\oplus\frac{1}{2}\oplus\frac{1}{2}$

(40)
$$\begin{cases} A_1 = \frac{1}{2}\begin{bmatrix} 2|i+i & 0 & 0 & 0 \\ 0 & -2|i+i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{bmatrix}, \\ A_2 = \frac{1}{2}\begin{bmatrix} j & 0 & 1|i & 1|i \\ 0 & j & 1|i & 1|i \\ 1|i & 1|i & j & 0 \\ 1|i & 1|i & 0 & j \end{bmatrix}, \\ A_3 = \frac{1}{2}\begin{bmatrix} k & 0 & -1 & -1 \\ 0 & k & 1 & 1 \\ 1 & -1 & k & 0 \\ \end{bmatrix}, \end{cases}$$

with eigenvectors (in decreasing order of s_x)

$$
\left\{
\begin{aligned}
& s=\frac{3}{2}:\begin{pmatrix}0\\ j\\ 0\\ 0\end{pmatrix},\quad \sqrt{\frac{1}{3}}\begin{pmatrix}0\\ i\\ j\\ j\end{pmatrix},\quad \sqrt{\frac{1}{3}}\begin{pmatrix}j\\ 0\\ i\\ i\end{pmatrix},\quad \begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix},\\
& s=\frac{1}{2}:\sqrt{\frac{2}{3}}\begin{pmatrix}0\\ -\frac{i}{2}\\ j\\ -\frac{j}{2}\end{pmatrix},\quad \sqrt{\frac{2}{3}}\begin{pmatrix}-\frac{j}{2}\\ 0\\ -\frac{i}{2}\\ i\end{pmatrix},\\
& s=\frac{1}{2}:\sqrt{\frac{1}{2}}\begin{pmatrix}0\\ i\\ 0\\ -j\end{pmatrix},\quad \sqrt{\frac{1}{2}}\begin{pmatrix}-j\\ 0\\ i\\ 0\end{pmatrix}.
\end{aligned}
\right.
\tag{41}
$$

The significant fact here is that there exists an invertible quaternion matrix S which transforms the above results for $\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2}$ into those of the previous section.

Explicitly,

$$
S=\begin{pmatrix}
\frac{1}{2}(1-i\,|\,i) & \frac{1}{2}(1+i\,|\,i) & 0 & 0\\
\frac{1}{2\sqrt{3}}k(i-1\,|\,i) & \frac{1}{2\sqrt{3}}k(i+1\,|\,i) & \frac{1}{\sqrt{3}}\,|\,i & \frac{1}{\sqrt{3}}\,|\,i\\
\frac{1}{2\sqrt{2}}k(1+i\,|\,i) & \frac{1}{2\sqrt{2}}k(1-i\,|\,i) & \frac{1}{2\sqrt{2}}(1-i\,|\,i) & \frac{1}{2\sqrt{3}}(-1-i\,|\,i)\\
-\frac{k\sqrt{2}}{4\sqrt{3}}(1+i\,|\,i) & -\frac{k\sqrt{2}}{4\sqrt{3}}(1-i\,|\,i) & \frac{\sqrt{2}}{4\sqrt{3}}(3+i\,|\,i) & \frac{\sqrt{2}}{4\sqrt{3}}(-3+i\,|\,i)
\end{pmatrix}
$$

with $S^{-1}=S^{\dagger}$.

Thus nothing new is found as far as the «fermionic» modes are concerned. On the other hand, we have a new and alternative set of «bosonic» states and corresponding generators (distinguished by the barred matrix elements) with the correct multiplicity. This obliges us to rediscuss the significance of the integer representations with anomalous solutions and this will be done in the final section.

4. – Conclusions.

The study of the representations of $U(1, q)$ has yielded results of interest, some of which are indeed surprising. We summarize the more important results found in the previous sections.

1) The groups $U(1, q)$ and $SU(2, c)$ are isomorphic, but the corresponding representation structures are not.

2) In order to reproduce the basic features of $SU(2, c)$, *e.g.* the number of eigenvectors for «spin» 1/2, one must adopt the right eigenvalue equation within $U(1, q)$.

3) Even then, the two solutions for spin 1/2 are not orthogonal unless one further adopts the complex scalar product.

4) The quaternion representations for «fermionic» states have half the dimensions of the equivalent $SU(2, c)$ cases. The total number of states remains the same because of the doubling due to the complex scalar product.

5) The known integer representations of $SU(2, c)$ are also representations of $U(1, q)$ characterized by the existence of anomalous solutions.

6) If these integer representations were unique, we would have difficulty in the multiplicity count for tensor product states. This can be claimed even in the absence of a specific definition of the tensor product.

7) Thanks to an explicit tensor product, suggested by the work of Horwitz and Biedenharn [7], a new set of integer spin representations have been found. These automatically satisfy the multiplicity counts. Nothing new in the fermionic sector appears.

8) These new representations have lower dimensions than those previously described and are without anomalous solutions. They are also characterized by the presence of barred operators in some matrix elements. As with the original set of integer spin representations, these new generator (and hence group) representations are not reducible into smaller block form but operate upon a reducible vector space (spin 1 plus 0 in the example treated).

The use of right eigenvalue equations and the need for a complex scalar product, which thus breaks the i, j, k symmetry of quaternions, are well-known results, even if, as we have already noted, the latter is not universally accepted. However, the derivation in this work is new and independent of any specific physical input such as the use of a particular quaternion version of the Dirac equation. Indeed our only objective is to reproduce the maximum of agreement between the representation theories of $U(1, q)$ and $SU(2, c)$. This has been achieved sufficiently to suggest that with quaternions the Glashow group could indeed be $U(1, q) \times U(1, c)$. The question that remains to be answered is what physical significance are we to ascribe to the alternative set of integer-spin representations *i.e.* those with the anomalous solutions. We recall that these appear automatically in the solution of all known, integer-spin equations such as the Maxwell equation.

We first give a heuristic argument why the two sets of integer-spin representations are not equivalent. Even the most general unitary similarity transformation ($S^{\dagger} = S^{-1}$) cannot alter the complex orthogonality of the standard and anomalous solutions. This is because the adjoint operator is only definable for matrices with at most $q|i$ elements (no $|j$ or $|k$ allowed). On the other hand, we have seen that with the alternative tensor product representations there are no anomalous solutions. Furthermore, we observe that the number of eigenvectors are different for the alternative sets of integer representations (but see appendix B).

The application of the group $U(1, q)$ within a quaternion version of the electro-wweak theory as an alternative to the weak isospin group is currently under study.

We conclude by observing that beyond the study of matrix groups with «simple» quaternion elements such as in $U(1, q)$, one can consider from the very beginning the more general groups with matrix elements of the form $q_1 + q_2 \,|\, i$. For example, the Lie algebra of the group $U(1, q_1 + q_2 \,|\, i)$ is isomorphic to that of the group $U(2, c)$ with four generators. The group $U(2, q_1 + q_2 \,|\, i)$ has sixteen generators and so forth. Mathematicians may even wish to generalize to elements containing $q \,|\, q'$, but these do not seem to be of any use within the physical models of interest to the present authors. To the best of our knowledge these more general matrix groups have not been studied in the literature.

Appendix A

Here we derive an explicit quaternion similarity transformation which reduces completely the generators $-i\sigma_m/2$. We have already noted in the text that these anti-Hermitian generators satisfy the same algebra as $Q_m/2$ where $(Q_1, Q_2, Q_3) = (i, j, k)$. Consequently, we expect that there exists an invertible matrix S such that

$$S(-i\sigma_m)S^{-1} = \begin{pmatrix} Q_m & 0 \\ 0 & Q_l \end{pmatrix} \tag{A.1}$$

(where, $m, l = 1, 2, 3$). It is not, *a priori*, necessary that $m = l$ in the above formula. However, any Q_l can always be transformed by another similarity transformation into Q_m. In fact, let T be such a transformation, *i.e.*

$$TiT^{-1} = j,$$

$$TjT^{-1} = k,$$

$$TkT^{-1} = i,$$

then it is straightforward to show that, up to an arbitrary constant,

$$T \sim \frac{1}{2}(1 + i + j + k).$$

Thus without loss of generality we can search for an S such that,

$$S(-i\sigma_m)S^{-1} = Q_m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.2}$$

As an example we derive this S explicitly. Let

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where, in principle, a, b, c and d may be barred quaternion numbers. Then eq. (A.2) implies that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(-i\sigma_m) = Q_m \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{A.3}$$

Now,

$$\{-i\sigma_m\}: \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Only two of these matrices need to be inserted in eq. (A.3) since the validity of the algebra will then guarantee the result for the third. Using $-i\sigma_1$ we find

$$i\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\begin{pmatrix} b & a \\ d & c \end{pmatrix} i,$$

i.e. $iai = b$ and $ici = d$. From $-i\sigma_2$ we obtain

$$\begin{pmatrix} b & -a \\ d & -c \end{pmatrix} = j\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

i.e. $b = ja$ and $d = jc$. These results limit the form of a, b, c and d to

$$\begin{aligned} a &= a_0(1+j) + a_1(i-k), \\ b &= -a_0(1-j) - a_1(i+k), \\ c &= c_0(1+j) + c_1(i-k), \\ d &= -c_0(1-j) - c_1(i+k), \end{aligned} \qquad (a_0, a_1, c_0, c_1 \in \boldsymbol{R})$$

We now require that S be a unitary operator $SS^\dagger = S^\dagger S = 1$. The off-diagona elements of this condition yields

$$a_0 c_0 + a_1 c_1 = 0,$$

while the diagonal elements yield

$$|a|^2 + |b|^2 = 4a_0^2 + 4a_1^2 = 1,$$

$$|c|^2 + |d|^2 = 4c_0^2 + 4c_1^2 = 1.$$

An acceptable solution to these equations is $c_0 = a_1 = 0$ and $a_0 = c_1 = 1/2$, whence

$$S = \frac{1}{2}\begin{pmatrix} 1+j & j-1 \\ i-k & -(i+k) \end{pmatrix}. \tag{A.4}$$

A similar reduction of $\sigma_m/2$ without the i factor *cannot* be achieved. The simplest way to demonstrate this is to recall that a similarity transformation leaves unaltered any algebra satisfied by the transformed quantities. Now, since the $\sigma_m/2$ are Hermitian, the eventual reduction should exhibit diagonal matrix elements of the J_m type ($i/2|i$, etc). However, $\sigma_m/2$ and J_m do not satisfy the same algebra. Indeed,

$$\left[\frac{\sigma_m}{2}, \frac{\sigma_n}{2}\right] = i\,\varepsilon_{mnp}\,\frac{\sigma_p}{2},$$

while

$$[J_m, J_n] = \varepsilon_{mnp}\, J_p\,|i$$

the diverse position of the i factors is essential and excludes the possibility of reduction.

Appendix B

In this appendix we consider the matrix representations for integer spin values, and, in particular, the spin-1 and spin-0 cases. For our convenience we will call with the adjective *old* the standard complex matrix representations (characterized by the presence of *anomalous* solutions) and with *new* the additional quaternionic matrix representations (the one for spin $1 \oplus 0$ is listed in sect. 3, eq. (37)). First we note that the *old* matrix representation for spin 1 alone is certainly irreducible. This follows from the fact that a reduction of a 3×3 matrix necessarily yields a one-dimensional representation of the algebra, and it is readily demonstrated that there is no such representation for spin 1. In fact, only spin 0 and spin 1/2 have one-dimensional representations.

Thus it seems that the only hope to reduce to blocks of smaller dimensions the *old* representations is to compare the *old* complex 4×4 matrix, which corresponds to spin $1 \oplus 0$, with a quaternionic matrix of the type

$$F_m = \begin{pmatrix} B_m^{\text{new}} & 0 \\ \bullet & C_m^{\text{new}} \end{pmatrix}, \tag{B.1}$$

where B_m^{new}, $\bullet$, C_m^{new} are quaternion 2×2 matrices.

We are thus searching for a 4×4 quaternion matrix S which operates upon the combined spin $1 \oplus 0$ space and such that

$$SA_m^{\text{old}}S^{-1} = F_m\,.$$

Whence, B_m^{new} and C_m^{new} satisfy the same algebra as A_m^{old}. It is sufficient to concentrate our attention upon $A^{2\,\text{old}}$ and use the fact that it is diagonal. Now,

$$SA^{2\,\text{old}}S^{-1} = F^2\,, \tag{B.2}$$

where $F^2 = F_1^2 + F_2^2 + F_3^2$ etc. This equation can be written as follows, since S

commutes with the identity

$$\text{(B.3)}\qquad -2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + 2S\begin{pmatrix}0 & 0\\ 0 & M\end{pmatrix}S^{-1} = \begin{pmatrix}B^{2\,\text{new}} & 0\\ \bullet & C^{2\,\text{new}}\end{pmatrix}$$

with

$$M = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

Now using for S and S^{-1} the *partial* expressions

$$S = \begin{bmatrix}\bullet & \bullet & \bullet & a\\ \bullet & \bullet & \bullet & b\\ \bullet & \bullet & \bullet & c\\ \bullet & \bullet & \bullet & d\end{bmatrix}, \qquad S^{-1} = \begin{bmatrix}\bullet & \bullet & \bullet & \bullet\\ \bullet & \bullet & \bullet & \bullet\\ \bullet & \bullet & \bullet & \bullet\\ \alpha & \beta & \gamma & \delta\end{bmatrix}$$

(where a, b, c, d, α, β, γ, δ are barred quaternion numbers), we obtain from eq. (B.3)

$$\text{(B.4)}\qquad \begin{bmatrix}a\alpha & a\beta & a\gamma & a\delta\\ b\alpha & b\beta & b\gamma & b\delta\\ c\alpha & c\beta & c\gamma & c\delta\\ d\alpha & d\beta & d\gamma & d\delta\end{bmatrix} = \begin{bmatrix}1+\dfrac{1}{2}B^{2\,\text{new}} & 0\\ \bullet & 1+\dfrac{1}{2}C^{2\,\text{new}}\end{bmatrix}$$

(where $1+(1/2)\,B^{2\,\text{new}}$, $\bullet$, $1+(1/2)\,C^{2\,\text{new}}$ are quaternion 2×2 matrices). Thus, in particular,

$$\begin{pmatrix}a\gamma & a\delta\\ b\gamma & b\delta\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

This last equation leads to a contradiction. From it, it follows necessarily that at least one of the couples (a, b) and (γ, δ) must be null. As a consequence,

$$(a, b) = (0, 0) \Rightarrow B^{2\,\text{new}} = -2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

$$(\gamma, \delta) = (0, 0) \Rightarrow C^{2\,\text{new}} = -2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

and this is not possible because $B^{2\,\text{new}}$ and $C^{2\,\text{new}}$ represent spin $1\oplus 0$ and thus they cannot be proportional to the identity.

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