Definition 1. Let $A$ be an $n \times p$ matrix and $B$ an $m \times q$ matrix. The $mn \times pq$ matrix

$$A \otimes B = \begin{bmatrix}
a_{1,1}B & a_{1,2}B & \cdots & a_{1,p}B \\
a_{2,1}B & a_{2,2}B & \cdots & a_{2,p}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{n,1}B & a_{n,2}B & \cdots & a_{n,p}B
\end{bmatrix}$$

is called the Kronecker product of $A$ and $B$. It is also called the direct product or the tensor product.

Some properties of the Kronecker product:

1. $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ \hspace{0.5cm} (associativity),

2. $A \otimes (B + C) = (A \otimes B) + (A \otimes C)$ \hspace{0.5cm} (distributivity)

   $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$,

3. For scalar $a$, $a \otimes A = A \otimes a = aA$,

4. For scalars $a$ and $b$, $aA \otimes bB = ab \ A \otimes B$,

5. For conforming matrices, $(A \otimes B)(C \otimes D) = AC \otimes BD$,

6. $(A \otimes B)^T = A^T \otimes B^T$, \hspace{0.5cm} $(A \otimes B)^H = A^H \otimes B^H$,

7. For vectors $a$ and $b$, $a^T \otimes b = ba^T = b \otimes a^T$

   (note $aa^T = a \otimes a^T$),
8. For partitioned matrices, \([A_1, A_2] \otimes B = [A_1 \otimes B, A_2 \otimes B]\), but \(A \otimes [B_1, B_2] \neq [A \otimes B_1, A \otimes B_2]\).

9. For square nonsingular matrices \(A\) and \(B\):
\[
(A \otimes B)^{-1} = A^{-1} \otimes B^{-1},
\]

10. For \(m \times m\) matrix \(A\) and \(n \times n\) matrix \(B\):
\[
|A \otimes B| = |A|^n |B|^m,
\]

11. \(\text{tr}(A \otimes B) = \text{tr}(A)\text{tr}(B)\),

12. \(\text{rank}(A \otimes B) = \text{rank}(A) \text{ rank}(B)\).

**Definition 2.** The \(\text{vec}\) operator creates a column vector from a matrix \(A\) by stacking the column vectors of \(A = [a_1 a_2 \cdots a_n]\) below one another:

\[
\text{vec}(A) = \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{bmatrix}.
\]

**Theorem 1.**

\[
\text{vec}(AXB) = (B^T \otimes A) \text{vec}(X).
\]

**Proof.** Let \(B = [b_1 \ b_2 \cdots b_n]\) (of size \(m \times n\)) and \(X = \)
\[ [x_1 \ x_2 \cdots \ x_m]. \] Then, the \( k \)th column of \( AXB \) is

\[
(AXB)_{:,k} = AXb_k = A \sum_{i=1}^{m} x_i b_{i,k}
\]

\[
= [b_{1,k}A \ b_{2,k}A \cdots b_{m,k}A] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}
\]

\[
= \left( [b_{1,k}, b_{2,k}, \ldots, b_{m,k}] \otimes A \right) \text{vec}(X)
\]

Stacking the columns together

\[
\text{vec}(AXB) = \begin{bmatrix}
(AXB)_{:,1} \\
(AXB)_{:,2} \\
\vdots \\
(AXB)_{:,n}
\end{bmatrix} = \begin{bmatrix}
b_1^T \otimes A \\
b_2^T \otimes A \\
\vdots \\
b_n^T \otimes A
\end{bmatrix} \text{vec}(X)
\]

\[
= \left( B^T \otimes A \right) \text{vec}(X).
\]

\[
\square
\]

Corollary: \( \text{vec}(AB) = (I \otimes A) \text{vec}(B) = (B^T \otimes I) \text{vec}(A). \)
Some Properties of the $\text{vec}$ Operator

\[ \text{tr}(ABC') = \text{vec}(A^T)^T (I \otimes B) \text{vec}(C), \]

and its corollary

\[ \text{tr}(AB) = \text{vec}(A^T)^T \text{vec}(B). \]

Also

\[ \text{vec}(aa^T) = a \otimes a, \]
\[ \text{tr}(A^T BCD^T) = \text{vec}(A)^T (D \otimes B) \text{vec}(C). \]