REDUCED TRIANGULAR FORM

THE RTF CAN BE USEFUL TO DETERMINE THE LINEAR COMBINATION OF VECTORS, SEE FOR EXAMPLE THE FOLLOWING SET OF LINEAR DEPENDET VECTORS

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 2 & 1 \end{matrix}$$

FIND THE LINEAR VECTOR AND WRITE THE LINEAR COMBINATION

$$\begin{matrix} -2 & -2 & -2 & -2 \\ 2 & 1 & 3 & 0 \\ \hline 0 & -1 & 1 & -2 \end{matrix} \rightarrow 0 \; 1 \; -1 \; 2$$

$$\begin{matrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 2 & 1 \\ \hline 0 & -1 & 1 & 0 \end{matrix} \rightarrow 0 \; 1 \; -1 \; 0$$

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 0 \end{matrix}$$

$$\begin{matrix} 0 & -1 & 1 & -2 \\ 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & -2 \end{matrix} \rightarrow 0 \; 0 \; 0 \; 1$$

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{matrix}$$

LET US NOW CALCULATE THE REDUCED FORM

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -2 \\ \hline 1 & 0 & 2 & -1 \end{matrix} \rightarrow \begin{matrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 2 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ \hline 0 & 1 & -1 & 0 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \checkmark & \checkmark & \times & \checkmark \end{matrix} \qquad \text{RTF}$$

$$2\begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$2\begin{pmatrix}1\\2\\1\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\3\\2\end{pmatrix} \checkmark$$

CHECK

$$2V_1 - V_2 = V_3$$

LET US NOW SEE HOW WE CAN USE THE RTF TO FIND THE INVERSE OF A MATRIX
THE IDEA IS REPEAT THE OPERATIONS DONE TO OBTAIN THE RTF STARTING FROM THE IDENTITY MATRIX

$$M \xrightarrow[M-1]{RTF} I \qquad I \xrightarrow[M-1]{RTF} M^{-1}$$

$$\begin{matrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 2 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{matrix} \qquad \begin{matrix} 2 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & 2 \end{matrix} \qquad \begin{matrix} 1-2 \\ 1-3 \end{matrix} \qquad \begin{matrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix} \qquad \begin{matrix} 2 & 0 & 0 \\ 2 & -1 & 0 \\ 2 & 0 & -2 \end{matrix}$$

$$\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}$$

$$\begin{array}{ccc|ccc} 2 & 2 & 2 & 2 & 0 & 0 \\ 0 & 1 & 2 & 2 & -1 & 0 \\ 0 & 2 & 0 & 2 & 0 & -2 \end{array}$$

$$\begin{array}{ccc|c|ccc} 2 & 2 & 2 & & 2 & 0 & 0 \\ 0 & 2 & 4 & 2 & 4 & -2 & 0 \\ 0 & 2 & 0 & & 2 & 0 & -2 \end{array}$$

$$\begin{array}{ccc|c|ccc} 2 & 0 & 2 & 1-3 & 0 & 0 & 2 \\ 0 & 2 & 4 & & 4 & -2 & 0 \\ 0 & 0 & 4 & 2-3 & 2 & -2 & 2 \end{array}$$

$$\begin{array}{ccc|c|ccc} 4 & 0 & 4 & 2 & 0 & 0 & 4 \\ 0 & 2 & 4 & & 4 & -2 & 0 \\ 0 & 0 & 4 & & 2 & -2 & 2 \end{array}$$

$$\begin{array}{ccc|c|ccc} 4 & 0 & 0 & 1-3 & -2 & 2 & 2 \\ 0 & 2 & 0 & 2-3 & 2 & 0 & -2 \\ 0 & 0 & 4 & & 2 & -2 & 2 \end{array}$$

$$\begin{array}{ccc|c|ccc} 1 & 0 & 0 & 1/4 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1/4 & 1/2 & -1/2 & 1/2 \end{array}$$

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad M^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

CHECK $\quad \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$

FIRST LINE $\quad \frac{1}{2}\left(-1+2+1,\ 1-1,\ 1-2+1\right) = (1,0,0)$

SECOND LINE $\quad \frac{1}{2}\left(-2+2,\ 2,\ 2-2\right) = (0,1,0)$

THIRD LINE $\quad \frac{1}{2}\left(-1+1,\ 1-1,\ 1+1\right) = (0,0,1)$

LET US NOW SOLVE A PROBLEM OF LINEAR DEPENDENCE, COMPONENTS IN A NEW BASIS, ORTOGONALIZATION AND INVERSE

$$\begin{array}{ccc|c|ccc} 2 & a & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 & 1 \end{array} \qquad \begin{array}{ccc|c|ccc} 2 & a & 1 & 3 & 1 & 0 & 0 \\ 0 & a & a & a & 0 & a & 0 \\ 0 & a & -3 & 1 & 1 & 0 & -2 \end{array}$$

$$\begin{array}{ccc|c|ccc} 2 & a & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 2 & 0 & 0 & 2 \end{array} \qquad \begin{array}{cccc|c|ccc} & 2 & a & 1 & 3 & 1 & 0 & 0 \\ & 0 & a & a & a & 0 & a & 0 \\ (2-3) & 0 & 0 & a+3 & a+1 & -1 & a & 2 \end{array}$$

$$\begin{array}{cccc|c|ccc} & 2 & a & 1 & 3 & 1 & 0 & 0 \\ & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ (1-3) & 0 & a & -3 & 1 & 1 & 0 & -2 \end{array} \qquad \begin{array}{ccc|c|ccc} 2 & a & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & a+3 & a+1 & -1 & a & 2 \end{array}$$

LINEAR DEPENDENCE $a = -3$ $\qquad$ x_3 FREE

NON TRIVIAL LINEAR COMBINATION

$(-2,-1,1)\, x_3$

$2x_1 - 3x_2 + x_3 = 0 \qquad x_2 + x_3 = 0$

$x_1 = (3x_2 - x_3)/2 \qquad x_2 = -x_3$

$= -2x_3$

COMPONENTS OF $\begin{pmatrix}3\\1\\1\end{pmatrix}$ IN THE NEW BASIS $\begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}a\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\2\end{pmatrix} \qquad a \neq -3$

$$\begin{aligned} 2x_1 + a x_2 + x_3 &= 3 \\ x_2 + x_3 &= 1 \\ (a+3)x_3 &= a-1 \end{aligned} \qquad \begin{aligned} x_1 &= 3 - a x_2 - x_3 = (3a+9-4a-a+1)/2(a+3) \\ x_2 &= (\cancel{a}+3-\cancel{a}+1)/(a+3) = 4/(a+3) \\ x_3 &= (a-1)/(a+3) \end{aligned}$$

COMPONENTS IN THE NEW BASIS $(5-a,\, 4,\, a-1)/(a+3)$

CHECKING THE SOLUTIONS:

$a=-3 \qquad -2\begin{pmatrix}2\\0\\1\end{pmatrix} - \begin{pmatrix}-3\\1\\0\end{pmatrix} + \begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}-4+3+1\\-1+1\\-2+2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ ✓

$a \neq -3 \qquad \frac{5-a}{a+3}\begin{pmatrix}2\\0\\1\end{pmatrix} + \frac{4}{a+3}\begin{pmatrix}a\\1\\0\end{pmatrix} + \frac{a-1}{a+3}\begin{pmatrix}1\\1\\2\end{pmatrix}$

$\to \frac{1}{a+3}\begin{pmatrix}10-2a+4a+a-1\\4+a-1\\5-a+2a-2\end{pmatrix} = \frac{1}{a+3}\begin{pmatrix}3a+9\\a+3\\a+3\end{pmatrix} = \begin{pmatrix}3\\1\\1\end{pmatrix}$ ✓

FOR $a=1$ ORTOGONALIZE AND FIND THE INVERSE

$$\left(\begin{array}{ccc|ccc} 2&1&1&1&0&0\\ 0&1&1&0&1&0\\ 1&0&2&0&0&1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 2&1&1&1&0&0\\ 0&1&1&0&1&0\\ 0&0&4&-1&1&2 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 2&0&0&1&-1&0\\ 0&1&0&\frac14&\frac34&-\frac12\\ 0&0&1&-\frac14&\frac14&\frac12 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1&0&1&\frac12&-\frac12&0\\ 0&1&0&+\frac14&\frac34&-\frac12\\ 0&0&1&-\frac14&\frac14&\frac12 \end{array}\right) \qquad M = \begin{pmatrix}2&1&1\\0&1&1\\1&0&2\end{pmatrix} \qquad M^{-1} = \frac14\begin{pmatrix}2&-2&0\\+1&3&-2\\-1&1&2\end{pmatrix}$$

checking first line $(4+1-1)/4=1,\ (-4+3+1)/4=0,\ (-2+2)/4=0$

second line $(1-1)/4=0,\ (3+1)/4=1,\ (2+2)/4=0$

third line $(2-2)/4=0,\ (-2+2)/4=0,\ 4/4=1$ ✓

GS ORTOGONALIZATION PROCESS $\quad S_1 = \begin{pmatrix}2\\0\\1\end{pmatrix}$

$$S_2 = \begin{pmatrix}1\\1\\0\end{pmatrix} - \frac{(2\,0\,1)\begin{pmatrix}1\\1\\0\end{pmatrix}}{(2\,0\,1)\begin{pmatrix}2\\0\\1\end{pmatrix}}\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} - \frac25\begin{pmatrix}2\\0\\1\end{pmatrix} = \frac15\begin{pmatrix}5-4\\5\\-2\end{pmatrix} = \frac15\begin{pmatrix}1\\5\\-2\end{pmatrix} \Rightarrow S_2 = \begin{pmatrix}1\\5\\-2\end{pmatrix}$$

$S_1 \cdot S_2 = 2-2=0$ ✓

$$S_3 = \begin{pmatrix}1\\1\\2\end{pmatrix} - \frac{(2\,0\,1)\begin{pmatrix}1\\1\\2\end{pmatrix}}{(2\,0\,1)\begin{pmatrix}2\\0\\1\end{pmatrix}}\begin{pmatrix}2\\0\\1\end{pmatrix} - \frac{(1\,5\,-2)\begin{pmatrix}1\\1\\2\end{pmatrix}}{(1\,5\,-2)\begin{pmatrix}1\\5\\-2\end{pmatrix}}\begin{pmatrix}1\\5\\-2\end{pmatrix}$$

$$= \begin{pmatrix}1\\1\\2\end{pmatrix} - \frac45\begin{pmatrix}2\\0\\1\end{pmatrix} - \frac{2}{30}\begin{pmatrix}1\\5\\-2\end{pmatrix} = \frac{1}{30}\begin{pmatrix}30-48-2\\30-10\\60-24+4\end{pmatrix} = \frac{1}{30}\begin{pmatrix}-20\\20\\40\end{pmatrix} \Rightarrow S_3 = \begin{pmatrix}-1\\1\\2\end{pmatrix}$$

$S_1 \cdot S_3 = -2+2=0$ ✓

$S_2 \cdot S_3 = -1+5-4=0$ ✓

NEW ORTOGONALIZED SET $\begin{pmatrix}2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\5\\-2\end{pmatrix}, \begin{pmatrix}-1\\1\\2\end{pmatrix}$