Autovalores complexos

$$M = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$$

$(3-\lambda)(-1-\lambda)+8=0$

$\lambda^2-2\lambda+5=0$

$\lambda_{1/2} = 1 \pm \sqrt{-4}$

$= 1 \pm 2i$

$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix}$$

$3a-2c = (1+2i)a \qquad 4a-c = (1+2i)c$

$3b-2d = (1-2i)b \qquad 4b-d = (1-2i)d$

$a=1 \quad 3-1-2i = 2c \rightarrow c = 1-i$

$4 \leftarrow 4a = 2(1+i)c \rightarrow 4$ ok!

$b=1 \quad \cancel{2}(1+i) = \cancel{2}d \rightarrow d = 1+i$

$4b = 2(1-i)d$ ✓ ok!

$$S = \begin{pmatrix} 1 & 1 \\ 1-i & 1+i \end{pmatrix} \qquad S^{-1} = \frac{1}{2i}\begin{pmatrix} 1+i & -1 \\ i-1 & 1 \end{pmatrix}$$

$$\exp\left[\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}\right] = \frac{1}{2i}\begin{pmatrix} 1 & 1 \\ 1-i & 1+i \end{pmatrix}\begin{pmatrix} e^{1+2i} & 0 \\ 0 & e^{1-2i} \end{pmatrix}\begin{pmatrix} 1+i & -1 \\ i-1 & 1 \end{pmatrix}$$

$$= \frac{e}{2i}\begin{pmatrix} e^{2i} & e^{-2i} \\ (1-i)e^{2i} & (1+i)e^{-2i} \end{pmatrix}\begin{pmatrix} 1+i & -1 \\ i-1 & 1 \end{pmatrix}$$

$M_{11} = \frac{e}{2i}\left[(1+i)e^{2i} + (i-1)e^{-2i}\right] = e(\cos 2 + \sin 2)$

$M_{12} = \frac{e}{2i}\left(e^{-2i} - e^{2i}\right) = -e\sin 2$

$M_{21} = \frac{e}{2i}\left(2e^{2i} - 2e^{-2i}\right) = 2e\sin 2$

$M_{22} = \frac{e}{2i}\left[(i-1)e^{2i} + (1+i)e^{-2i}\right] = e(\cos 2 - \sin 2)$

EXERCÍCIOS

$\begin{pmatrix} 3-\lambda & -a \\ 2 & -3-\lambda \end{pmatrix} \quad \lambda^2 - 9 + 2a = 0 \quad a = 5 \quad \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix} \quad \alpha = i, \ \beta = -i$

$\rightarrow 4$

$\lambda^2 - \lambda - 12 + 2a = 0$

$\Delta = 1 - 4(2a-12) = -9$ POR EXEMPLO $\quad a = \frac{29}{4} \quad \begin{pmatrix} 4 & -\frac{29}{4} \\ 2 & -3 \end{pmatrix}$

$1 - 8a + 48 = -9 \rightarrow 58 = 8a$

AUTOVALORES $\frac{1 \pm 3i}{2}$

CHECK! $(4-\lambda)(-3-\lambda) + \frac{29}{2} = 0$

$\lambda^2 - \lambda - 12 + 29/2 = 0 \rightarrow \lambda^2 - \lambda + \frac{5}{2} = 0 \rightarrow \lambda_{1/2} = \frac{1 \pm \sqrt{1-10}}{2}$

Forma de Jordan com autovalores complexos ?

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \rightarrow \lambda^2-(a+d)\lambda+ad-bc=0 \ !!!$$

M

$$\lambda_{1,2}=\frac{a+d\pm\sqrt{(a+d)^2-4(ad-bc)}}{2}$$

$a+d=\mathrm{Tr}\,M$
$ad-bc=\det M$

$$\lambda_{1,2}=\frac{\mathrm{Tr}\,M\pm\sqrt{(\mathrm{Tr}\,M)^2-4\det M}}{2}$$

FORMA DE JORDAN REQUER $(\mathrm{Tr}\,M)^2=4\det M$

TR M REAL IMPLICA AUTOVALORES REAIS

CONSIDERIAMOS AGORA A MATRIZ

$$\boxed{M=M^{+}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}^t=\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

$\boxed{a,d\in\mathbb{R}}$ $\quad M=\begin{pmatrix} a & b \\ b^* & d \end{pmatrix}$ $\quad \mathrm{Tr}\,M=a+d$, $\det M=ad-|b|^2$

$$\lambda_{1,2}=\frac{a+d\pm\sqrt{(a+d)^2-4ad+4|b|^2}}{2}$$

$(a-d)^2+4|b|^2 \in\mathbb{R}$

$$\boxed{\text{MATRIZES HERMITIANAS : AUTOVALORES REAIS}}$$

$$\exp\left[\begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}\right]=\frac{1}{3}S\begin{pmatrix} e^3 & 0 \\ 0 & 1 \end{pmatrix}S=\frac{1}{3}\begin{bmatrix} e^3+2 & (1+i)(e^3-1) \\ (1-i)(e^3-1) & 2e^3+1 \end{bmatrix}$$

SOL:

$$\begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix} \quad (1-\lambda)(2-\lambda)-2=0 \quad \lambda^2-3\lambda=0 \rightarrow \lambda_1=0 \quad \lambda_2=3$$

$$\begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$$

$\cancel{a}+(1+i)c=\cancel{3}^{2}a \qquad b+(1+i)d=0$
$(1-i)a+\cancel{2c}=\cancel{3}c \qquad (1-i)b+2d=0$ $\quad d=-1$, $b=1+i$

$a=1 \quad c=1-i$

$$S=\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix} \qquad S^{-1}=\frac{1}{3}S \qquad \frac{1}{3}\underbrace{\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix}$$

ok! $\quad \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 0 \\ 1-i & 0 \end{pmatrix}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix}$

CÁLCULANDO DIRETAMENTE EXP [M]

$$M = \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix} \quad M^2 = \begin{pmatrix} 3 & 3(1+i) \\ 3(1-i) & 6 \end{pmatrix} \quad M^3 = \begin{pmatrix} 9 & 9(1+i) \\ 9(1-i) & 18 \end{pmatrix} \quad M^4 = \begin{pmatrix} 27 & 27(1+i) \\ 27(1-i) & 54 \end{pmatrix}$$

11) $1 + 1 + \frac{3}{2!} + \frac{9}{3!} + \frac{27}{4!} + \cdots = \frac{1}{3}\left(2 + 1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots\right) = \frac{2+e^3}{3}$

22) $1 + 2 + \frac{6}{2!} + \frac{18}{3!} + \frac{54}{4!} + \cdots = 1 + 2\left(1 + \frac{3}{2!} + \frac{3^2}{3!} + \frac{3^3}{4!} + \cdots\right)$

$= 1 + \frac{2}{3}\left(3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots\right)$

$= 1 + \frac{2}{3}(e^3 - 1)$

$= \frac{2e^3+1}{3}$

12) $1 + (1+i)\left(1 + \frac{3}{2!} + \frac{3^2}{3!} + \frac{3^3}{4!} + \cdots\right) = 0 + \frac{1+i}{3}\left(3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots\right)$

$= \frac{1+i}{3}(e^3 - 1)$

21) $12\,[i \to -i] = \frac{1-i}{3}(e^3 - 1)$

$$M = \begin{pmatrix} 1 & 1+i \\ 1-i & 2 \end{pmatrix} \qquad \exp M = \frac{1}{3}\begin{bmatrix} e^3+2 & (1+i)(e^3-1) \\ (1-i)(e^3-1) & 2e^3+1 \end{bmatrix}$$

$$= \frac{e^3}{3}\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 & -(1+i) \\ -(1-i) & 1 \end{bmatrix}$$

$$= \frac{e^3}{3} M - \frac{1}{3}\begin{bmatrix} -2 & 1+i \\ 1-i & -1 \end{bmatrix} \quad \checkmark$$