Redução à forma canônica

Dada a equação

$$\boldsymbol{\widetilde{A}\,x^2 + \widetilde{B}\,x\,y + \widetilde{C}\,y^2 + \widetilde{D}\,x + \widetilde{E}\,y + \widetilde{F} = 0}$$

consideramos a seguinte trasformação de coordenadas

$$x = \cos\alpha\,X - \sin\alpha\,Y \quad , \quad y = \cos\alpha\,Y + \sin\alpha\,X \ .$$

Substituindo na conica dada, obtemos

$$\begin{array}{lcl}
\widetilde{A}\,x^2 & \Rightarrow & \widetilde{A}\,\cos^2\alpha\;X^2 + \widetilde{A}\,\sin^2\alpha\;Y^2 - 2\,\widetilde{A}\,\sin\alpha\,\cos\alpha\,X\,Y \\
\widetilde{B}\,x\,y & \Rightarrow & \widetilde{B}\,\sin\alpha\,\cos\alpha\,\left(X^2 - Y^2\right) + \widetilde{B}\,(\cos^2\alpha - \sin^2\alpha)\,X\,Y \\
\widetilde{C}\,y^2 & \Rightarrow & \widetilde{C}\,\sin^2\alpha\;X^2 + \widetilde{C}\,\cos^2\alpha\;Y^2 + 2\,\widetilde{C}\,\sin\alpha\,\cos\alpha\,X\,Y \\
\widetilde{D}\,x & \Rightarrow & \widetilde{D}\,\cos\alpha\;X - \widetilde{D}\,\sin\alpha\;Y \\
\widetilde{E}\,y & \Rightarrow & \widetilde{E}\,\sin\alpha\;X + \widetilde{E}\,\cos\alpha\;Y \\
\widetilde{F} & \Rightarrow & \widetilde{F}
\end{array}$$

Para eliminar o termo cruzado em XY temos que impor

$$2\left(\widetilde{A} - \widetilde{C}\right)\sin\alpha\,\cos\alpha - \widetilde{B}\,(\cos^2\alpha - \sin^2\alpha) = 0 \quad \Rightarrow \quad \widetilde{B}\,\tan^2\alpha + 2\left(\widetilde{A} - \widetilde{C}\right)\tan\alpha - \widetilde{B} = 0\ ,$$

que implica

$$\boldsymbol{\tan\alpha = \frac{\widetilde{C} - \widetilde{A} \pm \sqrt{(\widetilde{C} - \widetilde{A})^2 + \widetilde{B}^2}}{\widetilde{B}}}$$

É importante observar que podemos escolher para a nossa rotação um ângulo α no primeiro quadrante (valor de $\tan\alpha$ positivo).

$$\boldsymbol{\sin\alpha = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}} \qquad \boldsymbol{\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}}}$$

Classificação :

$$\boldsymbol{(\widetilde{A}\,\cos^2\alpha + \widetilde{B}\,\sin\alpha\,\cos\alpha + \widetilde{C}\,\cos^2\alpha)\,(\widetilde{A}\,\sin^2\alpha - \widetilde{B}\,\sin\alpha\,\cos\alpha + \widetilde{C}\,\sin^2\alpha)} \quad \begin{array}{ll} \boldsymbol{> 0} & \textbf{tipo elíptico} \\ \boldsymbol{= 0} & \textbf{tipo parabólico} \\ \boldsymbol{< 0} & \textbf{tipo hiperbólico} \end{array}$$

Determinar o tipo de cônica, reduzir à forma canônica e esboçar o gráfico:

1) $3\,x^2 + 10\,x\,y + 3\,y^2 - 2\,x - 14\,y - 13 = 0$

2) $25\,x^2 - 14\,x\,y + 25\,y^2 + 64\,x - 64\,y - 224 = 0$

3) $4\,x\,y + 3\,y^2 + 16\,x + 12\,y - 36 = 0$

4) $9\,x^2 - 24\,x\,y + 16\,y^2 - 20\,x + 110\,y - 50 = 0$

1) $\boldsymbol{3\,x^2 + 10\,x\,y + 3\,y^2 - 2\,x - 14\,y - 13 = 0}$

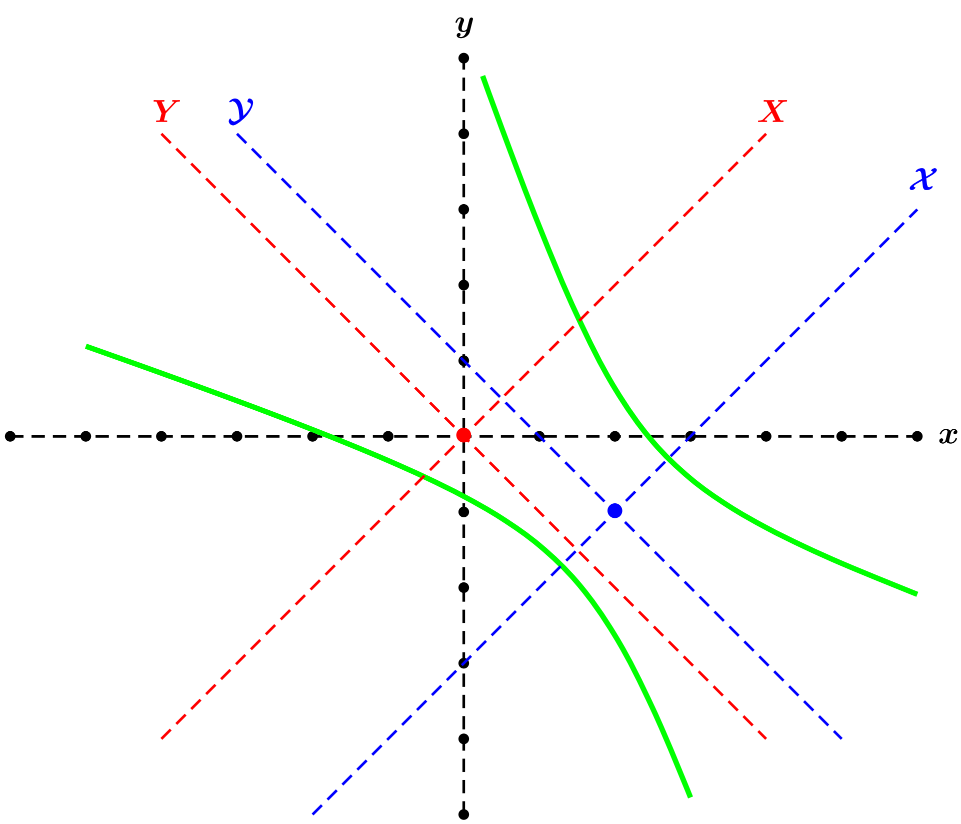
$$\tan\alpha = 1 \quad \Rightarrow \quad \sin\alpha = \cos\alpha = \frac{1}{\sqrt{2}}, \quad \Rightarrow \quad x = \frac{X - Y}{\sqrt{2}} \quad , \quad y = \frac{X + Y}{\sqrt{2}}$$

$$8\,X^2 - 2\,Y^2 - \frac{16}{\sqrt{2}}\,X - \frac{12}{\sqrt{2}}\,Y - 13 = 0$$

$$\left(X - \frac{1}{\sqrt{2}}\right)^2 - \frac{\left(Y + \frac{3}{\sqrt{2}}\right)^2}{4} = 1$$

$$(x_0\,,\,y_0) = (\,2\,,\,-1\,)$$

$$\mathcal{X}^2 - \frac{\mathcal{Y}^2}{4} = 1$$



2) $\mathbf{25\,x^2 - 14\,x\,y + 25\,y^2 + 64\,x - 64\,y - 224 = 0}$

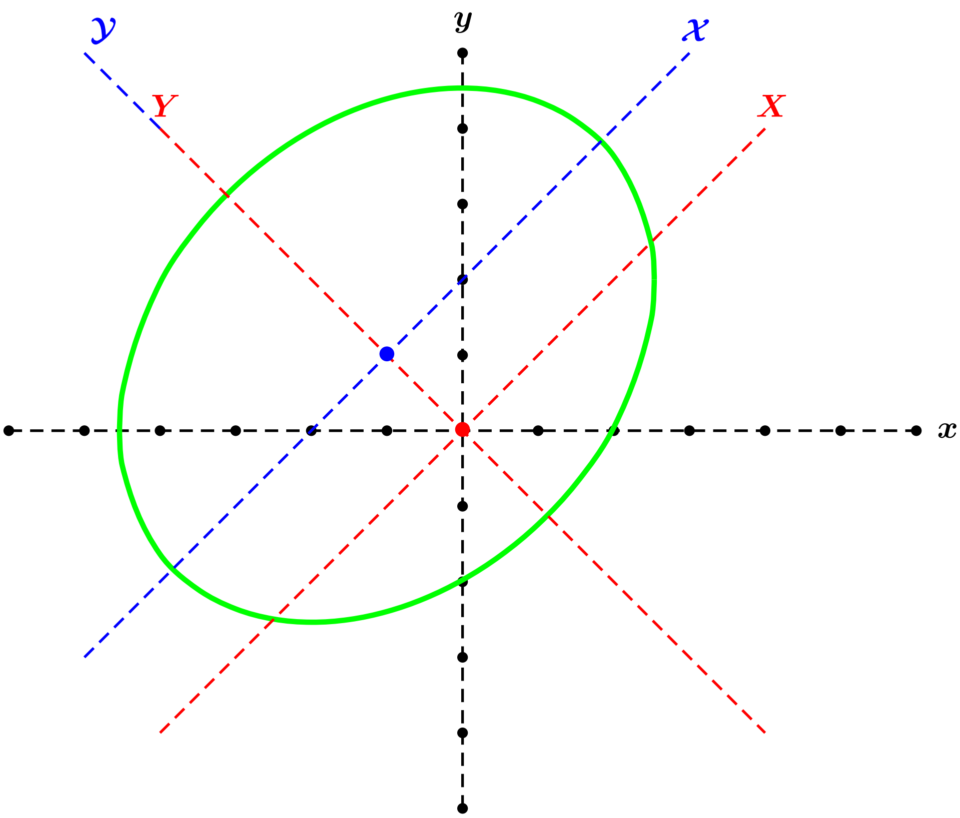
$$\tan\alpha = 1 \quad \Rightarrow \quad \sin\alpha = \cos\alpha = \frac{1}{\sqrt{2}}, \quad \Rightarrow \quad x = \frac{X - Y}{\sqrt{2}} \quad , \quad y = \frac{X + Y}{\sqrt{2}}$$

$$18\,X^2 + 32\,Y^2 - \frac{128}{\sqrt{2}}\,Y - 224 = 0$$

$$\frac{X^2}{16} + \frac{\left(Y - \sqrt{2}\right)^2}{9} = 1$$

$$(x_0\,,\,y_0) = (\,-1\,,\,1\,)$$

$$\frac{\mathcal{X}^2}{16} + \frac{\mathcal{Y}^2}{9} = 1$$



3) $\boldsymbol{4\,x\,y + 3\,y^2 + 16\,x + 12\,y - 36 = 0}$

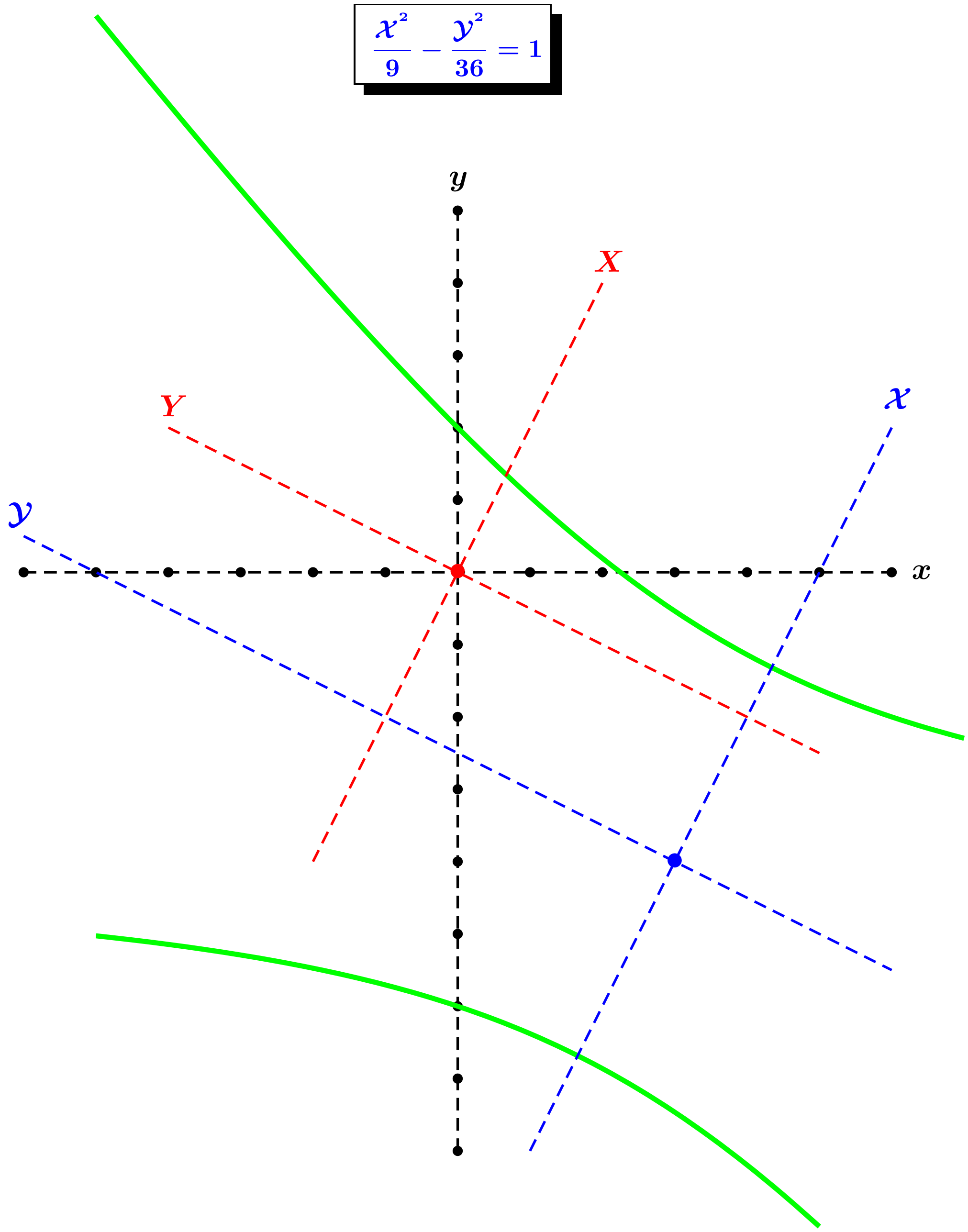
$$\tan\alpha = 2 \quad\Rightarrow\quad \sin\alpha = \frac{2}{\sqrt{5}}\ ,\quad \cos\alpha = \frac{1}{\sqrt{5}},\quad\Rightarrow\quad x = \frac{X - 2\,Y}{\sqrt{5}}\ ,\quad y = \frac{Y + 2\,X}{\sqrt{5}}$$

$$4\,X^2 - Y^2 + 8\sqrt{5}\,X - 4\sqrt{5}\,Y - 36 = 0$$

$$\boldsymbol{\frac{\left(X + \sqrt{5}\right)^2}{9} - \frac{\left(Y + 2\sqrt{5}\right)^2}{36} = 1}$$

$$\boldsymbol{(x_0\,,\ y_0) = (\,3\,,\ -4\,)}$$

$$\boldsymbol{\frac{\mathcal{X}^2}{9} - \frac{\mathcal{Y}^2}{36} = 1}$$



4) $9\,x^2 - 24\,x\,y + 16\,y^2 - 20\,x + 110\,y - 50 = 0$

$$\tan\alpha = \frac{3}{4} \quad \Rightarrow \quad \sin\alpha = \frac{3}{5} \;, \quad \cos\alpha = \frac{4}{5}, \quad \Rightarrow \quad x = \frac{4\,X - 3\,Y}{5} \;, \quad y = \frac{4\,Y + 3\,X}{5}$$

$$25\,Y^2 + 50\,X + 100\,Y - 50 = 0$$

$$(Y + 2)^2 = -\,2\;(X - 3)$$

$$(x_0\,,\,y_0) = \left(\frac{18}{5}\,,\,\frac{1}{5}\right)$$

$$\mathcal{Y}^2 = -\,2\,\mathcal{X}$$