3d rotations

ROTATION OF (x,y,z) AROUND THE AXIS (α,β,γ) OF AN ANGLE θ

$\alpha^2+\beta^2+\gamma^2=1$

$$(x,y,z) \Rightarrow \begin{pmatrix} iz & x+iy \\ -x+iy & -iz \end{pmatrix}$$

PAULI MATRICES

$$\begin{pmatrix} iZ & X+iY \\ -X+iY & -iZ \end{pmatrix} = M\left(\frac{\theta}{2}\right) \begin{pmatrix} iz & x+iy \\ -x+iy & -iz \end{pmatrix} M\left(-\frac{\theta}{2}\right)$$

WHERE $$M(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} + i\gamma\sin\frac{\theta}{2} & (\alpha+i\beta)\sin\frac{\theta}{2} \\ (-\alpha+i\beta)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} - i\gamma\sin\frac{\theta}{2} \end{pmatrix}$$

EXAMPLE

$(x,y,z) = (1,0,0)$ $\quad \theta=\pi$ $\quad (\alpha,\beta,\gamma) = (0,0,1)$

$$M\left(\tfrac{\pi}{2}\right) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad M\left(-\tfrac{\pi}{2}\right) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$\begin{pmatrix} iZ & X+iY \\ -X+iY & -iZ \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\Rightarrow (X,Y,Z) = (-1,0,0)$

$\theta = \pi/2$ $\quad \cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$M\left(\tfrac{\pi}{4}\right) = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}/\sqrt{2}$$

$$M\left(-\tfrac{\pi}{4}\right) = \begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}/\sqrt{2}$$

$$\begin{pmatrix} iZ & X+iY \\ -X+iY & -iZ \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1-i & 0 \\ 0 & 1-i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 1+i \\ -1+i & 0 \end{pmatrix}\begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$\Rightarrow (X,Y,Z) = (0,1,0)$

$\theta = \pi$ $\quad (\alpha,\beta,\gamma) = (1,1,1)/\sqrt{3}$

$$\begin{pmatrix} iZ & X+iY \\ -X+iY & -iZ \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} i & 1+i \\ -1+i & -i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{1}{\sqrt{3}}\begin{pmatrix} -i & -1-i \\ 1-i & i \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} -1-i & i \\ i & -1+i \end{pmatrix}\begin{pmatrix} -i & -1-i \\ 1-i & i \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2i & -1+2i \\ 1+2i & -2i \end{pmatrix}$$

$\Rightarrow (X,Y,Z) = \frac{1}{3}(-1,2,2)$

{INITIAL VECTOR}

$\{1, 1, 1\}$

{MATRIX FORM OF THE INITIAL VECTOR}

$$\left\{\begin{pmatrix} i & 1+i \\ -1+i & -i \end{pmatrix}\right\}$$

{ROTATION ANGLE}

$\{180\}$

{ROTATION AXIS}

$$\left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\}$$

{ROTATION MATRIX}

$$\left\{\begin{pmatrix} 0 & \frac{1+i}{\sqrt{2}} \\ -\frac{1-i}{\sqrt{2}} & 0 \end{pmatrix}\right\}$$

{MATRIX FORM OF THE ROTATED VECTOR}

$$\left\{\begin{pmatrix} 0 & \frac{1+i}{\sqrt{2}} \\ -\frac{1-i}{\sqrt{2}} & 0 \end{pmatrix} \cdot \begin{pmatrix} i & 1+i \\ -1+i & -i \end{pmatrix} \cdot \begin{pmatrix} 0 & -\frac{1+i}{\sqrt{2}} \\ \frac{1-i}{\sqrt{2}} & 0 \end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix} -i & 1+i \\ -1+i & i \end{pmatrix}\right\}$$

{ROTATED VECTOR}

$\{\{1, 1, -1\}\}$

{INITIAL VECTOR}

$\{1, 1, 1\}$

{MATRIX FORM OF THE INITIAL VECTOR}

$$\left\{\begin{pmatrix} i & 1+i \\ -1+i & -i \end{pmatrix}\right\}$$

{ROTATION ANGLE}

$\{180\}$

{ROTATION AXIS}

$$\left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\}$$

{ROTATION MATRIX}

$$\left\{\begin{pmatrix} 0 & \frac{1-i}{\sqrt{2}} \\ -\frac{1+i}{\sqrt{2}} & 0 \end{pmatrix}\right\}$$

{MATRIX FORM OF THE ROTATED VECTOR}

$$\left\{\begin{pmatrix} 0 & \frac{1-i}{\sqrt{2}} \\ -\frac{1+i}{\sqrt{2}} & 0 \end{pmatrix} \cdot \begin{pmatrix} i & 1+i \\ -1+i & -i \end{pmatrix} \cdot \begin{pmatrix} 0 & -\frac{1-i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & 0 \end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix} -i & -1-i \\ 1-i & i \end{pmatrix}\right\}$$

{ROTATED VECTOR}

$\{\{-1, -1, -1\}\}$

{INITIAL VECTOR}

$$\{1, 1, 1\}$$

{MATRIX FORM OF THE INITIAL VECTOR}

$$\left\{\begin{pmatrix} i & 1+i \\ -1+i & -i \end{pmatrix}\right\}$$

{ROTATION ANGLE}

$$\{90\}$$

{ROTATION AXIS}

$$\{1, 0, 0\}$$

{ROTATION MATRIX}

$$\left\{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\right\}$$

{MATRIX FORM OF THE ROTATED VECTOR}

$$\left\{\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} i & 1+i \\ -1+i & -i \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix} i & 1-i \\ -1-i & -i \end{pmatrix}\right\}$$

{ROTATED VECTOR}

$$\{\{1, -1, 1\}\}$$

{INITIAL VECTOR}

$$\{1, -1, 0\}$$

{MATRIX FORM OF THE INITIAL VECTOR}

$$\left\{\begin{pmatrix} 0 & 1-i \\ -1-i & 0 \end{pmatrix}\right\}$$

{ROTATION ANGLE}

$$\{90\}$$

{ROTATION AXIS}

$$\left\{-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right\}$$

{ROTATION MATRIX}

$$\left\{\begin{pmatrix} \frac{i\sqrt{2}+\sqrt{3}}{\sqrt{6}} & -\frac{\frac{1}{2}+\frac{i}{2}}{\sqrt{3}} \\ \frac{\frac{1}{2}-\frac{i}{2}}{\sqrt{3}} & \frac{-i\sqrt{2}+\sqrt{3}}{\sqrt{6}} \end{pmatrix}\right\}$$

{MATRIX FORM OF THE ROTATED VECTOR}

$$\left\{\begin{pmatrix} \frac{i\sqrt{2}+\sqrt{3}}{\sqrt{6}} & -\frac{\frac{1}{2}+\frac{i}{2}}{\sqrt{3}} \\ \frac{\frac{1}{2}-\frac{i}{2}}{\sqrt{3}} & \frac{-i\sqrt{2}+\sqrt{3}}{\sqrt{6}} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1-i \\ -1-i & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{-i\sqrt{2}+\sqrt{3}}{\sqrt{6}} & \frac{\frac{1}{2}+\frac{i}{2}}{\sqrt{3}} \\ -\frac{\frac{1}{2}-\frac{i}{2}}{\sqrt{3}} & \frac{i\sqrt{2}+\sqrt{3}}{\sqrt{6}} \end{pmatrix}\right\}$$

$$\left\{\begin{pmatrix} \frac{1}{6}\left(i\sqrt{2}\left(-i\sqrt{2}+\sqrt{3}\right)+i\sqrt{2}\left(i\sqrt{2}+\sqrt{3}\right)\right) & \frac{1}{6}\left((-1+i)+(1-i)\left(i\sqrt{2}+\sqrt{3}\right)^2\right) \\ \frac{1}{6}\left((1+i)-(1+i)\left(-i\sqrt{2}+\sqrt{3}\right)^2\right) & \frac{1}{6}\left(-i\sqrt{2}\left(-i\sqrt{2}+\sqrt{3}\right)-i\sqrt{2}\left(i\sqrt{2}+\sqrt{3}\right)\right) \end{pmatrix}\right\}$$

{ROTATED VECTOR}

$$\left\{\left\{\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right\}\right\}$$