Pre-aggregation functions constructed by CO-integrals applied in classification problems

Giancarlo Lucca[1], José Antonio Sanz[1,2], Graçaliz Pereira Dimuro[3], Benjamín Bedregal[4], and Humberto Bustince[1,2]

[1] Depto. of Automática y Computación, Universidad Publica de Navarra
Campus Arrosadía s/n, P.O. Box 31006, Pamplona, Spain
lucca.112793@e.unavarra.es
[2] Institute of Smart Cities, Universidad Publica de Navarra
Campus Arrosadía, Navarra, 31006, Spain
{joseantonio.sanz,bustince}@unavarra.es
[3] Centro de Ciências Computacionais, Universidade Federal do Rio Grande
Av. Itália km 08, Campus Carreiros, Rio Grande, 96201-900, Brazil
gracaliz@gmail.com
[4] Departamento de Informática e Matemática Aplicada, Universidade Federal do Rio Grande do Norte. Campus Universitário, Natal, 59072-970, Brazil
bedregal@dimap.ufrn.br

Abstract. Pre-aggregation functions follow the basic concept of aggregation functions with the difference that they are just directional increasing. In this work, we present a new pre-aggregation function by generalizing the Choquet integral in such a way that the product operator is substituted by overlap functions. We call this new concept as Choquet-like overlap based integral (CO-integral). Since Lucca et al. presented the concept of pre-aggregation having an excellent performance in Fuzzy Rule-Based Classification Systems (FRBCSs), we also apply a particular CO-integral in FRBCSs and compare our new generalization with the best pre-aggregation function proposed by them. We demonstrate that this CO-integral can be used as an alternative of the application of pre-aggregations by offering new possibilities for defining another aggregation operator in the fuzzy reasoning method (FRM) of FRBCSs.

Keywords: pre-aggregation functions, Choquet integral, overlap functions, fuzzy rule-based classification systems, fuzzy reasoning method

1 Introduction

Pre-aggregations functions is a new concept presented by Lucca et al. [1], where the required concept of aggregation function is partially satisfied, that is, in spite of the boundary condition be totally fulfilled the monotonicity is used along some a direction (directional increasing [2]) but not for all directions. As done in [1], in this paper, we present an application of our approach in Fuzzy Rule-Based Classification Systems (FRBCSs) [3] since this kind of system deals with interpretable models by using linguist labels in their rules. FRBCSs have been applied in severals real world problems, including industry [4], health [5], economy [6] and many others.

The Fuzzy Reasoning Method (FRM) is a key component in FRBCS since it determines how the information learned in form of fuzzy rules is used to classify new instances. Barrenechea et al. [7] introduced a new FRM that takes into account the information given by all fired fuzzy rules when classifying a new instance. To do so, they considered the Choquet integral [8] as the aggregation operator, along with a fuzzy measure called power measure, which raises the standard cardinal measure to the power q, which is learnt using an genetic algorithm [9] in order to adapt this parameter for each class. Later, Lucca et al generalized the Choquet integral by substituting the product by a t-norm [1].

In this work, based on the idea proposed in [1], we propose a new generalization of the standard Choquet integral. More precisely we propose to change the product operator of the Choquet integral by an overlap function, and thus, introducing the concept of Choquet-like overlap-based integral (CO-integral). We also apply this new concept in FRBCSs to test the suitability of the new aggregation operator.

In order to demonstrate the quality of our approach, we have selected 30 datasets that are accessible in KEEL[5] database repository [10]. We analyze the behavior of our CO-integrals with respect to the best generalization presented by Lucca et al. in [1]. Our conclusions are supported by the well-known Wilcoxon signed-rank test [11].

The paper is organized as follows. Section 2 presents some preliminary concepts that are necessary to develop the paper. The new concept of CO-integrals is presented in Section 3. We explain the experimental framework, the results achieved by the application of CO-integral in FRBCSs and the analyze of these results in Section 4. The main conclusions are drawn in Section 5.

2 Theoretical Framework

In this section we introduce some preliminaries concepts necessary to understand the paper.

Definition 1. *A function $A : [0,1]^n \to [0,1]$ is said to be an n-ary aggregation function if the following conditions hold:*

(A1) *A is increasing[6] in each argument: for each $i \in \{1,\dots,n\}$, if $x_i \le y$, then $A(x_1,\dots,x_n) \le A(x_1,\dots,x_{i-1},y,x_{i+1},\dots,x_n)$;*
(A2) *A satisfies the boundary conditions: $A(0,\dots,0) = 0$ and $A(1,\dots,1) = 1$.*

Let $\boldsymbol{r} = (r_1,\dots,r_n)$ be a real n-dimensional vector, $\boldsymbol{r} \neq \mathbf{0}$. A function $F : [0,1]^n \to [0,1]$ is directionally increasing[2] with respect to $\boldsymbol{r}$ ($\boldsymbol{r}$-increasing, for

[5] http://www.keel.es

[6] In this paper, an increasing (decreasing) function does not need to be strictly increasing (decreasing).

short) if for all $(x_1, \ldots, x_n) \in [0,1]^n$ and $c > 0$ such that $(x_1+cr_1, \ldots, x_n+cr_n) \in [0,1]^n$ it holds that

$$F(x_1 + cr_1, \ldots, x_n + cr_n) \geq F(x_1, \ldots, x_n). \tag{1}$$

Similarly, one defines an $\boldsymbol{r}$-decreasing function.

Definition 2. *[1] Let $\boldsymbol{r} = (r_1, \ldots, r_n)$ be a real n-dimensional vector, $\boldsymbol{r} \neq \mathbf{0}$. A function $F : [0,1]^n \to [0,1]$ is is said to be an n-ary pre-aggregation function if the following conditions hold:*

(PA1) *F is directionally increasing with respect to some $\boldsymbol{r}$, i.e., it is $\boldsymbol{r}$-increasing;*
(PA2) *F satisfies the boundary conditions: $F(0, \ldots, 0) = 0$, $F(1, \ldots, 1) = 1$.*

If F is a pre-aggregation function with respect to a vector $\boldsymbol{r}$ we just say that F is an $\boldsymbol{r}$-pre-aggregation function. See also the works by Mesiar et al.[12] and Dimuro et al.[13]

Definition 3. *A function $O : [0,1]^2 \to [0,1]$ is said to be an overlap function if it satisfies the following conditions:*

(O1) *O is commutative;*
(O2) *$O(x,y) = 0$ if and only if $xy = 0$;*
(O3) *$O(x,y) = 1$ if and only if $xy = 1$;*
(O4) *O is increasing;*
(O5) *O is continuous.*

Observe that the definitions of overlap functions can be easily extended to n-ary functions. [14,15].

Example 1. An example of an overlap function is $O_b : [0,1]^2 \to [0,1]$, defined as [16]:

$$O_b(x,y) = \min\{x\sqrt{y}, y\sqrt{x}\} \tag{2}$$

Now, we recall the notion of fuzzy measure [8,17], which is going to be a key tool for constructing our pre-aggregation functions. In the following, consider the set $N = \{1, \ldots, n\}$ for an arbitrary positive integer n.

Definition 4. *A function $\mathfrak{m} : 2^N \to [0,1]$ is a fuzzy measure if, for all $X, Y \subseteq N$, it satisfies the following properties:*

($\mathfrak{m}$1) *Increasing: if $X \subseteq Y$, then $\mathfrak{m}(X) \leq \mathfrak{m}(Y)$;*
($\mathfrak{m}$2) *Boundary conditions: $\mathfrak{m}(\emptyset) = 0$ and $\mathfrak{m}(N) = 1$.*

The fuzzy measure considered in this paper is the power measure $\mathfrak{m}_{PM} : 2^N \to [0,1]$, defined, for all $X \subseteq N$, by

$$\mathfrak{m}_{PM}(X) = \left(\frac{|X|}{n}\right)^q, \text{ with } q > 0, \tag{3}$$

where the exponent q is learned genetically. The choice for this fuzzy measure was based on the results obtained by Barrenechea et al. [7], who introduced an evolutionary algorithm to define the most suitable power measure definition to be used for each class. See also [1,18,19]

The Choquet integral is defined with respect to fuzzy measures. In this paper, we consider only the discrete Choquet integral [8], related to fuzzy measures, which are defined on finite spaces:

Definition 5. *[20, Definition 1.74] Let $\mathfrak{m} : 2^N \rightarrow [0,1]$ be a fuzzy measure. The discrete Choquet integral of $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$ with respect to $\mathfrak{m}$ is defined as a function $C_\mathfrak{m} : [0,1]^n \rightarrow [0,1]$, given by*

$$C_\mathfrak{m}(\boldsymbol{x}) = \sum_{i=1}^{n} \left(x_{(i)} - x_{(i-1)}\right) \cdot \mathfrak{m}\left(A_{(i)}\right), \tag{4}$$

where $\left(x_{(1)}, \ldots, x_{(n)}\right)$ is an increasing permutation on the input x, that is, $0 \leq x_{(1)} \leq \ldots \leq x_{(n)}$, with the convention that $x_{(0)} = 0$, and $A_{(i)} = \{(i), \ldots, (n)\}$ is the subset of indices of $n - i + 1$ largest components of $\boldsymbol{x}$.

The Choquet integral combines the inputs in such a way that the importance of the different groups of inputs (coalitions) may be taken into account. Allowing to assign importance to all possible groups of criteria, the Choquet integral offers greater flexibility in the aggregation modelling.

3 Choquet-like overlap-based integral (CO-integral)

In this section we introduce the idea of the generalization of the standard Choquet integral, by the overlap function shown in Equation (2), introducing the new concept of Choquet-like overlap-based integral. Specifically, in this section the theoretical study and the new FRM are presented.

3.1 Theoretical development of CO-integrals

The proof that our CO-integral is a pre-aggregation function, satisfying the boundary condition along with the directional increasing is presented in this subsection.

Definition 6. *Let $\mathfrak{m} : 2^N \rightarrow [0,1]$ be a fuzzy measure and $O : [0,1]^2 \rightarrow [0,1]$ be a bivariate overlap function. The Choquet-like overlap-based integral with respect to $\mathfrak{m}$ is defined as a function $\mathfrak{C}_\mathfrak{m}^{O_b} : [0,1]^n \rightarrow [0,1]$, given, for all $\mathbf{x} \in [0,1]^n$, by*

$$\mathfrak{C}_\mathfrak{m}^{O_b}(\boldsymbol{x}) = \sum_{i=1}^{n} O_b\left(x_{(i)} - x_{(i-1)}, \mathfrak{m}\left(A_{(i)}\right)\right), \tag{5}$$

where $\left(x_{(i)}, \ldots, x_{(n)}\right)$ is an increasing permutation on the input x, that is, $0 \leq x_{(1)} \leq \ldots \leq x_{(n)}$, with the convention that $x_{(0)} = 0$, and $A_{(i)} = \{(i), \ldots, (n)\}$ is the subset of indices of $n - i + 1$ largest components of $\boldsymbol{x}$.

Consider a fuzzy measure $\mathfrak{m} : 2^N \to [0,1]$ and the overlap functions that is presented in Equation (2). The Choquet-like overlap-based integral with respect to $\mathfrak{m}$, assume the following form:

$$\mathfrak{C}_{\mathfrak{m}}^{O_b}(\boldsymbol{x}) = \sum_{i=1}^{n} \min \left\{ \left(x_{(i)} - x_{(i-1)}\right) \sqrt{\mathfrak{m}\left(A_{(i)}\right)}, \mathfrak{m}\left(A_{(i)}\right) \sqrt{x_{(i)} - x_{(i-1)}} \right\} \quad (6)$$

Proposition 1. *For the overlap function $O_b : [0,1]^2 \to [0,1]$ and fuzzy measure $\mathfrak{m} : 2^N \to [0,1]$, $\mathfrak{C}_{\mathfrak{m}}^{O_b}$ satisfies the boundary conditions.*

Proof. Considering $\mathbf{0} = (0, \ldots, 0) \in [0,1]^n$ and $\mathbf{1} = (1, \ldots, 1) \in [0,1]^n$, one has that:

$$\mathfrak{C}_{\mathfrak{m}}^{O_b}(\mathbf{0}) = \sum_{i=1}^{n} \min \left\{ (0-0)\sqrt{\mathfrak{m}\left(A_{(i)}\right)}, \mathfrak{m}\left(A_{(i)}\right) \sqrt{(0-0)} \right\} = 0,$$

and

$$\mathfrak{C}_{\mathfrak{m}}^{O_b}(\mathbf{1}) = \min \left\{ (1-0)\sqrt{\mathfrak{m}\left(A_{(1)}\right)}, \mathfrak{m}\left(A_{(i)}\right) \sqrt{(1-0)} \right\}$$

$$+ \sum_{i=2}^{n} \min \left\{ (1-1)\sqrt{\mathfrak{m}\left(A_{(i)}\right)}, \mathfrak{m}\left(A_{(i)}\right) \sqrt{(1-1)} \right\} = 1.$$

□

Remark 1. Consider the following ordered vectors:

$$\boldsymbol{x} = (x_1, 0.1, 0.7, 0.9, x_5), \quad \boldsymbol{x'} = (x_{(1)}, 0.1, 0.8, 0.9, x_{(5)}) \in [0,1]^5.$$

Observe that $x_{(2)} = x'_{(2)}$, $x_{(3)} = 0.7 < 0.8 = x'_{(3)}$ and $x_{(4)} = x'_{(4)}$. Obviously $\boldsymbol{x'} > \boldsymbol{x}$, however $\mathfrak{C}_{\mathfrak{m}}^{O_b}(\boldsymbol{x}) > \mathfrak{C}_{\mathfrak{m}}^{O_b}(\boldsymbol{x'})$ as follows:

$$\mathfrak{C}_{\mathfrak{m}}^{O_b}(x_{(1)}, 0.1, 0.7, 0.9, x_{(5)}) = 0.58$$

whereas

$$\mathfrak{C}_{\mathfrak{m}}^{O_b}(x_{(1)}, 0.1, 0.8, 0.9, x_{(5)}) = 0.55$$

Therefore, $\mathfrak{C}_{\mathfrak{m}}^{O_b}$ is not increasing, which is a key propriety of aggregation functions.

Proposition 2. *Consider the vector $\boldsymbol{r} = (k, \ldots, k) \in \mathbb{R}^n$, with $k > 0$. For the overlap function $O_b : [0,1]^2 \to [0,1]$ and fuzzy measure $\mathfrak{m} : 2^N \to [0,1]$, $\mathfrak{C}_{\mathfrak{m}}^{O_b}$ is $\boldsymbol{r}$-increasing.*

Proof. Since the coordinates of $\boldsymbol{r}$ are all equal to some $k > 0$, then is sufficient to consider the case in which $\boldsymbol{x}$ is an ordered input, that is, $x_i = x_{(i)}$. For all

$(x_1, \ldots, x_n) \in [0,1]^n$ and for all $c > 0$ such that $(x_1 + ck, \ldots, x_n + ck) \in [0,1]^n$, it holds that:

$$
\begin{aligned}
&\mathfrak{C}_{\mathfrak{m}}^{O_b}(x_1 + ck, \ldots, x_n + ck) \\
&= \min\left\{(x_{(1)} + ck - 0)\sqrt{\mathfrak{m}\left(A_{(1)}\right)}, \mathfrak{m}\left(A_{(1)}\right)\sqrt{x_{(1)} + ck - 0}\right\} \\
&+ \sum_{i=2}^{n} \min\left\{(x_{(i)} + ck - (x_{(i-1)} + ck))\sqrt{\mathfrak{m}\left(A_{(i)}\right)}, \mathfrak{m}\left(A_{(i)}\right)\sqrt{x_{(i)} + ck - (x_{(i-1)} + ck)}\right\} \\
&= \min\left\{(x_{(1)} + ck)\sqrt{\mathfrak{m}\left(A_{(1)}\right)}, \mathfrak{m}\left(A_{(1)}\right) * \sqrt{x_{(1)} + ck}\right\} \\
&+ \sum_{i=2}^{n} \min\left\{(x_{(i)} - x_{(i-1)})\sqrt{\mathfrak{m}\left(A_{(i)}\right)}, \mathfrak{m}\left(A_{(i)}\right) * \sqrt{x_{(i)} - x_{(i-1)}}\right\} \\
&> \min\left\{x_{(1)}\sqrt{\mathfrak{m}\left(A_{(1)}\right)}, \mathfrak{m}\left(A_{(1)}\right) * \sqrt{x_{(1)}}\right\} \\
&+ \sum_{i=2}^{n} \min\left\{(x_{(i)} - x_{(i-1)})\sqrt{\mathfrak{m}\left(A_{(i)}\right)}, \mathfrak{m}\left(A_{(i)}\right) * \sqrt{x_{(i)} - x_{(i-1)}}\right\}
\end{aligned}
$$

and, thus, $\mathfrak{C}_{\mathfrak{m}}^{O_b}$ is $\boldsymbol{r}$-increasing. □

Theorem 1. *For the overlap function $O_b : [0,1]^2 \to [0,1]$ and fuzzy measure $\mathfrak{m} : 2^N \to [0,1]$, $\mathfrak{C}_{\mathfrak{m}}^{O_b}$ is a pre-aggregation function.*

Proof. It follows from Propositions 1 and 2.

3.2 The Fuzzy Reasoning Method Using CO-Integrals

In this subsection we present the new FRM generalized by the overlap function O_b, as presented in Equation (6). For the following, consider that a classification problem, consists of m training examples $\mathbf{x}_p = (x_{p1}, \ldots, x_{pn}, y_p)$, with $p = 1, \ldots, m$, where x_{pi}, with $i = 1, \ldots, n$, is the value of the i-th attribute and $y_p \in \mathbb{C} = \{C_1, C_2, ..., C_M\}$ is the label of the class of the p-th training example.

In this work we use FRBCSs to deal with classification problems. Specifically, we have selected FARC-HD [19] to accomplish the learning process and the form of the fuzzy rules used by this algorithm is:

$$
\text{Rule } R_j : \text{ If } x_{p1} \text{ is } A_{j1} \text{ and } \ldots \text{ and } x_{pn} \text{ is } A_{jn} \text{ then Class is } C_j \text{ with } RW_j, \quad (7)
$$

where $x_p = (x_{p1}, \ldots, x_{pn})$, is the n-dimensional vector of attribute values corresponding to an example $\mathbf{x}_p$. R_j is the label of the jth rule, A_{ji} is an antecedent fuzzy set modeling a linguistic term, C_j is the class of the j-th rule, and $RW_j \in [0,1]$ is the rule weight [21], which in this case is computed using the certainty factor.

Our proposal is a modification of the third step of the FRM in the FARC-HD fuzzy classifier. More precisely, we propose the usage of Choquet-like overlap-based integral (CO-integral) in order to obtain the information associated with each class of the problem. Specifically, the new classification soundness degree in the FRM is the following:

– **Example classification soundness degree for all classes.** In this step we apply our CO-functions to combine the association degrees obtained in the previous steps of the FRM, $b_i^k(x_p)$, as follows:

$$Y_k(x_p) = \mathfrak{C}_{\mathfrak{m}}^{O_b}\left(b_1^k(x_p), \ldots, b_L^k(x_p)\right), \text{ with } k = 1, \ldots, M, \tag{8}$$

where $\mathfrak{C}_{\mathfrak{m}}^{O_b}$ is the obtained CO-integral, which is the result of combining overlap functions $O : [0,1]^2 \to [0,1]$, the fuzzy measure $\mathfrak{m} : 2^N \to [0,1]$, x_p is the example to be classified, M is the number of classes of the problem and L is the number of fuzzy rules in the system. Specifically we use the overlap function shown in Equation (6), that is, O_b.

4 Application of CO-integrals in Fuzzy Rule-Based Classification Systems

In this section, firstly we present the 30 real world classification problems selected from the KEEL dataset repository [10] (Section 4.1). Furthermore, presenting the achieved results in test by the FRM generalized by our CO-integral, along with an analysis of these obtained results (Section 4.2).

4.1 Experimental framework

The properties of the datasets, containing for each dataset, the identifier (Id.), along with the name (Dataset), the number of instances (*#Inst*), the number of attributes (*#Att*) and the number of classes (*#Class*) are summarized in Table 1. The *magic*, *page-blocks*, *penbased*, *ring*, *shuttle*, *satimage* and *twonorm* datasets have been stratified sampled at 10% in order to reduce their size for training. Examples with missing values have been removed, e.g., in the *wisconsin* dataset.

As proposed in [7,18,22], we adopt the 5-fold cross-validation model, in other words, a dataset is splitted in five random partitions, where each partition have 20% of the examples, and a combination of four of them is used for training and the remainder one is used for testing. This process is repeated five times by using a different partition to test the created system each time. In order to measure the quality of each partition, the accuracy rate is calculated, that is, we divide the number of correctly classified examples divided by the total number of examples for each partition. Then, as the final result of the algorithm we consider the average of the achieved accuracy in this five partitions.

Table 1. Summary of the datasets used in this study.

Id.	Dataset	#Inst	#Att	#Class	Id.	Dataset	#Inst	#Att	#Class
App	Appendiciticis	106	7	2	Pho	Phoneme	5,404	5	2
Bal	Balance	625	4	3	Pim	Pima	768	8	2
Ban	Banana	5300	2	2	Rin	Ring	740	20	2
Bnd	Bands	365	19	2	Sah	Saheart	462	9	2
Bup	Bupa	345	6	2	Sat	Satimage	6,435	36	7
Cle	Cleveland	297	13	5	Seg	Segment	2,310	19	7
Eco	Ecoli	336	7	8	Shu	Shuttle	5,800	9	7
Gla	Glass	214	9	6	Spe	Spectfheart	267	44	2
Hab	Haberman	306	3	2	Tit	Titanic	2,201	3	2
Hay	Hayes-Roth	160	4	3	Two	Twonorm	740	20	2
Iri	Iris	150	4	3	Veh	Vehicle	846	18	4
Mag	Magic	1,902	10	2	Vow	Vowel	990	13	11
New	Newthyroid	215	5	3	Win	Wine	178	13	3
Pag	Pageblocks	5,472	10	5	Wis	Wisconsin	683	11	2
Pen	Penbased	1,099	16	10	Yea	Yeast	1,484	8	10

4.2 Experimental Results

This subsection present the results achieved in testing by the FRM generalized by our overlap function, O_b. In order to determine the quality of the approach, we also present the results achieved by this generalization along with the best pre-aggregation function proposed by Lucca et al. in [1] (named here as Hamacher$_{PA}$), since this generalization provides results with a mean accuracy superior than the standard Choquet Integral. Furthermore, this generalization is statistically superior when compared with the one which uses the FRM of the Winning Rule (WR), that is, using the maximum as aggregation function.

The results achieved in testing by these approaches are presented in Table 2 by columns, where the best result achieved among the different datasets if highlighted in **boldface**.

From the results shown in the Table 2, it is possible to observe that our generalization by the overlap function (O_b) presents a good performance, achieving a good mean, but, smaller than the one achieved by the pre-aggregation proposed in [1]. However, looking only at the mean is not enough to opine about the quality of this generalization, and for this reason, any conclusions should be constructed using an appropriate statistical study.

More specifically, we carried out a pair-wise comparison using the well known Wilcoxon signed-rank test [11]. Table 3 present the results of this comparisons, where R^+ indicates the ranks obtained by O_b and R^- represents the ranks achieved by Hamacher$_{PA}$.

According to the obtained statistical results presented in Table 3, we can affirm with a high level of confidence, that there is no statistical differences among these generalizations. Furthermore, if we look closer, it is possible to observe that CO-integrals achieves a better classification rate in 14 datasets under this study. Obviously, Hamacher$_{PA}$ achieved a mean superior than O_b in

Table 2. Results in testing provided by the CO-integral.

Dataset	Hamacher$_{PA}$	O$_b$
App	82.99	**83.03**
Bal	**82.72**	82.08
Ban	85.96	**86.81**
Bnd	**72.13**	71.83
Bup	**65.80**	65.51
Cle	55.58	**56.24**
Eco	**80.07**	76.20
Gla	63.10	**66.82**
Hab	72.21	**72.86**
Hay	**79.49**	78.72
Iri	93.33	**94.00**
Mag	79.76	**79.86**
New	**95.35**	94.88
Pag	94.34	**94.52**
Pen	90.82	**91.09**
Pho	**83.83**	82.92
Pim	73.44	**75.38**
Rin	88.78	**89.32**
Sah	**70.77**	69.48
Sat	**80.40**	78.54
Seg	**93.33**	92.51
Shu	97.20	**97.29**
Spe	76.02	**77.88**
Tit	**78.87**	**78.87**
Two	**85.27**	83.78
Veh	**68.20**	67.73
Vow	**68.18**	68.08
Win	**96.63**	94.97
Wis	**96.78**	96.63
Yea	56.53	**57.35**
Mean	**80.26**	80.17

15 datasets and they tie in the Titanic dataset. Therefore reinforcing the results obtained in the statistical test.

5 Conclusions

In this paper, we introduce the notion of Choquet-like overlap-based aggregation function (CO-integral). The CO-integral is defined in a similar way of a Choquet-like pre-aggregation function, introduced by Lucca et al. [1], also obtaining a directional increasing function, and so, producing an pre-aggregation function.

We applied the CO-integral based on the overlap function O_b in the FRM of FRBCSs. Moreover, we have to highlight that this CO-integral allows to conclude that this generalization is competitive with the one presented in [1]. Therefore,

Table 3. Wilcoxon Test to compare the CO-integral based on the O_b overlap versus the Hamacher$_{PA}$.

Comparison	R^+	R^-	p-value
O_b vs. Hamacher$_{PA}$	210	255	0.634

the approach presented in this work is an alternative to the function presented by Lucca et al, whereas an improvement in the quality of the new FRM was presented.

Future work is concerned with the usage of different overlap functions and also the study of the properties satisfied by the CO-integrals.

Acknowledgment

This work is supported by Brazilian National Counsel of Technological and Scientific Development CNPq (Proc. 232827/2014-1, 233950/2014-1, 481283/2013-7, 306970/2013-9, 307681/2012-2) and NSF (TIN2013-40765-P).

References

1. G. Lucca, J. Sanz, G. Pereira Dimuro, B. Bedregal, R. Mesiar, A. Kolesárová, H. Bustince Sola, Pre-aggregation functions: construction and an application, IEEE Transactions on Fuzzy Systems 24 (2) (2016) 260–272.
2. H. Bustince, J. Fernandez, A. Kolesárová, R. Mesiar, Directional monotonicity of fusion functions, European Journal of Operational Research 244 (1) (2015) 300–308.
3. H. Ishibuchi, T. Nakashima, M. Nii, Classification and Modeling with Linguistic Information Granules, Advanced Approaches to Linguistic Data Mining, Advanced Information Processing, Springer, Berlin, 2005.
4. S. R. Samantaray, K. El-Arroudi, G. Joos, I. Kamwa, A fuzzy rule-based approach for islanding detection in distributed generation, Power Delivery, IEEE Transactions on 25 (3) (2010) 1427–1433.
5. J. A. Sanz, M. Galar, A. Jurio, A. Brugos, M. Pagola, H. Bustince, Medical diagnosis of cardiovascular diseases using an interval-valued fuzzy rule-based classification system, Applied Soft Computing 20 (2014) 103 – 111.
6. J. Sanz, D. Bernardo, F. Herrera, H. Bustince, H. Hagras, A compact evolutionary interval-valued fuzzy rule-based classification system for the modeling and prediction of real-world financial applications with imbalanced data, Fuzzy Systems, IEEE Transactions on 23 (4) (2015) 973–990.
7. E. Barrenechea, H. Bustince, J. Fernandez, D. Paternain, J. A. Sanz, Using the Choquet integral in the fuzzy reasoning method of fuzzy rule-based classification systems, Axioms 2 (2) (2013) 208–223.
8. G. Choquet, Theory of capacities, Annales de l'Institut Fourier 5 (1953–1954) 131–295.
9. L. J. Eshelman, The CHC adaptive search algorithm: How to have safe search when engaging in nontraditional genetic recombination, in: G. J. E. Rawlings (Ed.), Foundations of Genetic Algorithms, Morgan Kaufmann, San Francisco, 1991, pp. 265–283.

10. J. Alcalá-Fdez, L. Sánchez, S. García, M. Jesus, S. Ventura, J. Garrell, J. Otero, C. Romero, J. Bacardit, V. Rivas, J. Fernández, F. Herrera, Keel: a software tool to assess evolutionary algorithms for data mining problems, Soft Computing 13 (3) (2009) 307–318.
11. F. Wilcoxon, Individual comparisons by ranking methods, Biometrics 1 (1945) 80–83.
12. R. Mesiar, A. Kolesárová, H. Bustince, G. P. Dimuro, B. C. Bedregal, Fusion functions based discrete Choquet-like integrals, European Journal of Operational Research 252 (2) (2016) 601 – 609.
13. G. P. Dimuro, B. Bedregal, H. Bustince, J. Fernandez, G. Lucca, R. Mesiar, New results on pre-aggregation functions(submitted).
14. E. P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publisher, Dordrecht, 2000.
15. D. Gómez, J. T. Rodríguez, J. Montero, H. Bustince, E. Barrenechea, n-dimensional overlap functions, Fuzzy Sets and Systems 287 (2016) 57 – 75.
16. G. Lucca, J. A. Sanz, G. P. Dimuro, B. Bedregal, J. Fernandez, H. Bustince, Cc-integrals: Choquet-like copula-based aggregation functions and their application to tackle classification problems(submitted).
17. T. Murofushi, M. Sugeno, M. Machida, Non-monotonic fuzzy measures and the Choquet integral, Fuzzy Sets and Systems 64 (1) (1994) 73 – 86.
18. G. Lucca, G. P. Dimuro, V. Mattos, B. Bedregal, H. Bustince, J. A. Sanz, A family of Choquet-based non-associative aggregation functions for application in fuzzy rule-based classification systems, in: 2015 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), IEEE, Los Alamitos, 2015, pp. 1–8.
19. J. Alcala-Fdez, R. Alcala, F. Herrera, A fuzzy association rule-based classification model for high-dimensional problems with genetic rule selection and lateral tuning, IEEE Transactions on Fuzzy Systems 19 (5) (2011) 857–872.
20. G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, Springer, Berlin, 2007.
21. H. Ishibuchi, T. Nakashima, Effect of rule weights in fuzzy rule-based classification systems, Fuzzy Systems, IEEE Transactions on 9 (4) (2001) 506–515.
22. J. Sanz, A. Fernández, H. Bustince, F. Herrera, IVTURS: A linguistic fuzzy rule-based classification system based on a new interval-valued fuzzy reasoning method with tuning and rule selection, IEEE Transactions on Fuzzy Systems 21 (3) (2013) 399–411.