Forma diagonal para calcular exp[M]

$$\exp[M] = \mathbb{1} + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots + \frac{M^n}{n!} + \cdots$$

$$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad M^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad M^3 = \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} \quad M^4 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

$$M^5 = \begin{pmatrix} 4 & 4 \\ 4 & -4 \end{pmatrix} \quad M^6 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} \quad M^7 = \begin{pmatrix} 8 & 8 \\ 8 & -8 \end{pmatrix}$$

$$\{\exp[M]\}_{11} = \underbrace{1 + 1 + \frac{2}{2!} + \frac{2}{3!} + \frac{4}{4!} + \frac{4}{5!}}_{3 + \frac{1}{3} + \frac{1}{3!}\left(1 + \frac{1}{5}\right)} + \underbrace{\frac{8}{6!} + \frac{8}{7!}}_{\frac{8}{6!}\left(1 + \frac{1}{7}\right)}$$

$$\frac{10}{3} + \frac{1}{5}$$

$$\frac{53}{15}$$

$$\frac{8}{6!}\left(1 + \frac{1}{7}\right) = \frac{\cancel{2} \cdot 4}{7 \cdot \cancel{6}^{3} \cdot 5 \cdot \cancel{4} \cdot 3 \cdot \cancel{2}} = \frac{4}{315}$$

$$\exp[M] \approx \begin{pmatrix} \frac{53}{15} & \frac{41}{30} \\ \frac{41}{30} & \frac{4}{5} \end{pmatrix}$$

Tentaremos achar a solução exata

SE $M = S\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} S^{-1} = SDS^{-1}$

$$\exp[M] = \mathbb{1} + SDS^{-1} + \frac{\cancel{SDS^{-1}}\cancel{SDS^{-1}}}{2!} + \frac{\cancel{SDS^{-1}}\cancel{SDS^{-1}}\cancel{SDS^{-1}}}{3!} + \cdots$$

$$= SS^{-1} + SDS^{-1} + S\frac{D^2}{2!}S^{-1} + S\frac{D^3}{3!}S^{-1} + \cdots$$

$$= S\left(\mathbb{1} + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots\right)S^{-1}$$

$$= S\begin{pmatrix} 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \cdots & 0 \\ 0 & 1 + \beta + \frac{\beta^2}{2!} + \frac{\beta^3}{3!} + \cdots \end{pmatrix} S^{-1}$$

$$= \boxed{S\begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{\beta} \end{pmatrix} S^{-1}}$$

α e β AUTOVALORES DE M

$$\det\begin{pmatrix} 1 - \lambda & 1 \\ 1 & -1 - \lambda \end{pmatrix} = 0$$

DETERMINA OS AUTOVALORES

$$-(1 - \lambda)(1 + \lambda) - 1 = 0$$

$$\lambda^2 = 2$$

$\alpha = \sqrt{2} \qquad \beta = -\sqrt{2}$

AGORA TEREMOS QUE ENCONTRAR A MATRIZ S!

$$M\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \longrightarrow$$

$$M\begin{pmatrix} a \\ c \end{pmatrix} = \alpha\begin{pmatrix} a \\ c \end{pmatrix}$$

$$M\begin{pmatrix} b \\ d \end{pmatrix} = \beta\begin{pmatrix} b \\ d \end{pmatrix}$$

AUTOVETORES

$$\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}a\\ c\end{pmatrix}=\sqrt{2}\begin{pmatrix}a\\ c\end{pmatrix} \rightarrow \begin{matrix}a+c=\sqrt{2}\,a\\ a-c=\sqrt{2}\,c\end{matrix}$$

ESCOLHEMOS POR EXEMPLO $a=1$ $\rightarrow$ $c=\sqrt{2}-1$

$$\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}b\\ d\end{pmatrix}=-\sqrt{2}\begin{pmatrix}b\\ d\end{pmatrix} \rightarrow \begin{matrix}b+d=-\sqrt{2}\,b\\ b-d=-\sqrt{2}\,d\end{matrix}$$

$b=-1$ $\rightarrow$ $d=1+\sqrt{2}$

$$S=\begin{pmatrix}1 & -1\\ \sqrt{2}-1 & 1+\sqrt{2}\end{pmatrix} \rightarrow S^{-1}=\frac{1}{2\sqrt{2}}\begin{pmatrix}1+\sqrt{2} & 1\\ 1-\sqrt{2} & 1\end{pmatrix}$$

$$\exp[M]=\frac{1}{2\sqrt{2}}\begin{pmatrix}1 & -1\\ \sqrt{2}-1 & 1+\sqrt{2}\end{pmatrix}\begin{pmatrix}e^{\sqrt{2}} & 0\\ 0 & e^{-\sqrt{2}}\end{pmatrix}\begin{pmatrix}1+\sqrt{2} & 1\\ 1-\sqrt{2} & 1\end{pmatrix}$$

$$=\frac{1}{2\sqrt{2}}\begin{pmatrix}e^{\sqrt{2}} & -e^{-\sqrt{2}}\\ (\sqrt{2}-1)e^{\sqrt{2}} & (1+\sqrt{2})e^{-\sqrt{2}}\end{pmatrix}\begin{pmatrix}1+\sqrt{2} & 1\\ 1-\sqrt{2} & 1\end{pmatrix}$$

(11) $\frac{1}{2\sqrt{2}}\left[e^{\sqrt{2}}(1+\sqrt{2})-e^{-\sqrt{2}}(1-\sqrt{2})\right]=\cosh\sqrt{2}+\frac{\sinh\sqrt{2}}{\sqrt{2}}$

(22) $\frac{1}{2\sqrt{2}}\left[(\sqrt{2}-1)e^{\sqrt{2}}+(1+\sqrt{2})e^{-\sqrt{2}}\right]=\cosh\sqrt{2}-\frac{\sinh\sqrt{2}}{\sqrt{2}}$

(12=21) $\frac{1}{2\sqrt{2}}\left(e^{\sqrt{2}}-e^{-\sqrt{2}}\right)=\frac{\sinh\sqrt{2}}{\sqrt{2}}$

$$\exp[M]=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{2}\cosh\sqrt{2}+\sinh\sqrt{2} & \sinh\sqrt{2}\\ \sinh\sqrt{2} & \sqrt{2}\cosh\sqrt{2}-\sinh\sqrt{2}\end{pmatrix}$$

NUM

$$\exp[M]\approx\begin{pmatrix}3.533 & 1.367\\ 1.367 & 0.8\end{pmatrix} \qquad \exp[M]\approx\begin{pmatrix}3.546 & 1.368\\ 1.368 & 0.810\end{pmatrix}$$

EXERCÍCIOS COM RESPOSTA

$M:\begin{pmatrix}7 & -2\\ 2 & 2\end{pmatrix}$ $\quad (\alpha,\beta):\ (3,6)$ $\quad S:\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$

$\begin{pmatrix}3 & -1\\ 2 & 0\end{pmatrix}$ $\quad (1,2)$ $\quad \begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}$

$\begin{pmatrix}7 & -1\\ -1 & 7\end{pmatrix}$ $\quad (6,8)$ $\quad \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$