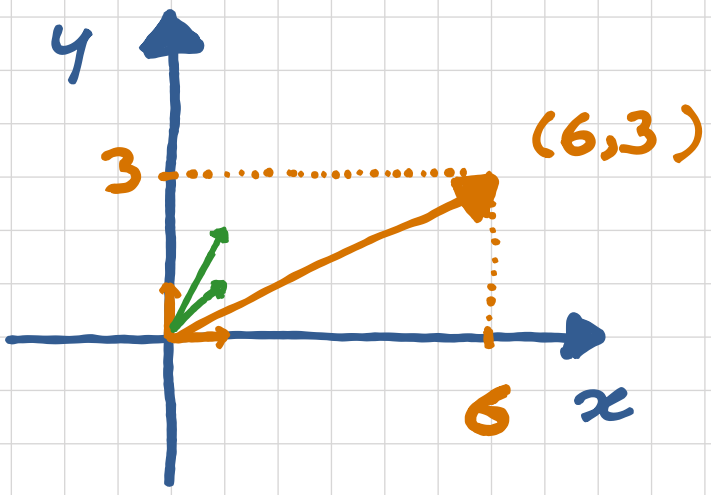
Vectors in the plane



CARTESIAN COORDINATES $\begin{bmatrix} 6 \\ 3 \end{bmatrix}$

WE WANT TO CALCULATE THE COORDINATES IN THE NEW BASE

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

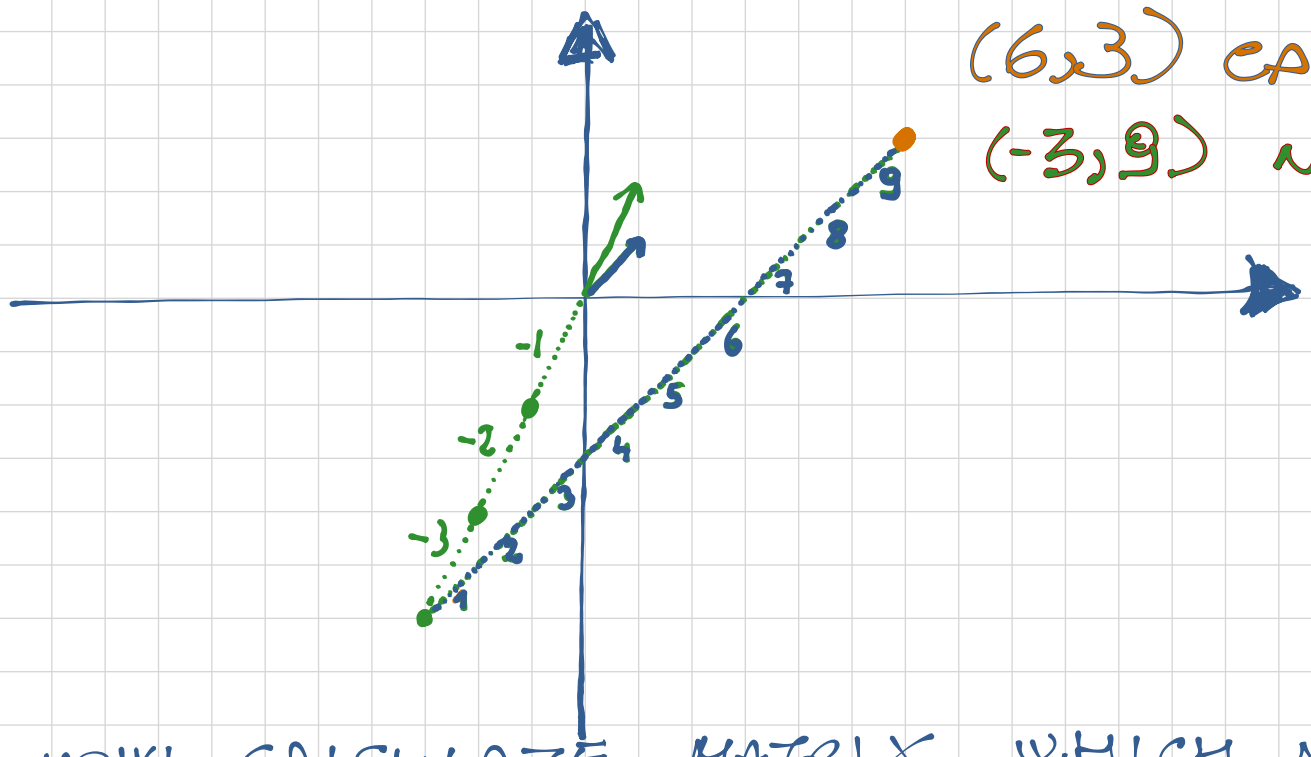
WE HAVE TO SOLVE THE FOLLOWING LINEAR SYSTEM

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$\begin{array}{r} x_1 + x_2 = 6 \\ 2x_1 + x_2 = 3 \\ \hline -x_1 \qquad = 3 \end{array}$$

$$-3 + x_2 = 6$$

THE SOLUTION IS $(-3, 9)$



(6,3) CARTESIAN BASE

(-3,9) NEW BASE

LET US NOW CALCULATE MATRIX WHICH ALLOWS TO FIND THE COMPONENTS OF A GIVEN VECTOR IN THE CARTESIAN SYSTEM IN THE NEW BASE

VECTOR IN THE CARTESIAN SYSTEM $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

NEW BASE $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$x_1 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + x_2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \Rightarrow \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

IF $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \neq c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{u_1 v_2 - u_2 v_1} \begin{bmatrix} v_2 & -v_1 \\ -u_2 & u_1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

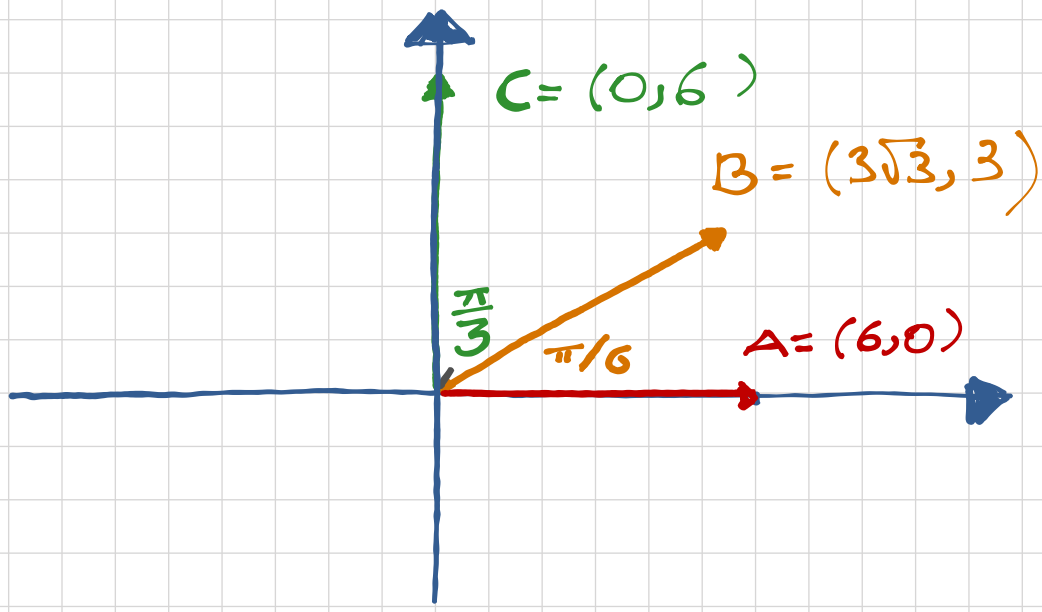
IN THE PREVIOUS EXAMPLE $u_1 = 1 \quad u_2 = 2 \quad v_1 = 1 \quad v_2 = 1$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$$

ROTATION MATRIX

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



$$A = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$B = \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 3\sqrt{3} \\ 3 \end{bmatrix}$$

$$C = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \begin{bmatrix} 3\sqrt{3} \\ 3 \end{bmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{bmatrix} 3\sqrt{3} \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

Vectors in the space

Algebraic Properties of Vectors

For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ and all scalars c and d,

(i) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ commutative

(ii) $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$ associative

(iii) $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{u} = \boldsymbol{u}$ $\vec{0}$ is an additive identity

(iv) $\boldsymbol{u} + (-\boldsymbol{u}) = (-\boldsymbol{u}) + \boldsymbol{u} = \boldsymbol{0}$, where "$-\boldsymbol{u}$" denotes $(-1)\boldsymbol{u}$ "$-\vec{u}$" is the additive inverse of $\vec{u}$

(v) $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$

(vi) $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$ } distributive properties

(vii) $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$ "associative"

(viii) $1\boldsymbol{u} = \boldsymbol{u}$ multiplicative identity

Vector Equations

- Given vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p \in \mathbb{R}^n$ and another vector $\boldsymbol{b} \in \mathbb{R}^n$, we can ask the question of whether $\boldsymbol{b}$ is a linear combination of the $\boldsymbol{v}_i$'s
- This is the same as asking whether the **vector equation**

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_p\boldsymbol{v}_p = \boldsymbol{b}$$

has a solution

An Example

- Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$
- Is $\boldsymbol{b}$ a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$?
- Equivalent question: Does the vector equation

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 = \boldsymbol{b}$$

have a solution?

- The vector equation is

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{array}{ccc} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}$$

$$\begin{matrix}1&2&7\\-2&5&4\\-5&6&-3\end{matrix}\qquad\begin{matrix}2&4&14\\-2&5&4\end{matrix}\qquad\begin{matrix}5&10&35\\-5&6&-3\end{matrix}$$

$$0\ 9\ 18 \qquad 0\ 16\ 32$$

$$\begin{matrix}1&2&7\\0&9&18\\0&16&32\end{matrix}\rightarrow\begin{matrix}1&2&7\\0&1&2\\0&1&2\end{matrix}\qquad\begin{matrix}0&-1&-2\\0&1&2\end{matrix}\qquad\begin{matrix}1&2&7\\0&-2&-4\end{matrix}$$

$$0\ 0\ 0 \qquad 1\ 0\ 3$$

$$\begin{matrix}1&0&3\\0&1&2\\0&0&0\end{matrix}$$

REDUCED FORM

- The corresponding augmented matrix is

$$\begin{bmatrix}1&2&7\\-2&5&4\\-5&6&-3\end{bmatrix}$$

- This matrix is row-equivalent to

$$\begin{bmatrix}1&0&3\\0&1&2\\0&0&0\end{bmatrix}$$

$$x_1 = 3$$
$$x_2 = 2$$

CHECKING THE SOLUTION

$$3\begin{bmatrix}1\\-2\\-5\end{bmatrix} + 2\begin{bmatrix}2\\5\\6\end{bmatrix} = \begin{bmatrix}7\\4\\-3\end{bmatrix} \checkmark\ \checkmark\ \checkmark$$

Span

- The **span** of a set of vectors is the set of all possible linear combinations of those vectors
- $\text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_p\}$ contains all vectors of the form

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p$$

for real numbers $c_1, c_2, \dots, c_p$

Definition

- A set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 + \cdots + x_p\boldsymbol{v}_p = \boldsymbol{0}$$

has only the trivial solution

LET US CONSIDER THE VECTORS OF THE PREVIOUS EXAMPLE

$$\begin{matrix}1&2&7&0\\-2&5&4&0\\-5&6&-3&0\end{matrix}$$

$$\begin{matrix}1&2&7&0\\-2&5&4&0\\-5&6&-3&0\end{matrix}\qquad\begin{matrix}2&4&14&0\\-2&5&4&0\end{matrix}\qquad\begin{matrix}5&10&35&0\\-5&6&-3&0\end{matrix}$$

$$0\ 9\ 18\ 0 \qquad 0\ 16\ 32\ 0$$

$$\begin{matrix}1&2&7&0\\0&9&18&0\\0&16&32&0\end{matrix}\qquad\begin{matrix}1&2&7&0\\0&1&2&0\\0&1&2&0\end{matrix}\qquad\begin{matrix}1&2&7&0\\0&1&2&0\\0&0&0&0\end{matrix}$$

$$x_1 + 2x_2 + 7x_3 = 0$$
$$x_2 + 2x_3 = 0$$

x_3 FREE $\qquad x_2 = -2x_3$

$$x_1 = -2x_2 - 7x_3 \qquad x_1 = -3x_3$$

$$-3x_3\begin{bmatrix}1\\-2\\-5\end{bmatrix} - 2x_3\begin{bmatrix}2\\5\\6\end{bmatrix} + x_3\begin{bmatrix}7\\4\\-3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \checkmark\ \checkmark\ \checkmark$$

LET US CONSIDER

$$\begin{bmatrix} 1 & 2 & 7 & 0 \\ -2 & 5 & 4 & 0 \\ -5 & 6 & -5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 7 & 0 \\ -2 & 5 & 4 & 0 \\ -5 & 6 & -5 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 4 & 14 & 0 \\ -2 & 5 & 4 & 0 \end{bmatrix} \quad \begin{bmatrix} 5 & 10 & 35 & 0 \\ -5 & 6 & -5 & 0 \\ 0 & 16 & 30 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 9 & 18 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 7 & 0 \\ 0 & 9 & 18 & 0 \\ 0 & 16 & 30 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 8 & 15 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -8 & -16 & 0 \\ 0 & 8 & 15 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 + 7x_3 = 0 \\ x_2 + 2x_3 = 0 \\ x_3 = 0 \end{cases} \Rightarrow (x_1, x_2, x_3) = (0,0,0)$$

THE VECTORS $\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \\ -5 \end{bmatrix}$ ARE LINEARLY INDEPENDENT

Example #1

Show that the set $\left\{\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 5 \end{bmatrix}\right\}$ is linearly independent.

Example #2

Is the set $\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \end{bmatrix}\right\}$ linearly independent?

EX 1

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 4 & 0 \\ 1 & -3 & 5 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 & 0 \\ -2 & 0 & 8 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 8 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & -5 & 10 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & -6 & 10 & 0 \\ 0 & -5 & 10 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & -5 & 10 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 1 & 8 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 5 & 0 & 0 \\ 0 & -5 & 10 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 8 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 10 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 10 & 0 \end{bmatrix}$$

$$2x_1 + x_2 = 0, \; x_2 = 0, \; x_3 = 0 \Rightarrow (x_1, x_2, x_3) = (0,0,0)$$

EX 2

$$\begin{bmatrix} 1 & -3 & 2 & -6 & 0 \\ -2 & 7 & 2 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & -6 & 4 & -12 & 0 \\ -2 & 7 & 2 & 0 & 0 \\ 0 & 1 & 6 & -12 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 2 & -6 & 0 \\ 0 & 1 & 6 & -12 & 0 \end{bmatrix}$$

$x_{3,4}$ FREE VARIABLES

$$x_1 = 3x_2 - 2x_3 + 6x_4 = -20x_3 + 42x_4$$
$$x_2 = -6x_3 + 12x_4$$
$$(x_1, x_2, x_3, x_4) = (-20x_3 + 42x_4, -6x_3 + 12x_4, x_3, x_4)$$

$x_3 = x_4 = 1$ $\quad 22\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 6\begin{bmatrix} -3 \\ 7 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓ ✓

Example #3

Find a linear dependence relation for

$$\boldsymbol{v}_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix}4\\5\\6\end{bmatrix}, \text{ and } \boldsymbol{v}_3 = \begin{bmatrix}2\\1\\0\end{bmatrix}.$$

$$\begin{matrix}1&4&2&0\\2&5&1&0\\3&6&0&0\end{matrix} \quad \begin{matrix}-2&-8&-4&0\\2&5&1&0\end{matrix} \quad \begin{matrix}-3&-12&-6&0\\3&6&0&0\end{matrix}$$

$$\begin{matrix}0&-3&-3&0\\0&-6&-6&0\end{matrix}$$

$$\begin{matrix}1&4&2&0\\0&-3&-3&0\\0&-6&-6&0\end{matrix} \quad \begin{matrix}0&6&6&0\\0&-6&-6&0\\0&0&0&0\end{matrix}$$

$$\begin{matrix}1&4&2&0\\2&5&1&0\\3&6&0&0\end{matrix} \Rightarrow \begin{matrix}1&4&2&0\\0&1&1&0\\0&0&0&0\end{matrix}$$

$x_1 = -4x_2 - 2x_3 = 2x_3$

$x_2 = -x_3$

x_3 FREE

SOLUTIONS

$(2, -1, 1)x_3$

$$2\begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}4\\5\\6\end{bmatrix} + \begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \checkmark\checkmark\checkmark$$