Funções geradoras e séries finitas

USAREMOS O CONCEITO DE FUNÇÃO GERADORA PARA OBTER A SOMA DE SÉRIES FINITAS

CALCULAREMOS COMO EXEMPLOS A SOMA DAS SEGUINTES SÉRIES

$1+2+3+\dots+n$

$1+2^2+3^2+\dots+n^2$

$1+2^3+3^3+\dots+n^3$

EXPLICAREMOS ANTES A IDÉIA: CONSIDERAMOS A SÉRIE INFINITA

$$\sum_0^{\infty}{}_n c_n x^n$$

$$\begin{aligned}
\sum_0^{\infty}{}_n x^n \cdot \sum_0^{\infty}{}_n c_n x^n &= (1+x+x^2+x^3+\dots)(c_0+c_1x+c_2x^2+c_3x^3+\dots) \\
&= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \\
&\quad\quad + c_0 x + c_1 x^2 + c_2 x^3 + \dots \\
&\quad\quad\quad + c_0 x^2 + c_1 x^3 + \dots \\
&\quad\quad\quad\quad + c_0 x^3 + \dots \\
&= c_0 + (c_0+c_1)x + (c_0+c_1+c_2)x^2 + \\
&\quad (c_0+c_1+c_2+c_3)x^3 + \dots \\
&= \sum_0^{\infty}{}_n \left(\sum_0^{n}{}_k c_k\right) x^n
\end{aligned}$$

O COEFICIENTE DO TERMO x^n DARÁ COMO RESULTADO $\boxed{\sum_0^n{}_k c_k}$

$$\frac{1}{1-x} = \sum_0^{\infty}{}_n x^n$$

$$\sum_0^{\infty}{}_n n x^n = x\left(\sum_0^{\infty}{}_n x^n\right)' = x\left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2}$$

$$\sum_0^{\infty}{}_n n^2 x^n = x\left(\sum_0^{\infty}{}_n n x^n\right)' = x\left[\frac{x}{(1-x)^2}\right]' = \frac{x(1+x)}{(1-x)^3}$$

$$\sum_0^{\infty}{}_n n^3 x^n = x\left(\sum_0^{\infty}{}_n n^2 x^n\right)' = x\left[\frac{x(1+x)}{(1-x)^3}\right]' = \frac{x(x^2+4x+1)}{(1-x)^4}$$

AGORA MULTIPLICAMOS PARA $\frac{1}{1-x}$ QUE REPRESENTA A SÉRIE $(1+x+x^2+\dots)$ E CALCULAMOS O COEFICIENTE DE x^n

$1+2+3+\dots+n$

$$\frac{1}{1-x}\,\frac{x}{(1-x)^2} = \frac{x}{(1-x)^3}$$

$$x\sum_0^{\infty}{}_n \binom{n+2}{2} x^n$$

$$\longrightarrow \binom{n-1+2}{2} = \frac{(n+1)\,n}{2}$$

$$\boxed{1+2+3+\dots+n = \frac{n(n+1)}{2}}$$

$1^2+2^2+3^2+\dots+n^2$

$$\frac{1}{1-x}\quad\frac{x(1+x)}{(1-x)^3}=\frac{x(1+x)}{(1-x)^4}$$

$$(x+x^2)\sum_0^\infty {}_r\binom{r+3}{3}x^r$$

COEFICIENTE DE x^r

$$\binom{n-1+3}{3}+\binom{n-2+3}{3}=\frac{(n+2)(n+1)n+(n+1)n(n-1)}{6}$$

$$=\frac{n+1}{6}\left(n^2+2n+n^2-n\right)$$

$$=\frac{n(n+1)(2n+1)}{6}$$

$$\boxed{1^2+2^2+3^2+\dots+n^2=\frac{n(n+1)(2n+1)}{6}}$$

$1^3+2^3+3^3+\dots+n^3$

$$\frac{1}{1-x}\quad\frac{x(x^2+4x+1)}{(1-x)^4}=\frac{x(x^2+4x+1)}{(1-x)^5}$$

$$(x^3+4x^2+x)\sum_0^\infty {}_r\binom{r+4}{4}x^r$$

$$\binom{n-3+4}{4}+4\binom{n-2+4}{4}+\binom{n-1+4}{4}$$

$$\frac{(n+1)n(n-1)(n-2)+4(n+2)(n+1)n(n-1)+(n+3)(n+2)(n+1)n}{24}$$

$$\frac{(n+1)n}{24}\left(n^2-3n+2+4n^2+4n-8+n^2+5n+6\right)$$

$$6n^2+6n$$

$$\boxed{1^3+2^3+3^3+\dots+n^3=\left[\frac{n(n+1)}{2}\right]^2}$$