Atanassov's Intuitionistic Fuzzy Entropy: Conjugation and Duality

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Abstract. This work extends the study of properties related to the Atanassov's Intuitionistic Fuzzy Entropy obtained as aggregation of Generalized Atanassov's Intuitionistic Fuzzy Index, by considering the concept of conjugate fuzzy implications and their dual constructions.

Keywords: Fuzzy Logic,Atanassov's Intuitionistic Fuzzy Logic, Genralized Atanassov's Intuitionistic Fuzzy Index, Entropy, Conjugation, Duality

1 Introduction

The Atanassov-intuitionistic fuzzy index ($A-IFIx$), also called as hesitancy or indeterminance degree of an element in an Atanassov-intuitionitic fuzzy set (A-IFS) [1], allows the expression related to the expert uncertainty in identifying a particular membership function. Thus, there are applications in which experts do not have precise knowledge. In addition, the A-IFIx provides a measure of the lack of information supporting or against a given proposition based on Atanassov-intuitionistic fuzzy logic (A-IFL) [2] .

In [3], $A-IFIx$ has been considered in order to calculate the Atanassov's intuitionistic fuzzy index of a hypergroupoid H, making evident some of its special properties connected with the intuitionistic fuzzy grade. In [4–6] principal component analysis for A-IFSs type data, $A-IFIx$ can be used to define correlation between A-IFS A and B.

Based on [7] and [8], a new concept the Generalized Atanassov's Intuitionistic Fuzzy Index associated with a strong intuitionistic fuzzy negation N_I ($A-GIFIx(N_I)$) is characterized in terms of fuzzy implication operators which is described by a construction method with automorphisms. In [8], by means of special aggregation functions applied to the A-GIFIx, the Atanassov's intuitionistic fuzzy entropy is introduced.

Following these previous researches, this work extends the study of properties related to A-GIFIx, by considering the concept of conjugate and dual fuzzy implications, mainly interested in the class of (S, N)-implications and (T, N)-coimplications. Additionally, $A-GIFIx$ associated with the standard negation together with known fuzzy implications are considered: Lukaziewicz, Reichenbach, Gaines-Rescher and I_{30} [9].

The preliminaries describe the basic properties of fuzzy connectives and basic concepts of A-IFL. The study of the $A - GIFIx(N_I)$ and general results in the analysis of its properties are stated in Section 2. Final remarks are reported in the conclusion.

2 Preliminaries

We firstly give a brief account on FL, keeping this paper self-contained by reporting basic concepts of automorphisms, fuzzy negations on $U = [0, 1]$ and main properties of fuzzy implications.

2.1 Fuzzy connectives

By [10, Def. 4.1], an **automorphism** $\phi : U \to U$ is a bijective, strictly increasing function (SIF) satisfying the monotonicity property:

A1: $x \leq y$ iff $\phi(x) \leq \phi(y), \forall x, y \in U$.

In [11], an automorphism $\phi : U \to U$ is a SIF satisfying the continuity property and the boundary conditions:

A2: $\phi(0) = 0$ and $\phi(1) = 1$.

The set $Aut(U)$ of all automorphisms are closed under composition:

A3: $\phi \circ \phi' \in Aut(U), \forall \phi, \phi' \in Aut(U)$.

In addition, there exists the inverse $\phi^{-1} \in U$, such that

A4: $\phi \circ \phi^{-1} = id_U, \forall \phi \in Aut(U)$.

Thus, $(Aut(U), \circ)$ is a group, with the identity function being the neutral element. The action of an automorphism $\phi : U \to U$ on a function $f : U^n \to U$, called **conjugate of** f, and given by

$$f^{\phi}(x_1, \ldots, x_n) = \phi^{-1}(f(\phi(x_1), \ldots, \phi(x_n))). \tag{1}$$

A function $N : U \to U$ is a *fuzzy negation* (FN) if

N1: $N(0) = 1$ and $N(1) = 0$; **N2**: If $x \geq y$ then $N(x) \leq N(y), \forall x, y \in U$.

FNs satisfying the involutive property **N3** are called *strong* fuzzy negations [11]:

N3: $N(N(x)) = x, \forall x \in U$.

Among several definitions, see [12] and [13, Definition 2], an aggregation is a function $A : U^n \to U$ demanding, for all $\boldsymbol{x}, \boldsymbol{y} \in U^n$, the following conditions::

Ag1: $A(\mathbf{0}) = A(0, 0, \ldots, 0) = 0$ and $A(\mathbf{1}) = A(1, 1, \ldots 1) = 1$;
Ag2: If $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \leq \boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ then $A(\boldsymbol{x}) \leq A(\boldsymbol{y})$;
Ag3: $A(\overrightarrow{x_\sigma}) = A(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}) = A(x_1, x_2, \ldots, x_n) = A(\boldsymbol{x})$.

A **triangular-(co)norm** (t-(co)norm) $T(S)$:$U^2 \to U$ is a binary aggregation with the identity element $T(1,x) = x$ ($S(0,x) = x$), for all $x \in U$.

By [14], a *fuzzy (co)implication* $I(J) : U^2 \to U$ satisfies the conditions:

I1: $x \leq z \Rightarrow I(x,y) \geq I(z,y)$; **J1**: $x \leq z \Rightarrow J(x,y) \geq J(z,y)$;
I2: If $y \leq z$ then $I(x,y) \leq I(x,z)$; **J2**: If $y \leq z$ then $J(x,y) \leq J(x,z)$;
I3: $I(0,x) = 1$; **J3**: $J(1,x) = 0$
I4: $I(x,1) = 1$; **J4**: $J(x,0) = 0$
I5: $I(1,0) = 0$; **J5**: $J(0,1) = 1$.

Several reasonable properties may be required for fuzzy (co)implications:

I6: $I(1,x) = x$; **J6**: $J(0,x) = x$;
I7: $I(x, I(y,z)) = I(y, I(x,z))$; **J7**: $J(x, J(y,z)) = J(y, J(x,z))$;
I8: $I(x,y) = 1 \Leftrightarrow x \leq y$; **J8**: $J(x,y) = 0 \Leftrightarrow x \geq y$;
I9: $I(x,y) = I(N(y), N(x))$, N is a SFN; **J9**: $J(x,y) = J(N(y), N(x))$, N is a SFN;
I10: $I(x,y) = 0 \Leftrightarrow x = 1$ and $y = 0$; **J10**: $J(x,y) = 1 \Leftrightarrow x = 0$ and $y = 1$.

If $I(J) : U^2 \to U$ is a fuzzy (co)implication satisfying **I1** (**J1**), then the function $N_I : U \to U$ defined by

$$N_I(x) = I(x,0) \text{ and } N_I(x) = J(x,1) \tag{2}$$

is a fuzzy negation [15, Lemma 2.1].

Let $T(S)$ be a t-(co)norm and N be a FN. An $(S,N)-$implication ($(T,N)-$coimplication) [11, 14, 15] is a fuzzy (co)implication $I_{S,N} : U^2 \to U$ defined by

$$I_{S,N}(x,y) = S(N(x), y); \quad J_{T,N}(x,y) = T(N(x), y). \tag{3}$$

In this paper, such S-implications are called *strong S-implications*.

In [16, Theorem 3.2] $I : U^2 \to U$ is a *strong S-implication* if and only if it satisfies **I1** – **I4**, and **I10**. In Baczynsky and Jayaram [15, Theorem 2.6]) introduced a characterization of strong S-implications considering **I1**, **I4** and **I7**. Strong S-implications satisfy **I8-I11** and properties below:
I12: $I(x,y) \geq N_I(x)$; **I13**: $I(x,y) = 0$ if and only if $x = 1$ and $y = 0$. Any S-implication $I_{S,N}$ satisfies the Properties **I1–I3**,**I8**,**I9**, and **I11**. Moreover, the strong S-implication $I_{S_M,N}$ also satisfies the Properties **I1–I11** and it is the only S-implication satisfying **I6**.

2.2 Intuitionistic Fuzzy Connectives

According with [2], an intuitionistic fuzzy set (IFS) A_I in a non-empty, universe χ, is expressed as

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in \chi, \mu_A(x) + \nu_A(x)) \leq 1\} \subseteq \mathcal{A}_I, \tag{4}$$

whenever $\mathcal{A}_I$ denotes the set of all Atanassov's intuitionistic fuzzy sets. Thus, an intuitionistic fuzzy truth value of an element $x \in A_I$ is related to the ordered pair $(\mu_A(x), \nu_A(x))$. Moreover, when $\mathcal{A}$ denotes the set of all fuzzy sets on U and

$$A = \{(x, \mu_A(x)) : x \in \chi, \mu_A(x) + \nu_A(x)) = 1\} \in \mathcal{A},$$

an IFS A_I generalizes a FS A and $\mathcal{A} \subset \mathcal{A}_I$ since $\nu_A(x)$, which means that the non-membership degree of an element x, is less than or equal to the complement of its membership degree $\mu_A(x)$. So, it is not necessarily equal to its complement $1 - \mu_A(x)$.

Let $\tilde{U} = \{(x_1, x_2) \in U^2 | x_1 \leq N_S(x_2)\}$ be the set of all intuitionistic fuzzy values and $l_{\tilde{U}}, r_{\tilde{U}} : \tilde{U} \to U$ be the projection functions on $\tilde{U}$, which are given by $l_{\tilde{U}}(\tilde{x}) = l_{\tilde{U}}(x_1, x_2) = x_1$ and $r_{\tilde{U}}(\tilde{x}) = r_{\tilde{U}}(x_1, x_2) = x_2$, respectively.

Thus, for all $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{U}^n$, such that $\tilde{x}_i = (x_{i1}, x_{i2})$ and $x_{i1} \leq N_S(x_{i2})$ when $1 \leq i \leq n$, considering $l_{\tilde{U}^n}, r_{\tilde{U}^n} : \tilde{U}^n \to U^n$ as the projections given by:

$$l_{\tilde{U}^n}(\tilde{\mathbf{x}}) = (l_{\tilde{U}}(\tilde{x}_1), l_{\tilde{U}}(\tilde{x}_2), \ldots, l_{\tilde{U}}(\tilde{x}_n)) = (x_{11}, x_{21}, \ldots x_{n1}); \quad (5)$$

$$r_{\tilde{U}^n}(\tilde{\mathbf{x}}) = (r_{\tilde{U}}(\tilde{x}_1), r_{\tilde{U}}(\tilde{x}_2), \ldots r_{\tilde{U}}(\tilde{x}_n)) = (x_{12}, x_{22}, \ldots x_{n2}). \quad (6)$$

By [2], for $\tilde{x}, \tilde{y} \in \tilde{U}$, the order relation $\leq_{\tilde{U}}$ is given as $\tilde{x} \leq_{\tilde{U}} \tilde{y} \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$, such that $\tilde{0} = (0, 1) \leq_{\tilde{U}} \tilde{x}$ and $\tilde{1} = (1, 0) \geq_{\tilde{U}} \tilde{x}$. Moreover, the following expression is known:

$$\tilde{x} \preceq_{\tilde{U}} \tilde{y} \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2. \quad (7)$$

Additonally, a function $\pi_A : \chi \to U$, called an **intuitionistic fuzzy index** (IFIx) of an element $x \in \chi$, related to an IFS A, is given as

$$\pi_A(x) = N_S(\mu_A(x) + \nu_A(x)) \quad (8)$$

Such function provides the hesitancy (indeterminance) degree of x in A. Based on this, the accuracy function $h_A : \chi \to U$ provides the accuracy degree of x in A, given as:

$$h_A(x) + \pi_A(x) = 1 \quad (9)$$

So, the largest $\pi_A(x)$ or $h_A(x)$, the higher the hesitancy (accuracy) degree of x in A.

A function $\Phi : \tilde{U} \to \tilde{U}$ is an **intuitionistic automorphism** on $\tilde{U}$ if it is bijective and $\tilde{x} \leq_{\tilde{U}} \tilde{y}$ iff $\Phi(\tilde{x}) \leq_{\tilde{U}} \Phi(\tilde{y})$. The action of $\Phi : \tilde{U} \to \tilde{U}$ on $f_I : \mathbb{U}^n \to \mathbb{U}$ is a function $f_I^{\Phi} : \tilde{U} \to \tilde{U}$, called conjugate function f_I, defined as follows

$$f_I^{\Phi}(\tilde{\mathbf{x}}) = \Phi^{-1}(f_I(\Phi(\tilde{x}_1), \ldots, \Phi(\tilde{x}_n))). \quad (10)$$

According with [**?**, Theorem 17], let $\phi : U \to U$ be an automorphism on U. Then, for all $x \in U$, the function $\Phi : \tilde{U} \to \tilde{U}$ defined by

$$\Phi(\tilde{x}) = (\phi(l_{\tilde{U}}(\tilde{x})), 1 - \phi(1 - r_{\tilde{U}}(\tilde{x}))); \quad (11)$$

is an intuitionistic automorphism on $\tilde{U}$ named as a **ϕ-representable intuitionistic automorphism** on $\tilde{U}$.

An **intuitionistic fuzzy negation** (IFN shortly) $N_I : \tilde{U} \to \tilde{U}$ satisfies, for all $\tilde{x}, \tilde{y} \in \tilde{U}$, the following properties:

N_I 1: $N_I(\tilde{0}) = N_I(0, 1) = \tilde{1}$ and $N_I(\tilde{1}) = N_I(1, 0) = \tilde{0}$;
N_I 2: If $\tilde{x} \geq \tilde{y}$ then $N_I(\tilde{x}) \leq N_I(\tilde{y})$.

Moreover, N_I is a **strong intuitionistic fuzzy negation** (SIFN) verifying the condition:

$\mathbf{N}_I 3$: $N_I(N_I(\tilde{x})) = \tilde{x}, \forall \tilde{x} \in \tilde{U}$.

Consider N_I as IFN and $\tilde{f} : \tilde{U}^n \to \tilde{U}$. For all $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{U}^n$, the **$N_I$-dual intuitionistic function of** $\tilde{f}$, denoted by $\tilde{f}_{N_I} : \tilde{U}^n \to \tilde{U}$, is given by:

$$\tilde{f}_{N_I}(\tilde{\mathbf{x}}) = N_I(\tilde{f}(N_I(\tilde{x}_1), \ldots, N_I(\tilde{x}_n))). \tag{12}$$

When $\tilde{N}_I$ is a SIFN, $\tilde{f}$ is a self-dual intuitionistic function. Additionally, by [17], taking a SFN $N : U \to U$, a IFN $N_I : \tilde{U} \to \tilde{U}$ such that

$$N_I(\tilde{x}) = (N(N_S(x_2)), N_S(N(x_1))), \tag{13}$$

is a SIFN generated by means of the standard negation N_S. Additionally, if $N = N_S$, Eq. 13 can be reduced to $N_I(\tilde{x}) = (x_2, x_1)$.

In this paper, we consider the complement of an IFS A given as

$$A_c = \{(x, N(N_S(\nu_A(x)), N_S(N(\mu_A(x)))) : x \in \chi, \mu_A(x) + \nu_A(x)) \leq 1\} \subseteq \mathcal{A}_I \tag{14}$$

3 (Co)Generalized Atanassov's Intuitionistic Fuzzy Index

In [8] and [7], the concept of generalized Atanassovs intuitionistic fuzzy index is characterized in terms of fuzzy implication operators and a construction method with automorphisms is also proposed together with some special properties of a GIFIx. In the following, we extend this concept in order to study its dual and conjugate constructions.

Definition 1. *[8, Definition 1], A function $\Pi : \tilde{U} \to U$ is called a generalized intuitionistic fuzzy index associated with a SIFN N_I ($A - GIFIx(N_I)$) if, for all $x_1, x_2, y_1, y_2 \in U$, it holds that:*

Π1: *$\Pi(x_1, x_2) = 1$ if and only if $x_1 = x_2 = 0$;*
Π2: *$\Pi(x_1, x_2) = 0$ if and only if $x_1 + x_2 = 1$;*
Π3: *if $(y_1, y_2) \preceq_{\tilde{U}} (x_1, x_2)$ then $\Pi(x_1, x_2) \leq \Pi(y_1, y_2)$*
Π4: *$\Pi(x_1, x_2) = \Pi(N_I(x_1, x_2))$ when N_I is a SIFN.*

Proposition 1. *[8, Theorem 3] Let N_I be a SIFN obtained by a SFN N, according with Eq.(13). A function $\Pi : \tilde{U} \to U$ is a $A - GIFIx(N_I)$ iff there exists a function $I : U^2 \to U$ verifying* **I1**, **I8**,**I9** *and* **I10** *such that*

$$\Pi_I(x_1, x_2) = N(I(1 - x_2, x_1)). \tag{15}$$

Remark 1. By Proposition 1, when $x_1 \leq x_2$ or equivalent $I(x_1, x_2)) = 1$, it holds that $(x_1, x_2) \in U$ since $\Pi_I(x_1, x_2) = 0$.

Proposition 2. *Let N_I be a SIFN obtained by a SFN N, according with Eq.(13). A function $\Pi : \tilde{U} \to U$ is a $A - co - GIFIx(N_I)$ iff there exists a function $J : U^2 \to U$ verifying* **J2**, **J8**, **J9** *and* **J10** *such that*

$$\Pi_J(x_1, x_2) = J(N(1 - x_2), N(x_1)). \tag{16}$$

Proof. (⇒) Let $J : U^2 \to U$ be a function verifying $\mathbf{J2}$, $\mathbf{J8}$, $\mathbf{J9}$ and $\mathbf{J10}$. It holds that:

$$\begin{aligned}
\mathbf{\Pi1} : \Pi_J(x_1, x_2) = 1 &\Leftrightarrow J(N(1-x_2), N(x_1)) = 1 \text{ (by Eq.(16))}\\
&\Leftrightarrow N(1-x_2) = 1 \, and \, N(x_1) = 0 \Leftrightarrow x_2 = 1 \, and \, x_1 = 1 \text{ (by } \mathbf{J10}\text{)}\\
\mathbf{\Pi2} : \Pi_J(x_1, x_2) = 0 &\Leftrightarrow J(N(1-x_2), N(x_1)) = 0 \text{ (by Eq.(16))}\\
&\Leftrightarrow N(1-x_2) \geq N(x_1)\\
&\Leftrightarrow x_1 + x_2 \leq 1 \text{ and } x_1 + x_2 \geq 1 \Leftrightarrow x_1 + x_2 = 1 \text{ (by } \mathbf{J8} \text{ and Eq.(4))}\\
\mathbf{\Pi3} : (y_1, y_2) \preceq (x_1, x_2) &\Rightarrow y_1 \leq x_1 \, and \, y_2 \leq x_2 \text{ by Eq.(7)}\\
&\Rightarrow N(x_1) \geq N(y_1) \, and \, N(1-x_2) \leq N(1-y_2) \text{ by } \mathbf{N2}\\
&\Rightarrow J(N(1-x_2), N(x_1)) \leq J(N(1-y_2), N(y_1)) \text{ by } \mathbf{I1}\\
&\Rightarrow \Pi_J(x_1, x_2) \leq \Pi_J(y_1, y_2) \text{ by Eq.(16)}
\end{aligned}$$

When N_I is a SIFN,

$$\begin{aligned}
\mathbf{\Pi4} : \Pi_J(N_I(x_1, x_2)) &= \Pi_J(N(N_S(x_2)), N_S(N(x_1))) \text{ by Eq.(13)}\\
&= (J(x_1, 1-x_2)) \text{ by Eq.(16)}\\
&= (J(N(1-x_2)), N(x_1)) \text{ by } \mathbf{I9}\\
&= \Pi_J(x_1, x_2) \text{ by Eq.(16)}
\end{aligned}$$

(⇐) Consider the function $J(x_1, x_2) = \begin{cases} 1, & \text{if } x_1 > x_2, \\ \Pi_J(N(x_2), 1 - N(x_1)), & \text{otherwise.} \end{cases}$
The following holds:

$$\begin{aligned}
\mathbf{J2} : y_1 \geq y_2 \Leftrightarrow J(x, y_1) &= \begin{cases} 1, & \text{if } x > y_1, \\ \Pi_J(N(y_1), 1 - N(x)), & \text{otherwise; by Eq.(16)} \end{cases}\\
&\geq \begin{cases} 1, & \text{if } x > y_2, \\ \Pi(N(y_2), 1 - N(x)), & \text{otherwise; by } \mathbf{\Pi3} \end{cases}\\
&= J(x, y_2); \text{ by Eq.(16).}
\end{aligned}$$

$\mathbf{J8}$: Strainghforward.

$$\begin{aligned}
\mathbf{J9} : J(N(x_2), N(x_1)) &= \begin{cases} 1, & \text{if } N(x_2) > N(x_1), \\ \Pi_J(x_1, 1 - x_2), & \text{otherwise; by Eqs.(16) and (13)} \end{cases}\\
&= \begin{cases} 1, & \text{if } x_1 \geq x_2, \\ \Pi_J(N_I(N(x_2), 1 - N(x_1)))), & \text{otherwise, by } \mathbf{\Pi4} \end{cases}\\
&= \begin{cases} 1, & \text{if } x_1 \geq x_2, \\ \Pi_J(N(x_2), 1 - N(x_1))), & \text{otherwise, by Eq.(16)} \end{cases}\\
&= J(x_1, x_2), \text{whenever } N \text{ is a SFN.}\\
\mathbf{J10} : J(x_1, x_2) = 1 &\Leftrightarrow \Pi_J(N(x_2), 1 - N(x_1)) = 1 \text{by Eq.(16)}\\
&\Leftrightarrow N(x_2) = 1 - N(x_1) = 0 \Leftrightarrow x_1 = 0 \, and \, x_2 = 1 \text{by } \mathbf{\Pi1}.
\end{aligned}$$

Therefore, Proposition 2 holds.

The next corollary follows straightforward from Proposition2:

Corollary 1. *Let $N_I = N_{SI}$ be a SIFN obtained by a SFN N, according with Eq.(13). A function $\Pi : \tilde{U} \to U$ is a $A\!-\!co\!-\!GIFIx(N_I)$ iff there exists a function $J : U^2 \to U$ verifying* $\mathbf{J2}$*,* $\mathbf{J8}$*,* $\mathbf{J9}$ *and* $\mathbf{J10}$ *such that*

$$\Pi_J(x_1, x_2) = J(x_2, 1 - x_1). \tag{17}$$

Proposition 3. *Let $I_N(J_N) : U^2 \to U$ be a N-dual (implication) coimplication of a fuzzy (co)implication $I(J) : U^2 \to U$. If $\Pi_I(\Pi_J) : \tilde{U} \to U$ is a $A - GIFIx(N)$ then $\Pi_{I_N}(\Pi_{J_N}) : \tilde{U} \to U$ is also a $A - GIFIx(N_I)$ given by*

$$\Pi_{I_N}(x_1, x_2) = \Pi_I(x_1, x_2), \text{ and } \Pi_{J_N}(x_1, x_2) = \Pi_J(x_1, x_2). \tag{18}$$

Proof. It follows from Eqs.(20) and (16) in Propositions 1 and 2 that $\Pi_{I_N}(x_1, x_2) = I_N(N(1-x_2), N(x_1)) = N(I(1-x_2, x_1)) = \Pi_I(x_1, x_2)$ and $\Pi_{J_N}(x_1, x_2) = J_N(J_N(1-x_2), x_1) = J(N(1-x_2), N(x_1))) = \Pi_J(x_1, x_2)$.

See Table 1 illustrating Proposition 1, considering $\tilde{x} = (x, y) \in \tilde{U}$, $x + y \leq 1$ in order to present examples of $A - GIFIx(N_{SI})$ associated with following fuzzy implications: R_0, Lukaziewicz, Reichenbach, Gaines-Rescher and I_{30} [9].

Table 1. Generalized intuitionistic fuzzy index associated with the standard negation.

Dual Functions Fuzzy	$A - GIFIx(N_{SI})$
$I_0(x,y)=\begin{cases}1, & \text{if } x \leq y,\\ \max(1-x, y), & \text{otherwise;}\end{cases}$ $J_0(x,y)=\begin{cases}0, & \text{if } x \geq y,\\ \min(1-x, y), & \text{otherwise;}\end{cases}$	$\Pi_0(x,y)=\begin{cases}0, & \text{if } x+y=1,\\ 1-\max(x,y), & \text{otherwise;}\end{cases}$
$I_{LK}(x,y)=\begin{cases}1, & \text{if } x \leq y,\\ 1-x+y, & \text{otherwise;}\end{cases}$ $J_{LK}(x,y)=\begin{cases}0, & \text{if } x \geq y,\\ y-x, & \text{otherwise;}\end{cases}$	$\Pi_{LK}(x,y)=\begin{cases}0, & \text{if } x+y=1,\\ 1-x-y, & \text{otherwise;}\end{cases}$
$I_{RB}(x,y)=\begin{cases}1, & \text{if } x \leq y,\\ 1-x+xy, & \text{otherwise;}\end{cases}$ $J_{RB}(x,y)=\begin{cases}0, & \text{if } x \geq y,\\ y-xy, & \text{otherwise;}\end{cases}$	$\Pi_{RB}(x,y)=\begin{cases}0, & \text{if } x+y=1,\\ 1-x-y+xy, & \text{otherwise;}\end{cases}$
$I_{GR}(x,y)=\begin{cases}1, \text{ if } x \leq y,\\ 0, \text{ otherwise;}\end{cases}$ $J_{GR}(x,y)=\begin{cases}0, \text{ if } x \geq y,\\ 1, \text{ otherwise;}\end{cases}$	$\Pi_{GR}(x,y)=\begin{cases}0, \text{ if } x+y=1,\\ 1, \text{ otherwise;}\end{cases}$
$I_{30}(x,y)=\begin{cases}\min(1-x, y, 0.5), & \text{if } 0<y<x<1,\\ \min(1-x, y), & \text{otherwise;}\end{cases}$ $J_{30}(x,y)=\begin{cases}\max(1-x, y, 0.5), & \text{if } 0<x<y<1,\\ \max(1-x, y), & \text{otherwise;}\end{cases}$	$\Pi_{30}(x,y)=\begin{cases}1-\min(x, y, 0.5), & \text{if } 0<x, y<1 \\ & \text{and } x+y=1,\\ 1-\min(x, y), & \text{otherwise;}\end{cases}$

3.1 Atanassov's Intuitionistic Fuzzy Index and Conjugate Operators

In this section we study the conjugation and duality property related to generalized Atanassov's Intuitionistic Fuzzy Index.

Proposition 4. *Let $\Phi \in Aut(\tilde{U})$ be a ϕ-representable automorphism, $N^{\phi} : U \to U$ be the ϕ-conjugate of a SFN $N : U \to U$. A function ${\Pi_G}^{\Phi} : \tilde{U} \to U$ given by*

$$ {\Pi_G}^{\Phi}(x_1, x_2) = \phi^{-1}(\Pi_G(\Phi(x_1, x_2))) = (\phi^{-1}(\Pi_G(\phi(x_1)), 1 - \phi(1 - x_2)), \quad (19)$$

is a $A - GIFIx(N_I)$ whenever $\Pi_G : \tilde{U} \to \tilde{U}$ is also a $A - GIFIx(N_I)$.

Proof. Let $\Phi : \tilde{U} \to U$ be a representable ϕ-automorphism and $\Pi_G : \tilde{U} \to \tilde{U}$ be a $A - GIFIx(N_I)$. It holds that:

$$\begin{aligned}
\mathbf{\Pi 1} : \Pi_G^{\Phi}(x_1, x_2) = 1 &\Leftrightarrow \phi^{-1}(\Pi_G(\phi(x_1), 1 - \phi(1 - x_2))) = 1 \,\text{(by Eq.(19))}\\
&\Leftrightarrow \Pi_G(\phi(x_1), 1 - \phi(1 - x_2)) = 1 \,\text{(by Eq.(10))}\\
&\Leftrightarrow \phi(x_1) = 0 \, and \, 1 - \phi(1 - x_2) = 0 \ \text{(by } \mathbf{\Pi 1}\text{)}\\
&\Leftrightarrow x_1 = 0 \, and \, x_2 = 0 \ \text{(by } \mathbf{A1}\text{)}\\
\mathbf{\Pi 2} : \Pi_G^{\Phi}(x_1, x_2) = 0 &\Leftrightarrow \phi^{-1}(\Pi_G(\phi(x_1), 1 - \phi(1 - x_2))) = 0 \,\text{(by Eq.(19))}\\
&\Leftrightarrow \Pi_G(\phi(x_1), 1 - \phi(1 - x_2)) = 0 \,\text{(by Eq.(10))}\\
&\Leftrightarrow \phi(x_1) + 1 - \phi(1 - x_2) = 1 \ \text{(by } \mathbf{\Pi 2}\text{)}\\
&\Leftrightarrow \phi(x_1) = \phi(1 - x_2) \Leftrightarrow x_1 = 1 - x_2 \Leftrightarrow x_1 + x_2 = 1\\
\mathbf{\Pi 3} : (x_1, x_2) \preceq (y_1, y_2) &\Rightarrow x_1 \leq y_1 \, and \, x_2 \leq y_2 \ \text{by Eq.(7)}\\
&\Rightarrow \phi(x_1) \leq \phi(y_1) \, and \ 1 - \phi(1 - x_2) \leq 1 - \phi(1 - y_2) \ \text{by } \mathbf{A1}\\
&\Rightarrow \Pi_G(\phi(x_1), 1 - \phi(1 - x_2)) \leq \Pi_G(\phi(y_1), 1 - \phi(1 - y_2)) \ \text{by } \mathbf{\Pi 3}\\
&\Rightarrow \phi^{-1}\left(\Pi_G(\phi(x_1), 1 - \phi(1 - x_2))\right) \leq\\
&\quad\ \phi^{-1}\left(\Pi_G(\phi(y_1), 1 - \phi(1 - y_2))\right) \ \text{by } \mathbf{A1}\\
&\Rightarrow \Pi_G^{\Phi}(x_1, x_2) \leq \Pi_G^{\Phi}(y_1, y_2) \ \text{by Eq.(19)}
\end{aligned}$$

Let N_I be a SIFN obtained by a SFN N, according with Eq.(13) and N_I^{Φ} be its $\Phi-$conjugate function. Therefore, it holds that:

$$\begin{aligned}
\mathbf{\Pi 4} : \Pi_G^{\Phi}\left(N_I^{\Phi}(x_1, x_2)\right) &= {\Pi_G}^{\Phi}\left(N_I^{\Phi}(x_1, x_2)\right) \ \text{(by Eq.(19))}\\
&= \phi^{-1}\left(\Pi_G(\Phi \circ \Phi^{-1}(N_I(\Phi(x_1, x_2))))\right) \ \text{(by Eqs.(19) and (10))}\\
&= \phi^{-1}\left(\Pi_G(N_I(\Phi(x_1, x_2))))\right) \ \text{(by } \mathbf{\Pi 4}\text{)}\\
&= \phi^{-1}\left(\Pi_G(\Phi(x_1, x_2))\right) = \Pi_G(x_1, x_2)
\end{aligned}$$

Proposition 5. *Let $\phi \in Aut(U)$ be an automorphism, $N^{\phi} : U \to U$ be a ϕ-conjugate of a SFN $N : U \to U$ and $I^{\phi} : U^2 \to U$ be a ϕ-conjugate of $I : U^2 \to U$. A function $\Pi_{I^{\phi}}(\Pi_{J^{\phi}}) : \tilde{U} \to U$ given by*

$$\Pi_{I^{\phi}}(x_1, x_2) = N^{\phi}(I^{\phi}(1 - x_2, x_1)), \quad (20)$$

$$\Pi_{J^{\phi}}(x_1, x_2) = J^{\phi}(N^{\phi}(1 - x_2), N^{\phi}(x_1)), \quad (21)$$

is a $A - GIFIx(N)$ whenever $\Pi_I(\Pi_J) : \tilde{U} \to \tilde{U}$ is also a $A - GIFIx(N)$.

Proof. It follows from Propositions 2 and 4.

See Table 2, considering $\tilde{x} = (x, y) \in \tilde{U}$ such that $x + y \leq 1$ and presenting the corresponding $A{-}GIFIx(N)$ associated with the conjugate fuzzy implications related to Table 1 .

Table 2. $A - GIFIx(N_{SI})$ associated with the automorphisms $\phi(x) = x^2$ *and* $\phi^{-1} = \sqrt{x}$.

Fuzzy Implications	$A - GIFIx(N_{SI})$
$I_0^\phi(x,y) = \begin{cases} 1, \text{ if } x \leq y, \\ \sqrt{\max((1-x)^2, y^2)}, \text{otherwise;} \end{cases}$ $J_0^\phi(x,y) = \begin{cases} 0, \text{ if } x \geq y, \\ \sqrt{\min((1-x)^2, y^2)}, \text{otherwise;} \end{cases}$	$\Pi_{I_0^\phi}(x,y) = \begin{cases} 0, \text{ if } x+y=1, \\ 1 - \sqrt{\max(y^2, x^2)}, \text{otherwise;} \end{cases}$
$I_{LK}^\phi(x,y) = \begin{cases} 1, \text{ if } x \leq y, \\ \sqrt{1 - x^2 + y^2}, \text{ otherwise;} \end{cases}$ $J_{LK}^\phi(x,y) = \begin{cases} 0, \text{ if } x \geq y, \\ \sqrt{1 - x^2 + y^2}, \text{ otherwise;} \end{cases}$	$\Pi_{I_{LK}^\phi}(x,y) = \begin{cases} 0, \text{ if } x+y=1, \\ 1 - \sqrt{2y - y^2 + x^2}, \text{otherwise;} \end{cases}$
$I_{RH}^\phi(x,y) = \begin{cases} 1, \text{ if } x \leq y, \\ \sqrt{1 - x^2 + x^2y^2}, \text{otherwise;} \end{cases}$ $J_{RH}^\phi(x,y) = \begin{cases} 0, \text{ if } x \geq y, \\ \sqrt{1 - x^2 + x^2y^2}, \text{otherwise;} \end{cases}$	$\Pi_{I_{RH}^\phi}(x,y) = \begin{cases} 0, \text{ if } x+y=1, \\ 1-\sqrt{x^2 + (1-x^2)(2y-y^2)}, \text{otherwise;} \end{cases}$
$I_{GR}^\phi(x,y) = \begin{cases} 1, \text{ if } x \leq y, \\ 0, \text{otherwise;} \end{cases}$ $J_{GR}^\phi(x,y) = \begin{cases} 0, \text{ if } x \geq y, \\ 1, \text{otherwise;} \end{cases}$	$\Pi_{I_{GR}^\phi}(x,y) = \begin{cases} 0, \text{ if } x+y=1, \\ 1, \text{otherwise;} \end{cases}$
$I_{30}^\phi(x,y) = \begin{cases} \sqrt{\min(1 - x^2, y^2, 0.5)}, \\ \quad \text{if } 0 < x < y < 1, \\ \sqrt{\min((1-x)^2, y^2)}, \text{otherwise;} \end{cases}$ $J_{30}^\phi(x,y) = \begin{cases} \sqrt{\max(1 - x^2, y^2, 0.5)}, \\ \quad \text{if } 0 < x < y < 1, \\ \sqrt{\max((1-x)^2, y^2)}, \text{otherwise;} \end{cases}$	$\Pi_{I_{30}^\phi}(x,y) = \begin{cases} 1-\sqrt{\min(1 - (1-y)^2, x^2, 0.5)}, \\ \quad \text{if } 0 < x, y < 1 \text{ and } x+y=1, \\ 1-\sqrt{\min(1-(1-y)^2, x^2)}, \text{otherwise;} \end{cases}$

3.2 A-GIFIx (S, N)-implications and (T, N)-coimplications

In the following, (S, N)-implications and (T, N)-coimplications are considered in order to obtain new expressions of A-GIFIx.

Proposition 6. *Let N be a SFN. A function $\Pi : \tilde{U} \to U$ is a $A - GIFIx(N)$ iff there exists an (S,N)-implication ((T,N)-coimplication) $I_{S,N}(J_{T,N}) : U^2 \to U$ such that*

$$\Pi_{I_{S,N}}(x_1, x_2) = S_N(N_S(x_2), N(x_1))); \tag{22}$$

$$\Pi_{J_{T,N}}(x_1, x_2) = T(N_S(x_2), N(x_1)). \tag{23}$$

Proof. $\Pi_{I_{S,N}}(x_1,x_2) = N(I_{S,N}(1-x_2,x_1)) = N(S(N(1-x_2),x_1)) = S_N(N_S(x_2),N(x_1)))$ and $\Pi_{J_{T,N}}(x_1,x_2) = J_{T,N}(x_2,1-x_1)) = T(x_2,N(1-x_1)) = T(N_S(x_2),N(x_1))$, for all $(x_1,x_2) \in \tilde{U}$.

Remark 2. When $N = N_S$, Eq.(22) can be expressed as $\Pi_{I_{S,N_S}}(x_1,x_2) = N_S(S(x_1,x_2))$ and $\Pi_{J_{T,N_S}}(x_1,x_2) = N_S(T_{N_S}(x_1,x_2))$.

4 Generation of Atanassov's Intuitionistic Fuzzy Entropy

By [8], the Atanassov's intuitionistic fuzzy entropy is discussed in the following.

Definition 2. *[8, Definition 2] A real function $E : \mathcal{A}_I \to U$ is called an Atanassov's intuitionistic fuzzy entropy if E satisfies, $\forall A, B \in \mathcal{A}_I$, the following properties:*

E1*: $E(A) = 0$ if and only if $A \in \mathcal{A}$,*
E2*: $E(A) = 1$ if and only if $\mu_A(x) = \nu_A(x) = 0$, $\forall x \in \chi$,*
E3*: $E(A) = E(A_c)$,*
E4*: if $A \preceq B$ then $E(A) \geq E(B)$.*

4.1 Atanassov's intuitionistic fuzzy entropy and Atanassov's generalized intuitionistic fuzzy index

This section we discuss properties related to the Atanassov's intuitionistic fuzzy entropy obtained by aggregation of Atanassov's generalized intuitionistic fuzzy index.

Proposition 7. *[8, Proposition 4]. Consider $\chi = \{x_1,\ldots,x_n\}$. Let Ag be an aggregation function, N be a strong negation, Π_G be an $A-GIFIx(N)$. Then, for all $A \in \mathcal{A}$, the mapping $E : \mathcal{A} \to U$ defined by*

$$E(A) = Ag_{i=1}^{n}\Pi_G(A(x_i)), \forall x_i \in \chi, \tag{24}$$

is an Atanassov's intuitionistic fuzzy entropy (A-IFE).

Proposition 8. *Consider $\chi = \{x_1,\ldots,x_n\}$ and $\Phi \in Aut(\tilde{U})$ given by Eq.(11). When Π_G is $A-GIFIx(N)$, for all $A \in \mathcal{A}$, the mapping $E^{\Phi} : \mathcal{A} \to U$ defined by*

$$E^{\Phi}(A) = Ag_{i=1}^{n}(\Pi_G)^{\Phi}(A(x_i)), \forall x_i \in \chi, \tag{25}$$

is an Atanassov's intuitionistic fuzzy entropy (A-IFE).

Proof. Straightforward Propositions 4 and 8.

The diagram below summarizes the main results related to the classes of $A-GIFIx(N)$ and $A-IFE$ denoted by $\mathcal{C}(\Pi_G)$ and $\mathcal{C}(E)$, respectively.

In the following, we extend the Atanassov's intuitionistic fuzzy entropy which is obtained not only from generalized intuitionistic fuzzy index as conceived in [11] but also from their dual and conjugate constructions.

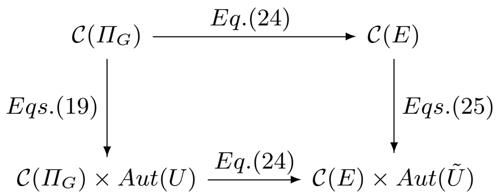

Fig. 1. Conjugate construction of $A - GIFIx(N)$ and $A - IFE$ on $Aut(\tilde{U})$

Proposition 9. *Consider $\phi \in Aut(U)$. Let $N : U \to U$ be a SFN, $Ag : U^n \to U$ be an aggregation function and $I_N : U^2 \to U$ be a N-dual operator of an implication $I : U^2 \to U$ which satisfies properties* **I1**, **I8**, **I9** *and* **I10**, *as discussed in Proposition 1. Then, for all $A \in \mathcal{A}$, the mappings $E_I, E_{I^\Phi} : \mathcal{A} \to U$ defined for by*

$$E_I(A) = Ag_{i=1}^n N(I(1 - \nu_A(x_i), \mu_A(x_i))), \tag{26}$$

$$E_{I^\phi}(A) = Ag_{i=1}^n N^\phi(I^\phi(1 - \nu_A(x_i), \mu_A(x_i))), \tag{27}$$

give new expressions of the Atanassov's intuitionistic fuzzy entropy related to (A-GIFIx(N)).

Proof. Eq.(26) is proved in [8, Corollary 5]. Other ones are straightforward from Propositions 1, 5 Eq.(20) and (24).

Proposition 10. *Consider $\phi \in Aut(U)$. Let $N : U \to U$ be a SFN, $Ag : U^n \to U$ be an aggregation function and $J_N : U^2 \to U$ be a N-dual operator of a coimplication $J : U^2 \to U$ satisfying properties* **J2**, **J8**, **J9** *and* **J10**, *according with Proposition 2. Then, for all $A \in \mathcal{A}$, the mappings $E_J, E_{J^\phi} : \mathcal{A} \to U$ defined by*

$$E_J(A) = Ag_{i=1}^n J\left(N(1 - \nu_A(x_i)), N(\mu_A(x_i))\right), \forall x_i \in \chi, \tag{28}$$

$$E_{J^\phi}(A) = Ag_{i=1}^n J^\phi\left(N^\phi(1 - \nu_A(x_i)), N^\phi(\mu_A(x_i))\right), \forall x_i \in \chi, \tag{29}$$

are also Atanassov's intuitionistic fuzzy entropies.

Proof. Straightforward from Propositions 2, 5 Eq.(16) and (24), Propositions 9 and 10.

Proposition 11. *$E_J, E_{J_N} : \mathcal{A} \to U$ be Atanassov's intuitionistic fuzzy entropies according with Propositions 9 and 10. Then, for all $A \in \mathcal{A}$, the following holds:*

$$E_{J_N}(A) = E_J(A) \text{ and } E_{I_N}(A) = E_I(A) \tag{30}$$

is an Atanassov's intuitionistic fuzzy entropy.

Proof. Straightforward from Proposition 3, Eqs.(20)a and (20)b and Proposition 10.

5 Conclusion

In this work, the concept of generalized Atanassov's intuitionistic fuzzy index was studied by dual and conjugate construction methods, in particular, by means of fuzzy (S, N)- and (T, N)-operators. We also extend the study of Atanassov's intuitionistic fuzzy entropy based on such two methodologies.

Further work considers the extension of such study related to properties verified by the $A - GIFIx(N)$ and $A - IFE$ to the interval-valued intuitionistic fuzzy approach.

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