INDUÇÃO MATEMÁTICA

A demonstração de veracidade de uma determinada proposição matemática $P(n)$, para todos os inteiros $n \geq n_0$, comporta dois passos:

(1) verifica-se a sua validade para um dado valor inteiro n_0 da variável de indução n,

(2) assume-se que é valida para um inteiro k e demonstra-se que é também vãlida para $k+1$.

• 1) $1+3+5+...+2n+1=(n+1)^2$ para $n \geq 0$

1.1)

$n=0$: $\quad 1=(0+1)^2$ ✓

1.2)

$1+3+5+...+2k+1=(k+1)^2$ ✓

$$\underbrace{1+3+5+...+2k+1}_{(k+1)^2}+2(k+1)+1=[(k+1)+1]^2 \text{ ?}$$

$k^2+2k+1+2k+3=(k+2)^2$ ✓

• 2) $2+4+6+...+2n=n\,(n+1)$ para $n \geq 1$

2.1)

$n=1$: $\quad 2=1\,(1+1)$ ✓

2.2)

$2+4+6+...+2k=k\,(k+1)$ ✓

$$\underbrace{2+4+6+...+2k}_{k\,(k+1)}+2(k+1)=(k+1)\,[(k+1)+1] \text{ ?}$$

$k^2+k+2k+2=(k+1)\,(k+2)$ ✓

• 3) $1+2+3+...+n=n\,(n+1)\,/2$ para $n \geq 1$

3.1)

$n=1$: $\quad 1=1\,(1+1)\,/2$ ✓

3.2)

$1+2+3+...+k=k\,(k+1)\,/2$ ✓

$$\underbrace{1+2+3+...+k}_{k\,(k+1)\,/2}+k+1=(k+1)\,[(k+1)+1]\,/2 \text{ ?}$$

$$\left(\frac{k}{2}+1\right)(k+1)=(k+1)\,\frac{k+2}{2} \;✓$$

• 4) $2^{n} > n$ para $n \geq 1$

4.1)

$n = 1$: $2^{1} > 1$ $\surd$

4.2)

$2^{k} > k$ $\surd$

$2^{k+1} > k+1$?

$2^{k+1} = 2 \cdot 2^{k} > 2\,k = k + k \geq k + 1 \Rightarrow 2^{k+1} > k + 1$ $\surd$

• 5) $n^{2} > 3\,n$ para $n \geq 4$

5.1)

$n = 4$: $4^{2} > 3 \cdot 4$ $\surd$

5.2)

$k^{2} > 3\,k$ $\surd$

$(k+1)^{2} > 3\,(k+1)$?

$(k+1)^{2} = k^{2} + 2k + 1 > 3k + \underbrace{2k+1}_{\geq 9} \geq= 3k + 9 > 3k + 3 \Rightarrow (k+1)^{2} > 3\,(k+1)$ $\surd$

• 6) $2^{n+1} < 3^{n}$ para $n \geq 2$

6.1)

$n = 2$: $2^{2+1} < 3^{2}$ $\surd$

6.2)

$2^{k+1} < 3^{k}$ $\surd$

$2^{(k+1)+1} < 3^{k+1}$?

$2^{k+2} = 2 \cdot 2^{k} < 2 \cdot 3^{k} < 3 \cdot 3^{k} = 3^{k+1} \Rightarrow 2^{(k+1)+1} < 3^{k+1}$ $\surd$

• 7) $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^n} = 1 - \frac{1}{2^n}$ para $n \geq 1$

7.1)

$n = 1$: $\frac{1}{2} = 1 - \frac{1}{2}$ $\surd$

7.2)

$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^k} = 1 - \frac{1}{2^k}$ $\surd$

$\underbrace{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^k}}_{1 - \frac{1}{2^k}} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}$?

$1 - \frac{2}{2^{k+1}} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}$ $\surd$

• 8) $(a+b)^n = \sum_{p=0}^{n} \binom{n}{p} a^p b^{n-p}$

8.1)

$n=0$: $(a+b)^0 = \binom{0}{0} a^0 b^{0-0}$ √

8.2)

$(a+b)^k = \sum_{p=0}^{k} \binom{k}{p} a^p b^{k-p}$ √

$(a+b)^{k+1} = \sum_{p=0}^{k+1} \binom{k+1}{p} a^p b^{k+1-p}$?

$$\begin{aligned}
(a+b)^{k+1} &= (a+b) \sum_{p=0}^{k} \binom{k}{p} a^p b^{k-p} \\
&= \sum_{p=0}^{k} \binom{k}{p} a^{p+1} b^{k-p} + \sum_{p=0}^{k} \binom{k}{p} a^p b^{k+1-p} \\
&= \sum_{p=1}^{k+1} \binom{k}{p-1} a^p b^{k-p+1} + \sum_{p=0}^{k} \binom{k}{p} a^p b^{k+1-p} \\
&= \binom{k}{k} a^{k+1} b^0 + \sum_{p=1}^{k} \binom{k}{p-1} a^p b^{k-p+1} + \binom{k}{0} a^0 b^{k+1} + \sum_{p=1}^{k} \binom{k}{p} a^p b^{k+1-p} \\
&= \binom{k+1}{0} a^0 b^{k+1} + \sum_{p=1}^{k} \underbrace{\left[\binom{k}{p-1} + \binom{k}{p} \right]}_{\binom{k+1}{p}} a^p b^{k+1-p} + \binom{k+1}{k+1} a^{k+1} b^0 \ \checkmark
\end{aligned}$$

• 9) $a^n - b^n = (a-b) \sum_{p=0}^{n-1} a^p b^{n-1-p}$

9.1)

$n=1$: $a-b = (a-b)\, a^0 b^{1-1-0}$ √

9.2)

$a^k - b^k = (a-b) \sum_{p=0}^{k-1} a^p b^{k-1-p}$ √

$a^{k+1} - b^{k+1} = (a-b) \sum_{p=0}^{k} a^p b^{k-p}$?

$$\begin{aligned}
a^{k+1} - b^{k+1} &= a \cdot a^k - b \cdot b^k - b \cdot a^k + b \cdot a^k \\
&= (a-b)\, a^k + b\,(a^k - b^k) \\
&= (a-b) \left[a^k + b \sum_{p=0}^{k-1} a^p b^{k-1-p} \right] = (a-b) \left[a^k + \sum_{p=0}^{k-1} a^p b^{k-p} \right] = (a-b) \sum_{p=0}^{k} a^p b^{k-p} \ \checkmark
\end{aligned}$$