Gram- Schmidt ortogonalization process

It is a procedure which takes a nonorthogonal set of vectors and constructs an orthogonal basis

$\vec{V_1}, \vec{V_2}, \vec{V_3}, \dots, \vec{V_n}, \dots$

$\vec{U_1} = \vec{V_1}$

to determine the second vector we use $\vec{U_2} = \vec{V_2} + \alpha \vec{U_1}$ and the condition $\vec{U_1} \cdot \vec{U_2} = 0 \Rightarrow \vec{U_1} \cdot \vec{V_2} + \alpha |\vec{U_1}|^2 = 0$

$$\vec{U_2} = \vec{V_2} - \frac{\vec{U_1} \cdot \vec{V_2}}{|\vec{U_1}|^2} \vec{U_1}$$

OBSERVE THAT IF $\vec{V_2} = k \vec{U_1}$ $\quad \vec{U_2} = \vec{0}$!!!

to determine the third vector we use $\vec{U_3} = \vec{V_3} + \beta \vec{U_1} + \gamma \vec{U_2}$ and the conditions $\vec{U_1} \cdot \vec{U_3} = \vec{U_2} \cdot \vec{U_3} = 0$ leading to $\vec{U_1} \cdot \vec{V_3} + \beta |\vec{U_1}|^2 = 0$ and $\vec{U_2} \cdot \vec{V_3} + \gamma |\vec{U_2}|^2 = 0$

$$\vec{U_3} = \vec{V_3} - \frac{\vec{U_1} \cdot \vec{V_3}}{|\vec{U_1}|^2} \vec{U_1} - \frac{\vec{U_2} \cdot \vec{V_3}}{|\vec{U_2}|^2} \vec{U_2}$$

OBSERVE THAT IF $\vec{V_3} = k \vec{U_1} + r \vec{U_2}$ $\quad \vec{U_3} = \vec{0}$!!!

FINALLY
$$\vec{U_n} = \vec{V_n} - \sum_{i=1}^{n-1} \frac{\vec{U_i} \cdot \vec{V_n}}{|\vec{U_i}|^2} \vec{U_i} \qquad n \geq 2$$

FIRST EXAMPLE

$$\begin{bmatrix}1\\2\\3\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\\0\end{bmatrix}, \begin{bmatrix}3\\4\\5\\0\end{bmatrix}, \begin{bmatrix}0\\3\\4\\5\end{bmatrix} \qquad \begin{bmatrix}1\\2\\3\\0\end{bmatrix}$$

$$\vec{U_2} = \begin{bmatrix}2\\3\\4\\0\end{bmatrix} - \frac{[1\ 2\ 3\ 0]\begin{bmatrix}2\\3\\4\\0\end{bmatrix}}{[1\ 2\ 3\ 0]\begin{bmatrix}1\\2\\3\\0\end{bmatrix}} \begin{bmatrix}1\\2\\3\\0\end{bmatrix} = \begin{bmatrix}2\\3\\4\\0\end{bmatrix} - \frac{\overset{10}{\cancel{20}}}{\underset{7}{\cancel{14}}}\!\!\!\!\!\!\!\!\quad \frac{2+6+12+0}{1+4+9+0} \begin{bmatrix}1\\2\\3\\0\end{bmatrix}$$

$$\vec{U_2} = \begin{bmatrix}2\\3\\4\\0\end{bmatrix} - \frac{10}{7}\begin{bmatrix}1\\2\\3\\0\end{bmatrix} = \frac{1}{7}\begin{bmatrix}14-10\\21-20\\28-30\\0\end{bmatrix} = \frac{1}{7}\begin{bmatrix}4\\1\\-2\\0\end{bmatrix} \Rightarrow \begin{bmatrix}4\\1\\-2\\0\end{bmatrix}$$

$$\begin{bmatrix}1\\2\\3\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\\0\end{bmatrix}, \begin{bmatrix}3\\4\\5\\0\end{bmatrix}, \begin{bmatrix}0\\3\\4\\5\end{bmatrix} \qquad \begin{bmatrix}1\\2\\3\\0\end{bmatrix} \begin{bmatrix}4\\1\\-2\\0\end{bmatrix}$$

$$\vec{U_3} = \begin{bmatrix}3\\4\\5\\0\end{bmatrix} - \frac{\overset{\cancel{26}\ 13}{3+8+15}}{\underset{\cancel{14}\ 7}{1+4+9}} \begin{bmatrix}1\\2\\3\\0\end{bmatrix} - \frac{\overset{\cancel{6}\ 2}{12+4-10}}{\underset{\cancel{21}\ 7}{16+1+4}} \begin{bmatrix}4\\1\\-2\\0\end{bmatrix} = \frac{1}{7}\begin{bmatrix}21-13-8\\28-26-2\\35-39+4\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

This means that $\vec{v_3}$ is a linear combination of $\vec{v_1}$ and $\vec{v_3}$

in this case $2\vec{v_2} - \vec{v_1} = \vec{v_3}$

$$\begin{bmatrix}1\\2\\3\\0\end{bmatrix} \quad \begin{bmatrix}2\\3\\4\\0\end{bmatrix} \quad \begin{bmatrix}3\\4\\5\\0\end{bmatrix} \quad \begin{bmatrix}0\\3\\4\\5\end{bmatrix} \qquad \begin{bmatrix}1\\2\\3\\0\end{bmatrix} \quad \begin{bmatrix}4\\1\\-2\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\vec{u_4} = \begin{bmatrix}0\\3\\4\\5\end{bmatrix} - \frac{\overset{\cancel{18}\ 9}{0+6+12+0}}{\underset{\cancel{14}\ 7}{1+4+9+0}} \begin{bmatrix}1\\2\\3\\0\end{bmatrix} - \frac{\overset{-5}{0+3-8+0}}{\underset{21}{16+1+4+0}} \begin{bmatrix}4\\1\\-2\\0\end{bmatrix} = \frac{1}{21} \begin{bmatrix}0-27+20\\63-54+5\\84-81-10\\105+0+0\end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix}-7\\14\\-7\\105\end{bmatrix} = \frac{1}{3} \begin{bmatrix}-1\\2\\-1\\15\end{bmatrix} \Rightarrow \begin{bmatrix}1\\-2\\1\\-15\end{bmatrix}$$

$$\begin{bmatrix}1\\2\\3\\0\end{bmatrix} \quad \begin{bmatrix}2\\3\\4\\0\end{bmatrix} \quad \begin{bmatrix}3\\4\\5\\0\end{bmatrix} \quad \begin{bmatrix}0\\3\\4\\5\end{bmatrix} \qquad \begin{bmatrix}1\\2\\3\\0\end{bmatrix} \quad \begin{bmatrix}4\\1\\-2\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad \begin{bmatrix}1\\-2\\1\\-15\end{bmatrix}$$

SECOND EXAMPLE

$$\begin{bmatrix}1\\2\\3\\0\end{bmatrix} \quad \begin{bmatrix}2\\3\\4\\0\end{bmatrix} \quad \begin{bmatrix}3\\4\\5\\0\end{bmatrix} \quad \begin{bmatrix}0\\3\\4\\5\end{bmatrix} \quad \begin{bmatrix}2\\1\\1\\1\end{bmatrix} \qquad \begin{bmatrix}1\\2\\3\\0\end{bmatrix} \quad \begin{bmatrix}4\\1\\-2\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad \begin{bmatrix}1\\-2\\1\\-15\end{bmatrix} \quad \begin{bmatrix}5\\-10\\5\\2\end{bmatrix}$$

PROOF

$$\vec{u_5} = \begin{bmatrix}2\\1\\1\\1\end{bmatrix} - \frac{\overset{\cancel{7}\ 1}{2+2+3}}{\underset{\cancel{14}\ 2}{1+4+9}} \begin{bmatrix}1\\2\\3\\0\end{bmatrix} - \frac{\overset{\cancel{7}\ 1}{8+1-2}}{\underset{\cancel{21}\ 3}{16+1+4}} \begin{bmatrix}4\\1\\-2\\0\end{bmatrix} - \frac{\overset{-\cancel{14}\ 2}{2-2+1-15}}{\underset{\cancel{231}\ 33}{1+4+1+225}} \begin{bmatrix}1\\-2\\1\\-15\end{bmatrix}$$

$$= \frac{1}{66} \begin{bmatrix}132-33-88+4\\66-66-22-8\\66-99+44+4\\66+0+0-60\end{bmatrix} = \frac{1}{66} \begin{bmatrix}15\\-30\\15\\6\end{bmatrix} \Rightarrow \begin{bmatrix}5\\-10\\5\\2\end{bmatrix}$$

PROVE THE FOLLOWING

$$\begin{bmatrix}1\\2\\3\\0\end{bmatrix} \quad \begin{bmatrix}2\\2\\4\\0\end{bmatrix} \quad \begin{bmatrix}3\\4\\5\\0\end{bmatrix} \quad \begin{bmatrix}0\\3\\4\\5\end{bmatrix} \quad \begin{bmatrix}2\\1\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\2\\3\\0\end{bmatrix} \quad \begin{bmatrix}5\\-4\\1\\0\end{bmatrix} \quad \begin{bmatrix}1\\1\\-1\\0\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\5\end{bmatrix} \quad \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

THEN FIND THE LINEAR COMBINATIO WHICH ALLOWS TO WRITE THE LAST VECTOR IN TERMS OF THE FIRST FOUR VECTORS

$$\begin{matrix} 1 & 2 & 3 & 0 & 2 \\ 2 & 2 & 4 & 3 & 1 \\ 3 & 4 & 5 & 4 & 1 \\ 0 & 0 & 0 & 5 & 1 \end{matrix} \qquad \begin{matrix} 2 & 4 & 6 & 0 & 4 \\ -2 & -2 & -4 & -3 & -1 \\ \\ 0 & 2 & 2 & -3 & 3 \end{matrix} \qquad \begin{matrix} 3 & 6 & 9 & 0 & 6 \\ -3 & -4 & -5 & -4 & -1 \\ \\ 0 & 2 & 4 & -4 & 5 \end{matrix}$$

$$\begin{matrix} 1 & 2 & 3 & 0 & 2 \\ 0 & 2 & 2 & -3 & 3 \\ 0 & 2 & 4 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \end{matrix} \qquad \begin{matrix} 0 & -2 & -2 & 3 & -3 \\ 0 & 2 & 4 & -4 & 5 \\ \\ 0 & 0 & 2 & -1 & 2 \end{matrix}$$

$$\begin{matrix} 1 & 2 & 3 & 0 & 2 \\ 0 & 2 & 2 & -3 & 3 \\ 0 & 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & 5 & 1 \end{matrix}$$

$$x_1 = -2x_2 - 3x_3 + 2$$
$$2x_2 = -2x_3 + 3x_4 + 3$$
$$2x_3 = x_4 + 2$$
$$5x_4 = 1$$

$$x_1 = -27/10$$
$$x_2 = 7/10$$
$$x_3 = 11/10$$
$$x_4 = 1/5$$

$$2x_2 = -\frac{11}{5} + \frac{3}{5} + 3 = \frac{7}{5}$$

$$x_1 = -\frac{7}{5} - \frac{33}{10} + 2 = \frac{-14-33+20}{10} = -\frac{27}{10}$$

$$\frac{1}{10}\begin{bmatrix} -27 + 14 + 33 + 0 \\ -54 + 14 + 44 + 6 \\ -81 + 28 + 55 + 8 \\ 0 + 0 + 0 + 10 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 20 \\ 10 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \checkmark$$

$$(x_1, x_2, x_3, x_4) = \frac{1}{10}(-27, 7, 11, 2)$$