Determinant Cofactors Transpose Adjoint Inverse

$$\begin{matrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{matrix}$$

M

$$\textcircled{1} \quad \begin{matrix} 1 & 1 \\ 2 & 1 \end{matrix} \quad (-1)^{1+1} \qquad \textcircled{2} \quad \begin{matrix} 2 & 1 \\ 1 & 1 \end{matrix} \quad (-1)^{1+2} \qquad \textcircled{0} \quad \begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix} \quad (-1)^{1+3}$$

$$1 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 0 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$-1 \qquad\qquad 1$$

$\det M = -3$

Let us now calculate the matrix of cofactors

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \qquad C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \qquad C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} \qquad C_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \qquad C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} \qquad C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \qquad C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$C = \begin{pmatrix} -1 & -1 & 3 \\ -2 & 1 & 0 \\ 2 & -1 & -3 \end{pmatrix}$$

The adjoint matrix is obtained from the C matrix by using the transpose operation

$$A = C^t = \begin{pmatrix} -1 & -2 & 2 \\ -1 & 1 & -1 \\ 3 & 0 & -3 \end{pmatrix}$$

Finally the inverse of M is given by

$$M^{-1} = \frac{1}{\det M} A = \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 1/3 & -1/3 & 1/3 \\ -1 & 0 & 1 \end{pmatrix}$$

Let us now consider the matrix

$$M = \begin{pmatrix} 1 & 2 & b \\ 1 & 1 & 0 \\ 3 & 1 & a \end{pmatrix} \qquad \det M = 1 \cdot \begin{vmatrix} 1 & 0 \\ 1 & a \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 0 \\ 3 & a \end{vmatrix} + b \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= a - 2a - 2b = -(a + 2b)$$

$\det M = -(a + 2b)$

$$C_{11} = \begin{vmatrix} 1 & 0 \\ 1 & a \end{vmatrix} \quad C_{12} = -\begin{vmatrix} 1 & 0 \\ 3 & a \end{vmatrix} \quad C_{13} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}$$

$$C_{21} = -\begin{vmatrix} 2 & b \\ 1 & a \end{vmatrix} \quad C_{22} = \begin{vmatrix} 1 & b \\ 3 & a \end{vmatrix} \quad C_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$C_{31} = \begin{vmatrix} 2 & b \\ 1 & 0 \end{vmatrix} \quad C_{32} = -\begin{vmatrix} 1 & b \\ 1 & 0 \end{vmatrix} \quad C_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

$$C = \begin{pmatrix} a & -a & -2 \\ b-2a & a-3b & 5 \\ -b & b & -1 \end{pmatrix}$$

$$M^{-1} = \frac{-1}{a+2b} \begin{pmatrix} a & b-2a & -b \\ -a & a-3b & b \\ -2 & 5 & -1 \end{pmatrix} \xrightarrow{a=b} \frac{1}{3} \begin{pmatrix} -1 & 1 & 1 \\ 1 & 2 & -1 \\ 2/a & -5/a & 1/a \end{pmatrix}$$

For example, let us solve

$$\begin{pmatrix} 1 & 2 & a \\ 1 & 1 & 0 \\ 3 & 1 & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 8 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 & 1 & 1 \\ 1 & 2 & -1 \\ 2/a & -5/a & \frac{1}{a} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 7 \\ -7 \\ 10/a \end{pmatrix}$$

For 4x4 matrices we use the decomposition in blocks of 2x2 matrices

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$$

$$A\tilde{A} + B\tilde{C} = \mathbb{1}$$
$$C\tilde{B} + D\tilde{D} = \mathbb{1}$$
$$A\tilde{B} + B\tilde{D} = C\tilde{A} + D\tilde{C} = 0$$

From $A\tilde{B} + B\tilde{D} = C\tilde{A} + D\tilde{C} = 0$ we get

$$\tilde{B} = -A^{-1}B\tilde{D} \qquad \tilde{D} = -B^{-1}A\tilde{B} \qquad \tilde{A} = -C^{-1}D\tilde{C} \qquad \tilde{C} = -D^{-1}C\tilde{A}$$

From $A\tilde{A} + B\tilde{C} = C\tilde{B} + D\tilde{D} = \mathbb{1}$ we obtain

$$A\tilde{A} - BD^{-1}C\tilde{A} = -AC^{-1}D\tilde{C} + B\tilde{C} = \mathbb{1}$$
$$C\tilde{B} - DB^{-1}A\tilde{B} = -CA^{-1}B\tilde{D} + D\tilde{D} = \mathbb{1}$$

$$\tilde{A} = (A - BD^{-1}C)^{-1} \qquad \tilde{A} = -C^{-1}D\tilde{C}$$
$$\tilde{B} = (C - DB^{-1}A)^{-1} \qquad \tilde{B} = -A^{-1}B\tilde{D}$$
$$\tilde{C} = (B - AC^{-1}D)^{-1} \qquad \tilde{C} = -D^{-1}C\tilde{A}$$
$$\tilde{D} = (D - CA^{-1}B)^{-1} \qquad \tilde{D} = -B^{-1}A\tilde{B}$$

$$M = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 2 & 3 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad \cancel{B^{-1}}$$

$$C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \quad \boxed{\tilde{B} = -A^{-1}B\tilde{D}}$$

$$\tilde{A} = (A - BD^{-1}C)^{-1} = -C^{-1}D\tilde{C} \qquad \tilde{C} = (B - AC^{-1}D)^{-1} = -D^{-1}C\tilde{A} \qquad \boxed{\tilde{D} = (D - CA^{-1}B)^{-1}}$$

$$A=\begin{pmatrix}1&1\\2&1\end{pmatrix}\qquad B=\begin{pmatrix}1&2\\1&2\end{pmatrix}\qquad C=\begin{pmatrix}1&2\\2&1\end{pmatrix}\qquad D=\begin{pmatrix}0&1\\2&3\end{pmatrix}$$

$$A^{-1}=\begin{pmatrix}-1&1\\2&-1\end{pmatrix}\qquad \cancel{B^{-1}}\qquad C^{-1}=\begin{pmatrix}-1/3&2/3\\2/3&-1/3\end{pmatrix}\qquad D^{-1}=\begin{pmatrix}-3/2&1/2\\1&0\end{pmatrix}$$

$$\tilde{A}=\left[\begin{pmatrix}1&1\\2&1\end{pmatrix}-\begin{pmatrix}1&2\\1&2\end{pmatrix}\begin{pmatrix}-3/2&1/2\\1&0\end{pmatrix}\begin{pmatrix}1&2\\2&1\end{pmatrix}\right]^{-1}$$

$$=\left[\begin{pmatrix}1&1\\2&1\end{pmatrix}-\begin{pmatrix}1/2&1/2\\1/2&1/2\end{pmatrix}\begin{pmatrix}1&2\\2&1\end{pmatrix}\right]^{-1}$$

$$=\left[\begin{pmatrix}1&1\\2&1\end{pmatrix}-\begin{pmatrix}3/2&3/2\\3/2&3/2\end{pmatrix}\right]^{-1}$$

$$=\begin{bmatrix}-1/2&-1/2\\1/2&-1/2\end{bmatrix}^{-1}=\left[-\frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\right]^{-1}=-2\begin{pmatrix}1&1\\-1&1\end{pmatrix}^{-1}$$

$$=-2\,\frac{1}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix}=\begin{pmatrix}-1&1\\-1&-1\end{pmatrix}$$

$$\tilde{A}=\begin{pmatrix}-1&1\\-1&-1\end{pmatrix}$$

$$\tilde{C}=-D^{-1}C\tilde{A}=-\begin{pmatrix}-3/2&1/2\\1&0\end{pmatrix}\begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}-1&1\\-1&-1\end{pmatrix}$$

$$=\begin{pmatrix}3/2&-1/2\\-1&0\end{pmatrix}\begin{pmatrix}-3&-1\\-3&1\end{pmatrix}=\begin{pmatrix}-9/2+3/2&-3/2-1/2\\3&1\end{pmatrix}$$

$$\tilde{C}=\begin{pmatrix}-3&-2\\3&1\end{pmatrix}$$

$$\tilde{D}=(D-CA^{-1}B)^{-1}=\left[\begin{pmatrix}0&1\\2&3\end{pmatrix}-\begin{pmatrix}1&2\\2&1\end{pmatrix}\begin{pmatrix}-1&1\\2&-1\end{pmatrix}\begin{pmatrix}1&2\\1&2\end{pmatrix}\right]^{-1}$$

$$=\left[\begin{pmatrix}0&1\\2&3\end{pmatrix}-\begin{pmatrix}3&-1\\0&1\end{pmatrix}\begin{pmatrix}1&2\\1&2\end{pmatrix}\right]^{-1}=\left[\begin{pmatrix}0&1\\2&3\end{pmatrix}-\begin{pmatrix}2&4\\1&2\end{pmatrix}\right]^{-1}$$

$$=\begin{bmatrix}-2&-3\\1&1\end{bmatrix}^{-1}=\begin{bmatrix}1&3\\-1&-2\end{bmatrix}$$

$$\tilde{D}=\begin{pmatrix}1&3\\-1&-2\end{pmatrix}$$

$$\tilde{B}=-A^{-1}B\tilde{D}=\begin{pmatrix}1&-1\\-2&1\end{pmatrix}\begin{pmatrix}1&2\\1&2\end{pmatrix}\begin{pmatrix}1&3\\-1&-2\end{pmatrix}=\begin{pmatrix}0&0\\-1&-2\end{pmatrix}\begin{pmatrix}1&3\\-1&-2\end{pmatrix}$$

$$=\begin{pmatrix}0&0\\1&1\end{pmatrix}$$

$$\tilde{B}=\begin{pmatrix}0&0\\1&1\end{pmatrix}$$

Finally

$$M=\begin{pmatrix}1&1&1&2\\2&1&1&2\\1&2&0&1\\2&1&2&3\end{pmatrix}\qquad M^{-1}=\begin{pmatrix}\tilde{A}&\tilde{B}\\\tilde{C}&\tilde{D}\end{pmatrix}=\begin{pmatrix}-1&1&0&0\\-1&-1&1&1\\-3&-2&1&3\\3&1&-1&-2\end{pmatrix}$$

Let us now solve the following system $(a \geq 0)$

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1+a & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} (1-a)N \\ 0 \\ aN \\ 0 \end{bmatrix}$$

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\cancel{B^{-1}} \qquad \cancel{C^{-1}}$$

$$D = \begin{pmatrix} -1 & 0 \\ 1+a & -1 \end{pmatrix} \qquad D^{-1} = \begin{pmatrix} -1 & 0 \\ -1-a & -1 \end{pmatrix}$$

$$\tilde{A} = (A - BD^{-1}C)^{-1} = \left[\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ -1-a & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right]^{-1}$$

$$= \left[\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 0 & -2-a \end{pmatrix}\right]^{-1} = \left[\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -4-2a \end{pmatrix}\right]^{-1}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & -3-2a \end{pmatrix}^{-1} = \frac{1}{-3-2a+1}\begin{pmatrix} -3-2a & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2(a+1)}\begin{pmatrix} 3+2a & -1 \\ 1 & -1 \end{pmatrix}$$

$$\tilde{A} = \frac{1}{2(a+1)}\begin{pmatrix} 3+2a & -1 \\ 1 & -1 \end{pmatrix}$$

$$\tilde{C} = -\begin{pmatrix} -1 & 0 \\ -1-a & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\frac{1}{2(a+1)}\begin{pmatrix} 3+2a & -1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2(a+1)}\begin{pmatrix} 1 & 0 \\ a+1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2(a+1)}\begin{pmatrix} 1 & -1 \\ a+2 & -a-2 \end{pmatrix}$$

$$\tilde{C} = \frac{1}{2(a+1)}\begin{pmatrix} 1 & -1 \\ a+2 & -(a+2) \end{pmatrix}$$

$$\tilde{D} = (D - CA^{-1}B)^{-1} = \left[\begin{pmatrix} -1 & 0 \\ 1+a & -1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\right]^{-1}$$

$$= \left[\begin{pmatrix} -1 & 0 \\ 1+a & -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ 0 & -2 \end{pmatrix}\right]^{-1} = \left[\begin{pmatrix} -1 & 0 \\ 1+a & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}\right]^{-1}$$

$$= \left[\begin{pmatrix} -1 & 0 \\ 1+a & -1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right]^{-1} = \begin{pmatrix} -1 & 1 \\ 1+a & 0 \end{pmatrix}^{-1} = \frac{1}{-(1+a)}\begin{pmatrix} 0 & -1 \\ -(1+a) & -1 \end{pmatrix}$$

$$\tilde{D} = \frac{1}{a+1}\begin{pmatrix} 0 & 1 \\ a+1 & 1 \end{pmatrix}$$

$$\tilde{B} = -A^{-1}B\tilde{D} = -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}\frac{1}{a+1}\begin{pmatrix} 0 & 1 \\ a+1 & 1 \end{pmatrix}$$

$$= \frac{1}{2(a+1)}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ a+1 & 1 \end{pmatrix} = \frac{1}{2(a+1)}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 2(a+1) & 2 \end{pmatrix}$$

$$= \frac{1}{2(a+1)}\begin{pmatrix} 2(a+1) & 2 \\ 2(a+1) & 2 \end{pmatrix}$$

$$\tilde{B} = \frac{1}{2(a+1)}\begin{pmatrix} 2(a+1) & 2 \\ 2(a+1) & 2 \end{pmatrix}$$

Solutions :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \frac{1}{2(a+1)}\begin{pmatrix} 3+2a & -1 & 2(a+1) & 2 \\ 1 & -1 & 2(a+1) & 2 \\ 1 & -1 & 0 & 2 \\ a+2 & -(a+2) & 2(a+1) & 2 \end{pmatrix}\begin{pmatrix} (1-a)N \\ 0 \\ aN \\ 0 \end{pmatrix}$$

$$= \frac{N}{2(a+1)}\begin{bmatrix} (3+2a)(1-a)+2a(a+1) \\ 1-a+2a(a+1) \\ 1-a \\ (a+2)(1-a)+2a(a+1) \end{bmatrix} = \frac{N}{2(a+1)}\begin{bmatrix} a+3 \\ 2a^2+a+1 \\ 1-a \\ a^2+a+2 \end{bmatrix}$$