1. Questão 1

   (a) Temos que:

$$
\begin{aligned}
f_{\boldsymbol{X}}(\boldsymbol{x};\lambda) &= \exp\left\{-n\lambda + \sum_{i=1}^{n} x_i \ln\lambda\right\}\left(\prod_{i=1}^{n} x_i\right)\mathbb{1}_A(\boldsymbol{x}) \\
&= \exp\{c(\theta)t(\boldsymbol{x}) + d(\theta)\}\, h(\boldsymbol{x}) \qquad (1)\\
&= \exp\{\eta t(\boldsymbol{x}) + d_0(\eta)\}\, h(\boldsymbol{x}) \qquad (2)
\end{aligned}
$$

   Em que $c(\theta) = \eta = \ln\lambda$, $t(\boldsymbol{x}) = \sum_{i=1}^{n} x_i$, $d(\theta) = -n\lambda$, $\lambda = e^{\eta}$, $d_0(\eta) = -ne^{\eta}$.
   Por (1) $\boldsymbol{X} \in FE_1(\lambda)$, T é suficiente, e, por (2), como $\Theta_\eta = (-\infty, 0)$ contem algum segmento de reta, T é completa e minimal, também.
   Como $\mathcal{E}(T^*) = \mathcal{E}(T/n) = \lambda$, T^* é o ENVUM de λ.

   (b) Pelo item anterior, temos que, $c(.)$ é uma função crescente e, assim, $\boldsymbol{X}$ tem RVMND em t. Portanto, um TUMP, para as hipóteses em questão, é dado por:

$$
\phi(\boldsymbol{x}) = \begin{cases} 1, \text{ se } t(\boldsymbol{x}) > c \\ \gamma, \text{ se } t(\boldsymbol{x}) = c \\ 0, \text{ se } t(\boldsymbol{x}) < c \end{cases}
$$

   em que $t(\boldsymbol{x}) = \sum_{i=1}^{n} x_i$, $\alpha = P_{\lambda_0}(T > c) + \gamma P_{\lambda_0}(T = c)$, e $T \sim \text{Poisson}(n\lambda_0)$, se $\lambda = \lambda_0$.

   (c) Temos, pelo item a), que:

$$
L(\theta) \propto e^{-n\lambda}\lambda^{n\overline{x}}\mathbb{1}_{(0,\infty)}(\lambda)
$$

   Assim, a família conjugada corresponde à $\lambda \sim \text{gama}(a, b^{-1})$. Portanto,

$$
\pi(\theta|\boldsymbol{x}) \propto e^{-(n+b)\lambda)}\lambda^{n\overline{+}a-1}
$$

   ou seja, $\theta|\boldsymbol{x} \sim \text{gama}(n\overline{+}a, (n+b)^{-1})$.

   (d) Pelo item anterior e o formulário, temos que: $2(n+b)\lambda|\boldsymbol{x} \sim \chi^2_{2a^*}$, em que $a^* = n\overline{+}a$. Portanto:

$$P(q_1 \leq 2(n+b)\lambda \leq q_2|\boldsymbol{x}) \leftrightarrow P\left(\frac{q_1}{2(n+b)} < \lambda < \frac{q_2}{2(n+b)}|\boldsymbol{x}\right)\gamma$$

Logo, $IC_B(\lambda, \gamma) = \left[\frac{q_1}{2(n+b)}; \frac{q_2}{2(n+b)}\right]$, em que $P\left(X \leq q_1\right) = \frac{1-\gamma}{2}$ e $P\left(X \leq q_2\right) = \frac{1+\gamma}{2}$.

2. Questão 2

   (a) Temos que:

$$\begin{aligned}\mathcal{E}(g(X)) &= \theta g(0) + 3\theta g(1) + (1-4\theta)g(2) = g(2) + (g(0) + 3g(1) - 4g(2))\theta \\ &\rightarrow \mathcal{E}(g(X)) = 0 \leftrightarrow g(2) + (g(0) + 3g(1) - 4g(2))\theta = 0 \leftrightarrow g(2) = 0; g(0) = -3g(1)\end{aligned}$$

   portanto, não é completa

   (b) Temos que;

$$L(x) = \frac{1}{-x + 3/2} < c \leftrightarrow -x > c^{-1} - 3/2 \leftrightarrow x < c^*$$

   Assim, temos que um TUMP é dado por:

$$\phi(x) = \begin{cases} 1, \text{se} x < c^*, \\ 0, \text{cc} \end{cases}$$

   em que $\alpha = P_{\theta_0}(X < c^*)$ e, se $\theta = 0$, temos que $X \sum U(0,1)$ e, assim, $c^* = \alpha$

3. Questão 3

   (a) Temos que

$$\mathcal{E}(T_a) = a\theta + (1-a)c\mathcal{E}(S)$$

Mas, por outro lado, temos que: $Y = \frac{(n-1)S^2}{\theta^2} = \frac{2S^2}{2\theta^2/(n-1)} \sim \chi^2_{2(n-1)/2}$. Portanto,

$$(S^2 =)V = \frac{\theta^2}{n-1}Y \sim \text{gama}((n-1)/2, 2\theta^2/(n-1)) \equiv \text{gama}(b,d)$$

Portanto ($k > 0$, $b = (n-1)/2$, $d = 2\theta^2/(n-1)$)

$$\begin{aligned}\mathcal{E}(V^k) &= \int_0^\infty \frac{1}{\Gamma(b)d^b} e^{-v/d} v^{b+k-1} dv = \frac{\Gamma(a+k)b^{a+k}}{\Gamma(a)b^a} \\ &= \frac{\Gamma(b+k)d^k}{\Gamma(b)}\end{aligned}$$

Assim,

$$\mathcal{E}(V^{1/2}) = \frac{\Gamma(\frac{n-1}{2} + \frac{1}{2})\frac{\theta\sqrt{2}}{\sqrt{n-1}}}{\Gamma\left(\frac{n-1}{2}\right)} = \frac{\Gamma\left(\frac{n}{2}\right)\theta\sqrt{2}}{\sqrt{n-1}\Gamma\left(\frac{n-1}{2}\right)}$$

Portanto, vem que:

$$\mathcal{E}(T_a) = a\theta + (1-a)c\frac{1}{c}\theta = \theta$$

   (b) Neste caso, temos que:

$$\mathcal{V}(T_a) = a^2\frac{\theta^2}{n} + (1-a)^2c^2\mathcal{V}(S)$$

mas

$$\mathcal{V}(S) = \mathcal{E}(S^2) - \mathcal{E}^2(S) = \theta^2 - \frac{\theta^2}{c^2} = \frac{\theta^2}{c^2}(c^2-1)$$

Logo:

$$\begin{aligned}
\mathcal{V}(T_a) &= \frac{\theta^2}{n}\left(a^2 + n(c^2-1) - 2n(c^2-1)a + a^2n(c^2-1)\right)\\
&= \frac{\theta^2}{n}\left(n(c^2-1) - 2n(c^2-1)a + a^2[n(c^2-1)+1]\right)\\
&= \frac{\theta^2}{n}\left(a^2b_1 + ab_2 + b_3\right)
\end{aligned}$$

em que $b_1 = n(c^2-1)+1$, $b_2 = -2n(c^2-1)$, $b_3 = n(c^2-1)$
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$$\Delta = b_2^2 - 4b_1b_3 = 4n^2(c^2-1)^2 - 4n^2(c^2-1)^2 - 4n(c^2-1)$$

Minimizar $g(a) = a^2b_1 + ab_2 + b_3$, $a' = -\dfrac{b_2}{2b_1} = 1$.
Assim, o estimar T_1 é o que apresenta a menor variância.

4. Questão 4

Temos que $F_X(x;\boldsymbol{\theta}) = \dfrac{x-\theta_1}{\theta_2-\theta_1}; S_X(x;\boldsymbol{\theta}) = \dfrac{\theta_2 - x}{\theta_2-\theta_1}$. Também,

$$f_{Y_1}(y;\theta_1) = n\frac{(\theta_2-y)^{n-1}}{(\theta_2-\theta_1)^n}\mathbb{1}_{(\theta_1,\theta_2)}(y); f_{Y_n}(y;\theta_2) = n\frac{(y-\theta_1)^{n-1}}{(\theta_2-\theta_1)^n}\mathbb{1}_{(\theta_1,\theta_2)}(y)$$

(a) Temos que:

$$f_{\boldsymbol{X}}(\boldsymbol{x};\theta_2) = \frac{1}{(\theta_2-\theta_1)^n}\mathbb{1}_{(\theta_1,y_n)}(y_1)\mathbb{1}_{(\theta_1,\theta_2)}(y_n) = \frac{1}{(\theta_2-\theta_1)^n}\mathbb{1}_{(\theta_1,y_n)}(y_1)\mathbb{1}_{(y_n,\infty)}(\theta_2)$$

Por outro lado, temos que:

$$\begin{aligned}
\mathcal{E}(g(Y_n)) &= \int_{\theta_1}^{\theta_2} g(y)n\frac{(y-\theta_1)^{n-1}}{(\theta_2-\theta_1)^n}dy = 0 \\
&\to \int_0^{\theta_2-\theta_1} g(z+\theta_1)z^{n-1}dz = 0 \to g(\theta_2) = 0, \forall\theta_2 \in \Theta = (\theta_1,\infty)
\end{aligned}$$

Logo, vale $\forall y_n \in (\theta_1,\theta_2)$. Logo Y_n também é completa. Por outro lado, temos que:

$$\begin{aligned}
\mathcal{E}(Y_n) &= \int_{\theta_1}^{\theta_2} yn\frac{(y-\theta_1)^{n-1}}{(\theta_2-\theta_1)^n}dy = \frac{n}{(\theta_2-\theta_1)^n}\int_{\theta_1}^{\theta_2} y(y-\theta_1)^{n-1}dy \\
&= \frac{n}{(\theta_2-\theta_1)^n}\int_0^{\theta_2-\theta_1}(z+\theta_1)z^{n-1}dz = \frac{n}{(\theta_2-\theta_1)^n}\left(\frac{(\theta_2-\theta_1)^{n+1}}{n+1} + \theta_1\frac{(\theta_2-\theta_1)^n}{n}\right) \\
&= \frac{n(\theta_2-\theta_1)}{n+1} + \theta_1 = \frac{n\theta_2 - n\theta_1 + n\theta_1 + \theta_1}{n+1} = \frac{n\theta_2+\theta_1}{n+1} \\
&\to \mathcal{E}\left(\frac{(n+1)Y_n - \theta_1}{n}\right) = \theta_2
\end{aligned}$$

Logo $\dfrac{(n+1)Y_n-\theta_1}{n}$ é o ENVUM de θ_1.

(b) Analogamente, temos que:

$$f_{\boldsymbol{X}}(\boldsymbol{x};\theta_2) = \frac{1}{(\theta_2-\theta_1)^n}\mathbb{1}_{(\theta_1,y_n)}(y_1)\mathbb{1}_{(\theta_1,\theta_2)}(y_n) = \frac{1}{(\theta_2-\theta_1)^n}\mathbb{1}_{(-\infty,y_1)}(\theta_1)\mathbb{1}_{(y_1,\theta_2)}(y_n)$$

Por outro lado, temos que:

$$
\begin{aligned}
\mathcal{E}(g(Y_1)) &= \int_{\theta_1}^{\theta_2} g(y) n \frac{(\theta_2 - y)^{n-1}}{(\theta_2 - \theta_1)^n} dy = 0 \\
&\rightarrow \int_{\theta_2-\theta_1}^{0} g(\theta_2 - z) z^{n-1} dz = 0 \rightarrow g(\theta_1) = 0, \forall \theta_1 \in \Theta = (-\infty, \theta_2)
\end{aligned}
$$

Logo, vale $\forall y_1 \in (\theta_1, \theta_2)$. Logo Y_1 também é completa. Por outro lado, temos que:

$$
\begin{aligned}
\mathcal{E}(Y_1) &= \int_{\theta_1}^{\theta_2} yn \frac{(\theta_2 - y)^{n-1}}{(\theta_2 - \theta_1)^n} dy = \frac{n}{(\theta_2 - \theta_1)^n} \int_{\theta_1}^{\theta_2} y(\theta_2 - y)^{n-1} dy \\
&= -\frac{n}{(\theta_2 - \theta_1)^n} \int_{\theta_2-\theta_1}^{0} (\theta_2 - z) z^{n-1} dz = \frac{n}{(\theta_2 - \theta_1)^n} \int_{0}^{\theta_2-\theta_1} (\theta_2 - z) z^{n-1} dz \\
&= \frac{n}{(\theta_2 - \theta_1)^n} \left(\frac{-(\theta_2 - \theta_1)^{n+1}}{n+1} + \theta_2 \frac{(\theta_2 - \theta_1)^n}{n} \right) \\
&= \frac{-n(\theta_2 - \theta_1)}{n+1} + \theta_2 \rightarrow \mathcal{E}(Y_1) = \frac{n\theta_1 + \theta_2}{n+1} \\
&\rightarrow \mathcal{E}\left(\frac{(n+1)Y_1 - \theta_2}{n} \right) = \theta_1
\end{aligned}
$$

Logo $\dfrac{(n+1)Y_1 - \theta_2}{n}$ é o ENVUM de θ_1.

(c) Temos que

$$
\begin{aligned}
\mathcal{E}(Y_n + Y_1) &= = \frac{n\theta_2 + \theta_1 + n\theta_1 + \theta_2}{n+1} = \theta_1 + \theta_2 \\
\mathcal{E}(Y_n - Y_1) &= \frac{(n-1)(\theta_2 - \theta_1)}{n+1} \rightarrow \mathcal{E}\left(\frac{n+1}{n-1}(Y_n - Y_1) \right) = \theta_2 - \theta_1
\end{aligned}
$$

Defina $V = Y_n + Y_1$ e $W = \dfrac{n+1}{n-1} Y_n - Y_1$. Assim, temos que:

$$
\begin{aligned}
\mathcal{E}\left(W+V\right) &= 2\theta_2 \rightarrow \mathcal{E}\left(\frac{W+V}{2}\right) = \theta_2 \\
\mathcal{E}\left(V-W\right) &= 2\theta_1 \rightarrow \mathcal{E}\left(\frac{V-W}{2}\right) = \theta_1
\end{aligned}
$$

Portanto, $\dfrac{W+V}{2}$ e $\frac{V-W}{2}$ são os ENVUM's de θ_2 e θ_1, respectivamente.