Solving systems of differential equations

BEFORE TO SOLVE SYSTEMS OF DIFFERENTIAL EQUATIONS (SDE) WE CALCULATE $\mathrm{Exp}[Mt]$. FOR 2×2 MATRICES WE HAVE THE FOLLOWING POSSIBILITIES $M = SDS^{-1}$ OR $M = SJS^{-1}$ WHERE $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ AND $J = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. CONSEQUENTLY

$$\mathrm{Exp}[Mt] = S\,\mathrm{Exp}\left[\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} t\right] S^{-1}$$
$$= S \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} S^{-1} \qquad \text{DIAGONAL FORM}$$

$$\mathrm{Exp}[Mt] = S\,\mathrm{Exp}\left[\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} t\right] S^{-1} \qquad \text{JORDAN FORM}$$

$$\mathrm{Exp}\left[\begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix}^2 + \frac{1}{3!}\begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix}^3 + \dots$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} \lambda^2 t^2 & 2\lambda t \\ 0 & \lambda^2 t^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} \lambda^3 t^3 & 3\lambda^2 t^2 \\ 0 & \lambda^3 t^3 \end{pmatrix} + \dots$$
$$= \begin{pmatrix} 1+\lambda t+\lambda^2t^2/2!+\lambda^3t^3/3!+\dots & t\,(1+\lambda t+\lambda^2t^2/2!+\dots) \\ 0 & 1+\lambda t+\lambda^2t^2/2!+\lambda^3t^3/3! \end{pmatrix}$$

$$\mathrm{Exp}\left[\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} t\right] = \mathrm{Exp}\left[\begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}\right]$$

LET US NOW SOLVE SOME EXAMPLES FOR SDE OF THE FORM

$$\alpha\,\dot{x}_1(t) + \beta\,\dot{x}_2(t) = a\,x_1(t) + b\,x_2(t)$$
$$\gamma\,\dot{x}_1(t) + \delta\,\dot{x}_2(t) = c\,x_1(t) + d\,x_2(t)$$

WHERE THE MATRIX $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ IS INVERTIBLE

WE SHALL HAVE THREE POSSIBILITIES : 2 REAL EIGENVALUES $\lambda_1 \neq \lambda_2$

2 COMPLEX EIGENVALUES λ, λ^*

1 REAL EIGENVALUE $\lambda_1 = \lambda_2$

LET US BEGIN BY CONSIDERING THE FOLLOWING SYSTEM

$$\dot{x}_1(t) - \dot{x}_2(t) = -x_1(t) + 5\,x_2(t)$$
$$\dot{x}_2(t) = 2\,x_1(t) - x_2(t)$$

$x_1(0) = 2$
$x_2(0) = 1$
INITIAL CONDITION

IN MATRIX FORM WE HAVE $\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{pmatrix}-1 & 5\\ 2 & -1\end{pmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$

$$\underbrace{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}}_{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 5\\ 2 & -1\end{pmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

$$=\begin{pmatrix}1 & 4\\ 2 & -1\end{pmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

SOLUTION: $\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\mathrm{Exp}\left[\begin{pmatrix}1 & 4\\ 2 & -1\end{pmatrix}t\right]\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}$

EIGENVALUES

$$\begin{vmatrix}1-\lambda & 4\\ 2 & -1-\lambda\end{vmatrix}=0 \Rightarrow \lambda^2-1-8=0 \Rightarrow \lambda=\pm 3$$

$$\begin{pmatrix}1 & 4\\ 2 & -1\end{pmatrix}\begin{pmatrix}S_1 & S_2\\ S_3 & S_4\end{pmatrix}=\begin{pmatrix}S_1 & S_2\\ S_3 & S_4\end{pmatrix}\begin{pmatrix}3 & 0\\ 0 & -3\end{pmatrix}$$

$$\begin{pmatrix}S_1+4S_3 & S_2+4S_4\\ 2S_1-S_3 & 2S_2-S_4\end{pmatrix}=\begin{pmatrix}3S_1 & -3S_2\\ 3S_3 & -3S_4\end{pmatrix} \qquad S_1=2S_3 \qquad S_2=-S_4$$

$$S=\begin{pmatrix}2 & -1\\ 1 & 1\end{pmatrix}$$

$$\begin{pmatrix}2 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}3 & 0\\ 0 & -3\end{pmatrix}\frac{1}{3}\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}=\frac{1}{3}\begin{pmatrix}6 & 3\\ 3 & -3\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}=\begin{pmatrix}1 & 4\\ 2 & -1\end{pmatrix} \checkmark$$

SOLUTION: $\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\frac{1}{3}\begin{pmatrix}2 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}e^{3t} & 0\\ 0 & e^{-3t}\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix}$

$$=\frac{1}{3}\begin{pmatrix}2e^{3t} & -e^{-3t}\\ e^{3t} & e^{-3t}\end{pmatrix}\begin{pmatrix}3\\ 0\end{pmatrix}$$

$$=\begin{pmatrix}2\\ 1\end{pmatrix}e^{3t}$$

INITIAL CONDITION OK!

$\dot{x}_1(t)-\dot{x}_2(t)=-x_1(t)+5x_2(t)$ $\qquad 6e^{3t}-3e^{3t}=-2e^{3t}+5e^{3t}$ $\checkmark$

$\dot{x}_2(t)=2x_1(t)-x_2(t)$ $\qquad 3e^{3t}=4e^{3t}-e^{3t}$ $\checkmark$

LET US NOW CONSIDER

IN MATRIX FORM WE HAVE $\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{pmatrix}-1 & 0\\ 2 & -1\end{pmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$

$$\underbrace{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}}_{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 2 & -1\end{pmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

$$=\begin{pmatrix}1 & -1\\ 2 & -1\end{pmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

SOLUTION: $\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\mathrm{Exp}\left[\begin{pmatrix}1 & -1\\ 2 & -1\end{pmatrix}t\right]\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 + 2 = 0 \Rightarrow \lambda = \pm i$$

$$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\begin{pmatrix} s_1 - s_3 & s_2 - s_4 \\ 2s_1 - s_3 & 2s_2 - s_4 \end{pmatrix} = \begin{pmatrix} i s_1 & -i s_2 \\ i s_3 & -i s_4 \end{pmatrix}$$

$(1-i)s_1 = s_3$ $\quad$ $(1+i)s_2 = s_4$

$2s_1 = (1+i)s_3$ $\quad$ $2s_2 = (1-i)s_4$

$$S = \begin{pmatrix} 1 & 1 \\ 1-i & 1+i \end{pmatrix}$$

$$S^{-1} = \frac{1}{2i}\begin{pmatrix} 1+i & -1 \\ i-1 & 1 \end{pmatrix}$$

SOLUTION $$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2i}\begin{pmatrix} 1 & 1 \\ 1-i & 1+i \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\begin{pmatrix} 1+i & -1 \\ i-1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \frac{1}{2i}\begin{pmatrix} e^{it} & e^{-it} \\ (1-i)e^{it} & (1+i)e^{-it} \end{pmatrix}\begin{pmatrix} 1+2i \\ 2i-1 \end{pmatrix}$$

$$= \frac{1}{2i}\begin{bmatrix} e^{it} - e^{-it} + 2i(e^{it} + e^{-it}) \\ 3(e^{it} - e^{-it}) + i(e^{it} + e^{-it}) \end{bmatrix}$$

$$= \begin{pmatrix} \sin t + 2\cos t \\ 3\sin t + \cos t \end{pmatrix}$$

INITIAL CONDITIONS OK!

$\dot{x}_1(t) = \cos t - 2\sin t$ $\quad$ $\dot{x}_2(t) = 3\cos t - \sin t$

$\dot{x}_1(t) - \dot{x}_2(t) = -x_1(t)$ $\quad$ $\cos t - 2\sin t - 3\cos t + \sin t = -\sin t - 2\cos t$ ✓

$\dot{x}_2(t) = 2x_1(t) - x_2(t)$ $\quad$ $3\cos t - \sin t = 2\sin t + 4\cos t - 3\sin t - \cos t$ ✓

FINALLY, LET US CONSIDER THE SDE $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{pmatrix} -1 & -\frac{7}{2} \\ 2 & 3 \end{pmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & -\frac{7}{2} \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1/2 \\ 2 & 3 \end{pmatrix}$$

SOLUTION: $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \operatorname{Exp}\left[\begin{pmatrix} 1 & -1/2 \\ 2 & 3 \end{pmatrix} t\right]\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

EIGENVALUES

$$\begin{vmatrix} 1-\lambda & -1/2 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 + 1 = 0 \Rightarrow (\lambda - 2)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 2$$

$$\begin{pmatrix} 1 & -1/2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} s_1 - s_3/2 & s_2 - s_4/2 \\ 2s_1 + 3s_3 & 2s_2 + 3s_4 \end{pmatrix} = \begin{pmatrix} 2s_1 & s_1 + 2s_2 \\ 2s_3 & s_3 + 2s_4 \end{pmatrix}$$

$s_1 = -s_3/2$ $\quad$ $s_3 = -2$ $\quad$ $s_2 - s_4/2 = 1 + 2s_2$ $\quad$ $s_4 = 0$ $\quad$ $s_2 = -1$

$2s_1 = -s_3$ $\quad$ $s_1 = 1$ $\quad$ $2s_2 + 3s_4 = -2 + s_4$

$$S = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \frac{e^{2t}}{2}\begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \frac{e^{2t}}{2}\begin{pmatrix} -1 & 1-t \\ 2 & 2t \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \frac{e^{2t}}{2}\begin{pmatrix} 4-5t \\ 2+10t \end{pmatrix}$$

INITIAL CONDITION OK

CHECKING THE SOLUTION

$$\dot{x}_1(t) = e^{2t}(4-5t) - \frac{5}{2}e^{2t} = e^{2t}\left(\frac{3}{2}-5t\right)$$

$$\dot{x}_2(t) = e^{2t}(2+10t) + 5e^{2t} = e^{2t}(7+10t)$$

$$\dot{x}_1(t) - \dot{x}_2(t) = -x_1(t) - 7/2\, x_2(t) \qquad \cancel{e^{2t}}$$

$$\frac{3}{2} - 5t - 7 - 10t = -2 + \frac{5}{2}t - \frac{7}{2}(1+5t)$$

$$-\frac{11}{2} - 15t = -\frac{11}{2} - \frac{30}{2}t \quad \checkmark$$

$$\dot{x}_2(t) = 2x_1(t) + 3x_2(t) \qquad \cancel{e^{2t}}$$

$$7 + 10t = 4 - 5t + 3 + 15t \quad \checkmark$$