Resolvendo equações dif erenciais

$x = x(t)$

CONSIDERAMOS $\ddot{x} - a\dot{x} - bx = 0$ $\quad x(0)=x_0 \quad \dot{x}(0)=v_0$

$$X = \begin{pmatrix}\dot{x}\\ x\end{pmatrix} \qquad \dot{X} = \begin{pmatrix} a & b\\ 1 & 0\end{pmatrix} X = MX \qquad \boxed{X(t) = \exp[Mt]\, X(0)}$$

teremos que calcular o exponencial de Mt

precisamos então achar os autolavores de M

$$(a-\lambda)(-\lambda) - b = 0 \rightarrow \lambda^2 - a\lambda - b = 0$$

$\lambda_1 \neq \lambda_2 \quad \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$

$\lambda_1 = \lambda_2 \quad \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$

primeiro caso

$$\boxed{X(t) = S\begin{pmatrix} e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{pmatrix} S^{-1} X(0)}$$

segundo caso

$$\mathbb{1} + \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} t + \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}^2 \frac{t^2}{2!} + \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}^3 \frac{t^3}{3!} + \dots$$

$$\mathbb{1} + \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} t + \begin{pmatrix}\lambda^2 & 2\lambda\\ 0 & \lambda^2\end{pmatrix} \frac{t^2}{2!} + \begin{pmatrix}\lambda^3 & 3\lambda^2\\ 0 & \lambda^3\end{pmatrix} \frac{t^3}{3!} + \dots$$

diagonal $e^{\lambda t}$ $\qquad$ off-diagonal (up) $\quad t\left[1 + \lambda t + \lambda^2 \frac{t^2}{2!} + \dots\right] = \boxed{t e^{\lambda t}}$

$$\boxed{X(t) = S\begin{pmatrix} e^{\lambda t} & t e^{\lambda t}\\ 0 & e^{\lambda t}\end{pmatrix} S^{-1} X(0)}$$

$\boxed{x(0) = 1 \quad \dot{x}(0) = 2}$

$(\lambda-1)(\lambda-3) = \lambda^2 - 4\lambda + 3 \qquad a=4,\ b=-3 \qquad \boxed{\ddot{x} - 4\dot{x} + 3x = 0}$

$$\begin{pmatrix}4 & -3\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}$$

$4\alpha - 3\gamma = 3\alpha \qquad \alpha = 3\gamma$

$4\beta - 3\delta = \beta \qquad \beta = \delta$

$\boxed{\beta = \delta = \gamma = 1 \quad \alpha = 3}$

$$S = \begin{pmatrix}3 & 1\\ 1 & 1\end{pmatrix} \qquad S^{-1} = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 3\end{pmatrix}$$

$$X(t) = \frac{1}{2}\begin{pmatrix}3 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}e^{3t} & 0\\ 0 & e^{t}\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 3\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3e^{3t} & e^{t}\\ e^{3t} & e^{t}\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}$$

$$\boxed{x(t) = \frac{e^{3t} + e^{t}}{2}}$$

$$\begin{pmatrix}4 & -4\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}$$

$(\lambda-2)^2 = \lambda^2 - 4\lambda + 4 \qquad a=4,\ b=-4$

$4(\alpha - \gamma) = 2\alpha \qquad \alpha = 2\gamma$

$4(\beta - \delta) = \alpha + 2\beta \qquad \beta = \gamma + 2\delta$

$\boxed{\gamma = \delta = 1 \quad \alpha = 2 \quad \beta = 3}$

$$S = \begin{pmatrix}2 & 3\\ 1 & 1\end{pmatrix} \qquad S^{-1} = \begin{pmatrix}-1 & 3\\ 1 & -2\end{pmatrix}$$

$$X(t) = \begin{pmatrix}2 & 3\\ 1 & 1\end{pmatrix}\begin{pmatrix}e^{2t} & te^{2t}\\ 0 & e^{2t}\end{pmatrix}\begin{pmatrix}-1 & 3\\ 1 & -2\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix} = e^{2t}\begin{pmatrix}2 & 2t+3\\ 1 & t+1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = e^{2t}\begin{pmatrix}2\\ 1\end{pmatrix}$$

$$\boxed{x(t) = e^{2t}}$$

Mudamos os condições da última equação $x(0)=1$
$\dot{x}(0)=1$

$$\mathcal{X}(t) = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} e^{2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= e^{2t} \begin{pmatrix} 2 & 2t+3 \\ 1 & t+1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$
$$= e^{2t} \begin{pmatrix} 1-2t \\ 1-t \end{pmatrix} = \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix}$$

$$\boxed{x(t) = e^{2t}(1-t)} \qquad x(0)=1$$
$$\dot{x}(t) = 2e^{2t}(1-t) - e^{2t}$$
$$= e^{2t}(1-2t) \checkmark \qquad \dot{x}(0)=1$$

$$\boxed{x(0) = x_0 \quad \dot{x}(0) = v_0}$$ NOVAS CONDIÇÕES

$$\mathcal{X}(t) = e^{2t} \begin{pmatrix} 2 & 2t+3 \\ 1 & t+1 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_0 \\ x_0 \end{pmatrix}$$
$$= e^{2t} \begin{pmatrix} 1+2t & -4t \\ t & 1-2t \end{pmatrix} \begin{pmatrix} v_0 \\ x_0 \end{pmatrix}$$

$$\boxed{x(t) = e^{2t}\left[(1-2t)x_0 + t v_0\right]} \qquad x(0) = x_0 \checkmark$$
$$\dot{x}(t) = e^{2t}\left\{ 2\left[(1-2t)x_0 + t v_0\right] - 2x_0 + v_0 \right\}$$
$$= e^{2t}\left[-4t x_0 + (1+2t) v_0\right] \qquad \dot{x}(0) = v_0 \checkmark$$

$x_0 = 1 \quad v_0 = 2 \qquad x(t) = e^{2t} \checkmark$

$x_0 = 1 \quad v_0 = 1 \qquad x(t) = e^{2t}(1-t) \checkmark$