Diagonalização : útil se queremos calcular M^n

$M = SDS^{-1}$ D Matriz diagonal com elementos nulos fora da diagonal principal

$$M^n = (SDS^{-1})^n = \underbrace{S D \overbrace{S^{-1} S}^{1} D S^{-1}}_{\text{"1" \quad "2"}} \cdots \underbrace{SDS^{-1}}_{\text{"n"}} = S D^n S^{-1}$$

Levamos a potência da matriz M para a matriz D !!!

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1\alpha & \lambda_2\beta \\ \lambda_1\gamma & \lambda_2\delta \end{pmatrix}$$

EQUAÇÃO DOS AUTOVALORES : $M\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \lambda_1 \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}$ $\quad M\begin{pmatrix} \beta \\ \delta \end{pmatrix} = \lambda_2 \begin{pmatrix} \beta \\ \delta \end{pmatrix}$

AUTOVALOR AUTOVETOR

$(M - \lambda I) V = 0$ SOLUÇÃO NÃO TRIVIAL QUANDO $\det(M - \lambda I) = 0$

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - bc = 0 \qquad \lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\boxed{\begin{aligned} \lambda_{1,2} &= \left[a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}\,\right]/2 \\ &= \left[a+d \pm \sqrt{(a-d)^2 + 4bc}\,\right]/2 \end{aligned}}$$

EXEMPLO $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$

$$\lambda_{1,2} = \left[6 \pm \sqrt{4+12}\,\right]/2 = (6 \pm 4)/2 \begin{cases} 1 \\ 5 \end{cases}$$

AUTOVALORES DE M $\boxed{\{1, 5\}}$ COMO CALCULAR OS AUTOVETORES?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 5\beta \\ \gamma & 5\delta \end{pmatrix}$$

$2\alpha + \gamma = \alpha \qquad 2\beta + \delta = 5\beta$

$3\alpha + 4\gamma = \gamma \qquad 3\beta + 4\delta = 5\delta$

$\alpha + \gamma = 0 \qquad \delta = 3\beta$ ✓

$\alpha + \gamma = 0 \qquad 3\beta = \delta$

$\gamma = -\alpha \qquad \alpha = 1 \qquad \gamma = -1$

$\delta = 3\beta \qquad \beta = 1 \qquad \delta = 3$

$$\boxed{S = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}}$$

Autovalor 1 Autovetor $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Autovalor 5 Autovetor $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

controlamos que as contas sejam corretas $\begin{pmatrix}2&1\\3&4\end{pmatrix} = \begin{pmatrix}1&1\\-1&3\end{pmatrix}\begin{pmatrix}1&0\\0&5\end{pmatrix}\begin{pmatrix}1&1\\-1&3\end{pmatrix}^{-1}$

$\begin{pmatrix}1&1\\-1&3\end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix}3&-1\\1&1\end{pmatrix}$ controle $\underbrace{\begin{pmatrix}1&1\\-1&3\end{pmatrix}\begin{pmatrix}1&0\\0&5\end{pmatrix}}\frac{1}{4}\begin{pmatrix}3&-1\\1&1\end{pmatrix}$

$\begin{pmatrix}1&5\\-1&15\end{pmatrix}\frac{1}{4}\begin{pmatrix}3&-1\\1&1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}8&4\\12&16\end{pmatrix} = \begin{pmatrix}2&1\\3&4\end{pmatrix}$

ok!

se quisermos calcular

$\begin{pmatrix}2&1\\3&4\end{pmatrix}^n$ podemos usar $\left[S\begin{pmatrix}1&0\\0&5\end{pmatrix}S^{-1}\right]^n = S\begin{pmatrix}1^n&0\\0&5^n\end{pmatrix}S^{-1}$

$= \begin{pmatrix}1&1\\-1&3\end{pmatrix}\begin{pmatrix}1&0\\0&5^n\end{pmatrix}\frac{1}{4}\begin{pmatrix}3&-1\\1&1\end{pmatrix} = \begin{pmatrix}1&5^n\\-1&3\cdot5^n\end{pmatrix}\frac{1}{4}\begin{pmatrix}3&-1\\1&1\end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix}5^n+3&5^n-1\\3 5^n-3&3\cdot5^n+1\end{pmatrix} \xrightarrow{n\gg1} \frac{5^n}{4}\begin{pmatrix}1&1\\3&3\end{pmatrix}$

NUM $n=2$ $\frac{25}{4}\begin{pmatrix}1.12&0.96\\2.88&3.04\end{pmatrix}$ $\begin{matrix}+12\%&-4\%\\-12\%&+4\%\end{matrix}$

$n=3$ $\frac{125}{4}\begin{pmatrix}1.024&0.992\\2.976&3.008\end{pmatrix}$ $\begin{matrix}+2.4\%&-0.8\%\\-2.4\%&+0.8\%\end{matrix}$

COMPORTAMENTO ASSINTÓTICO DE M^n

$\begin{pmatrix}3/5^n&-1/5^n\\-3/5^n&1/5^n\end{pmatrix}$ ERRO %

APLICAÇÕES CADEIA DE MARKOV E RESOLUÇÃO DE SISTEMAS DE EQUAÇÕES

$\exp[Mt]$

$\exp\begin{bmatrix}2&1\\3&4\end{bmatrix} = ?$ $\mathbb{1} + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots \frac{M^n}{n!} + \cdots$

$SS^{-1} + SDS^{-1} + \frac{SDS^{-1}SDS^{-1}}{2!} + \cdots + \frac{SD^nS^{-1}}{n!} + \cdots$

$$\boxed{\exp[M] = S\exp[D]S^{-1} = S\begin{pmatrix}e^{\lambda_1}&0\\0&e^{\lambda_2}\end{pmatrix}S^{-1}}$$

$e^{\begin{pmatrix}2&1\\3&4\end{pmatrix}} = \begin{pmatrix}1&1\\-1&3\end{pmatrix}\begin{pmatrix}e&0\\0&e^5\end{pmatrix}\frac{1}{4}\begin{pmatrix}3&-1\\1&1\end{pmatrix} = \begin{pmatrix}e&e^5\\-e&3e^5\end{pmatrix}\frac{1}{4}\begin{pmatrix}3&-1\\1&1\end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix}3e+e^5&e^5-e\\3e^5-e&3e^5+e\end{pmatrix} = \frac{e}{4}\begin{pmatrix}e^4+3&e^4-1\\3e^4-1&3e^4+1\end{pmatrix}$

AGORA UM EXEMPLO COM CADEIA DE MARKOV $\begin{pmatrix}70\%&30\%\\40\%&60\%\end{pmatrix}$

$\begin{pmatrix}SS&SC\\CS&CC\end{pmatrix}$

O QUE ACONTECE NO 10° DIA? $\begin{pmatrix} 70\% & 30\% \\ 40\% & 60\% \end{pmatrix}^{10}$

TEREMOS QUE DIAGONALIZAR A MATRIZ $\begin{pmatrix} 70\% & 30\% \\ 40\% & 60\% \end{pmatrix} = \begin{pmatrix} \frac{7}{10} & \frac{3}{10} \\ \frac{4}{10} & \frac{6}{10} \end{pmatrix}$

AUTOVALORES $\left(\frac{7}{10} - \lambda\right)\left(\frac{6}{10} - \lambda\right) - \frac{12}{100} = 0$

$$\lambda^2 - \frac{13}{10}\lambda + \frac{42}{100} - \frac{12}{100} = 0 \Rightarrow \lambda^2 - \frac{13}{10}\lambda + \frac{3}{10} = 0$$

$$\lambda_{1,2} = \left[\frac{13}{10} \pm \sqrt{\frac{169}{100} - \frac{120}{100}}\right]/2 = \left(\frac{13}{10} \pm \frac{7}{10}\right)/2 = \boxed{\left\{1, \frac{3}{10}\right\}}$$

AUTOVETORES

$$\cancel{\frac{1}{10}}\begin{pmatrix} 7 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}\cancel{\frac{1}{10}}$$

$7\alpha + 3\gamma = 10\alpha$ $\quad$ $7\beta + 3\delta = 3\beta$ $\quad \Rightarrow \quad \alpha = \gamma \quad 4\beta = -3\delta$

$4\alpha + 6\gamma = 10\gamma$ $\quad$ $4\beta + 6\delta = 3\delta$

$\alpha = \gamma = 1 \quad \delta = 4 \quad \beta = -3$

$$\boxed{S = \begin{pmatrix} 1 & -3 \\ 1 & 4 \end{pmatrix} \qquad S^{-1} = \frac{1}{7}\begin{pmatrix} 4 & 3 \\ -1 & 1 \end{pmatrix}}$$

$$\begin{pmatrix} 70\% & 30\% \\ 40\% & 60\% \end{pmatrix}^{n} = \begin{pmatrix} 1 & -3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \lambda^n \end{pmatrix}\frac{1}{7}\begin{pmatrix} 4 & 3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3\lambda^n \\ 1 & 4\lambda^n \end{pmatrix}\frac{1}{7}\begin{pmatrix} 4 & 3 \\ -1 & 1 \end{pmatrix}$$

$\lambda = \frac{3}{10}$

$$= \frac{1}{7}\begin{pmatrix} 4+3\lambda^n & 3-3\lambda^n \\ 4-4\lambda^n & 3+4\lambda^n \end{pmatrix}$$

$n = 10$ $\begin{pmatrix} 57.14\% & 42.86\% \\ 57.14\% & 42.86\% \end{pmatrix} = \begin{pmatrix} 4/7 & 3/7 \\ 4/7 & 3/7 \end{pmatrix}$ comportamento assintotico

$n = 3$ $\begin{pmatrix} 58.30\% & 41.70\% \\ 55.60\% & 44.40\% \end{pmatrix}$ calcular $n = 4, 5$

Resolver o problema para $\begin{pmatrix} 65\% & 35\% \\ 45\% & 55\% \end{pmatrix}$ $\begin{pmatrix} 60\% & 40\% \\ 30\% & 70\% \end{pmatrix}$