Adaptive Median and Wiener Filters as Reference Functions for Morphological Associative Memories in Complete Inf-Semilattices

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Abstract. Mathematical morphology (MM) is a theory for nonlinear image and signal processing that was originally based on complete lattices and is usually still conducted in this framework. Later, MM was extended from complete lattices to complete inf-semilattices (cisls) using reference functions. Recently an auto-associative memory model based on a cisl was introduced by Sussner and Medeiros who conducted experiments concerning gray-scale image restoration using the median filter as a reference function. The adaptive median and Wiener filters often exhibit a better performance regarding noise reduction of corrupted images. We employ these filters as reference functions of auto-associative memories based on cisls in this paper. In experiments regarding the recall of noisy gray-scale images, our approach outperformed both the aforementioned image filters as well as a number of associative memory models including the cisl-based one that uses the median filter as a reference function.

Keywords: Mathematical morphology, complete inf-semilattice, reference element, auto-associative memory, image filter, image restoration.

1 Introduction

Mathematical morphology was initiated in the late 1960s by G. Matheron and J. Serra, as the part of binary image processing that is concerned with image filtering and geometric analysis by means of structuring elements [1]. Afterwards, MM was extended to gray scale images using the umbra approach [2]. More recent approaches to MM include fuzzy MM [3–5] and $\mathbb{L}$-fuzzy MM [9]. Note that, in all approaches towards MM, the class of images represents a partially ordered set and in most cases including the ones mentioned before a complete lattice [16] on which morphological operators are defined [6–8].

The effect of a morphological operator is determined by the specific partial ordering on the underlying image space, the choice of what is foreground and what is background. This choice, which is never made explicit and for that reason usually goes unnoticed, causes morphological operators to usually come in dual pairs, for example dilation/erosion, opening/closing [6] etc. In contrast, an operator ψ is called self-dual when $\psi(f^*) = (\psi(f))^*$ for any input image f. Here, f^* denotes the dual image of the

image f. In many applications such as image filtering or image denoising, self-duality is a desirable property.
An algebraic approach towards self-dual mathematical morphology that is based on complete inf-semilattices was developed by Keshet and Heijmans [8, 10]. Specifically, by using self-dual partial orderings the image space becomes a complete inf-semilattice on which self-dual erosion operators can be defined that have many interesting properties and promising applications in nonlinear image analysis. A complete inf-semilattice is a set in which every arbitrary subset has an infimum (but not necessarily a supremum exist). A cisl can be derived from a conditionally complete lattice-ordered group $\mathbb{F}$ by defining a partial order $\leq_r$ that depends on an arbitrary reference element r of $\mathbb{F}$. The resulting cisl, denoted $\mathbb{F}_r$, has r as its least element. Given an arbitrary element x of a complete lattice-ordered group, a reference element r arises as the value $\rho(x)$ of a so called reference function.

In a previous paper [14], the median filter was selected as a reference function for an auto-associative morphological memory based on a cisl that we shall call semi-lattice associative memory (SLAM) in this paper. The choice of the reference function that is employed in a SLAM is an open research problem. In this paper, the adaptive median and Wiener filters are used as reference functions of auto-associative memory models based on cisls in the experimental section on gray-scale images retrieval since the adaptive median filter and the Wiener filter are respectively suitable for reducing the amount of salt and pepper and Gaussian noise in corrupted gray-scale images [22, 24]. The simulations conducted in this paper not only confirm the latter observation but also reveal that the SLAM models using the adaptive median and Wiener filters exhibit a better image restoration performance in terms of the normalized mean squared error than the respective filters alone. In addition, we compared the results obtained by the approach proposed in this paper with the ones produced by a number of fuzzy and neural AM models from the literature [25, 26, 29].

2 Some Mathematical Background

A partially ordered set or poset is a set in which a reflexive, antisymmetric, and transitive binary relation " $\leq$ " is defined. For simplicity, we assume that a partially ordered set is non-empty [15]. If we additionally have either $x \leq y$ or $y \leq x$ in a partially ordered set P, then P is said to be totally ordered and is called chain. An operator $\psi : P \to P$ is said to be increasing if $x \leq y$ implies that $\psi(x) \leq \psi(y)$. An operator $\psi : P \to P$ is called extensive (anti-extensive) if $x \leq \psi(x)$ ($\psi(x) \leq x$) $\forall\, x \in P$. Finally, ψ is idempotent if $\psi^2 = \psi$.
A partially ordered set $\mathbb{L}$ is called a *lattice* if every finite, non-empty subset of $\mathbb{L}$ has an infimum and a supremum in $\mathbb{L}$ [16]. In particular, every totally ordered set or *chain* such as $\mathbb{R}$ and $\mathbb{Z}$ is a lattice. For any $X \subseteq \mathbb{L}$, we denote the infimum of X using the symbol $\bigwedge X$ and the supremum of X using the symbol $\bigvee X$. If $X = \{x_j \in \mathbb{L} : j \in J\}$ for some index set J, then we write $\bigwedge_{j \in J} x_j$ instead of $\bigwedge X$ and $\bigvee_{j \in J} x_j$ instead of $\bigvee X$. A lattice $\mathbb{L}$ is *complete* if every subset of $\mathbb{L}$ has an infimum and a supremum in $\mathbb{L}$. A lattice $\mathbb{L}$ is called *conditionally complete* if every bounded subset of $\mathbb{L}$ has an infimum and a supremum in $\mathbb{L}$. In particular, the *set of finite elements* of a complete

lattice $\mathbb{L}$, i.e., $\mathbb{L}\backslash\{\bigwedge\mathbb{L}, \bigvee\mathbb{L}\}$, is conditionally complete. If every finite, non-empty subset of a partially ordered set $\mathbb{L}$ has an infimum in $\mathbb{L}$, then $\mathbb{L}$ constitutes an *inf-semilattice*. If every subset of $\mathbb{L}$ has an infimum in $\mathbb{L}$ then $\mathbb{L}$ is called a *complete inf-semilattice* or *cisl*. If $\mathbb{L}$ is a (complete) lattice or an inf-semilattice then the direct product $\mathbb{L}^n$ is also a (complete) lattice or an inf-semilattice, respectively. These ideas can be further extended by considering $\mathbb{L}^X$, the class of functions from a set $X \neq \emptyset$ to $\mathbb{L}$. A function $\phi : \mathbb{L} \to \mathbb{M}$, where $\mathbb{L}$ and $\mathbb{M}$ are cisls, is called a *cisl homomorphism* or (algebraic) *erosion* if we have the following equation for all index sets J and all $x_j \in \mathbb{L}$ (the concept of (algebraic) *dilation* is defined in a similar way):

$$\phi(\bigwedge_{j\in\mathbb{L}} x_j) = \bigwedge_{j\in\mathbb{L}} \phi(x_j)\,. \tag{1}$$

The chains $\mathbb{R}$ and $\mathbb{Z}$ also constitute examples of *lattice-ordered groups*, for short l-groups i.e lattices that also form a group in which every group translation $x \to a + x + b$ is isotone [16]. Note that in this paper the group operation is denoted using the symbol "+" for addition.

If $\mathbb{G}$ is a complete lattice whose set of finite element forms a group with isotone group translation then we refer to $\mathbb{G}$ as a *complete l-group extension* [7]. Of course, a *conditionally complete lattice* $\mathbb{F}$ can form a group at the same time and, in this case $\mathbb{F}$ is simply called a conditionally complete l-group [16]. For example, the l-groups $\mathbb{R}$ and $\mathbb{Z}$ are conditionally complete.

From now on, let $\mathbb{F}$ stand for an arbitrary conditionally complete l-group. Note that $\mathbb{F}$ induces conditionally complete l-groups $\mathbb{F}^n$, $\mathbb{F}^{m\times n}$, and $\mathbb{F}^X$. Given a matrix $A \in \mathbb{F}^{m\times p}$ and a matrix $B \in \mathbb{F}^{p\times n}$, the matrix $C = A \boxdot\!\!\vee B$, called the max-product of A and B, and the matrix $D = A \boxdot\!\!\wedge B$, called the min-product of A and B are defined by the following equation for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$:

$$c_{ij} = \bigvee_{\xi=1}^{k} (a_{i\xi} + b_{\xi j}), \quad d_{ij} = \bigwedge_{\xi=1}^{k} (a_{i\xi} + b_{\xi j}). \tag{2}$$

The cone $\mathbb{F}^+$ is defined as $\{x \in \mathbb{F} : 0 \le x\}$, where 0 denotes the neutral element with respect to the group operation of addition. Note that $(\mathbb{F}^+, \le)$ represents a cisl. The positive part x^+ and the negative part x^- of an element x of $\mathbb{F}$ are respectively given by $x^+ = x \vee 0$ and $x^- = -x \vee 0$, where 0 denotes the neutral element of the group $\mathbb{F}$. Every $x \in \mathbb{F}$ can be written as $x = x^+ - x^-$. The element x^+ and x^- of the cone $\mathbb{F}^+$ are said to be disjoint because $x^+ \wedge x^- = 0$. Defining the following partial order $\preceq$ on $\mathbb{F}$ turns $\mathbb{F}$ into a cisl [6].

Proposition 1. *Consider the binary relation $\preceq_0$ on $\mathbb{F}$ that is defined as follows:*

$$x \preceq_0 y \Leftrightarrow x^+ \le y^+ \text{ and } x^- \le y^- \tag{3}$$

We have that $(\mathbb{F}, \preceq_0)$ is a cisl whose least element is 0. The infimum of an arbitrary subset $\{x_i : i \in I\}$ of $\mathbb{F}$ is given by

$$\bigcurlywedge_{i\in I} x_i = \bigwedge_{i\in I} (x_i)^+ - \bigwedge_{i\in I} (x_i)^-. \tag{4}$$

In particular the infimum operation in the cisl $(\mathbb{F}, \preceq_0)$ *satisfies*

$$(\curlywedge_{i \in I} x_i)^+ = \bigwedge_{i \in I} (x_i)^+ \; and \; (\curlywedge_{i \in I} x_i)^- = \bigwedge_{i \in I} (x_i)^- \tag{5}$$

The cisl $(\mathbb{F}, \preceq_0)$ is also denoted using the symbol $\mathbb{F}_0$. The neutral element of addition 0 plays an important role in $\mathbb{F}_0$ whose construction is based on the fact that 0 represents a reference element of the lattice $(\mathbb{F}, \leq)$. Recall that an arbitrary element r of a lattice $\mathbb{L}$ is called a reference element if the following statement is satisfied for all $x, y \in \mathbb{L}$:

$$x \wedge r = y \wedge r \; and \; x \vee r = y \vee r \Leftrightarrow x = y. \tag{6}$$

If $\mathbb{F}$ is conditionally complete l-group then every $r \in \mathbb{F}$ is reference element of the lattice $(\mathbb{F}, \leq)$ and a cisl arises via the following definition of "$\preceq_r$" which constitutes a partial order on $\mathbb{F}$. The resulting cisl $(\mathbb{F}, \preceq_r)$ can be denoted using the symbol $\mathbb{F}_r$.

$$x \preceq_r y \Leftrightarrow x \vee r \leq y \vee r \; and \; y \wedge r \leq x \wedge r \tag{7}$$

For an arbitrary $X \subseteq \mathbb{F}$, the infimum of X in the cisl $\mathbb{F}_r$ is denoted using the symbol $\curlywedge_r X$. In the special case where $X = \{x_j \in \mathbb{L} : j \in J\}$ for some index set J, $\curlywedge_r X$ is also denoted

$$\curlywedge_{j \in J}{}_{\mathrm{r}} \, x_j. \tag{8}$$

Moreover, (z_r) denotes $z - r$ for all $z, r \in \mathbb{F}$. Note that $x \preceq_r y$ is equivalent to having both $(x_r)^+ \leq (y_r)^+$ and $(x_r)^- \leq (y_r)^-$. This observation leads to the following expression:

$$\curlywedge_{j \in J}{}_{\mathrm{r}} \, x_j = \bigwedge_{j \in J} (x_j)_r^+ - \bigwedge_{j \in J} (x_j)_r^- + r. \tag{9}$$

3 Semilattice Associative Memories

Associative memories (AMs) are designed to store a finite set of pattern associations $(\mathbf{x}^\xi, \mathbf{y}^\xi)$, where $\xi = 1, \ldots, k$, called set of fundamental memories [19]. Moreover, an AM should permit the retrieval of a desired output upon presentation of a possibly noisy or incomplete version of a input pattern.

In this paper, the focus is on auto-associative memories, i.e., the case where $\mathbf{y}^\xi = \mathbf{x}^\xi$ for all $\xi = 1, \ldots, k$. Furthermore, the patterns $\mathbf{x}^\xi$ are assumed to be in $\mathbb{F}^n$, where $\mathbb{F}$ is a conditionally complete l-group. Hence, the auto-associative memory described in this paper corresponds to a mapping $\mathcal{M} : \mathbb{F}^n \to \mathbb{F}^n$. Ideally, $\mathcal{M}$ exhibits perfect recall of the original pattern, that is, $\mathcal{M}(\mathbf{x}^\xi) = \mathbf{x}^\xi$ for all $\xi \in \mathcal{K}$ and some tolerance with respect to noise, that is, $\mathcal{M}\left(\widetilde{\mathbf{x}}^\xi\right) = \mathbf{x}^\xi$ for noisy or incomplete versions $\widetilde{\mathbf{x}}^\xi$ of $\mathbf{x}^\xi$.

The classical auto-associative morphological memories (AMMs) [20], [21], also referred to as lattice auto-associative memories, are defined in terms of the min- and max-products. Originally, defined as mappings $\mathbb{F}^n \to \mathbb{F}^n$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{Z}$, AMMs can also be viewed as mappings $\mathbb{G}^n \to \mathbb{G}^n$, where $\mathbb{G} = \mathbb{R} \cup \{-\infty, +\infty\}$ or $\mathbb{G} = \mathbb{Z} \cup \{-\infty, +\infty\}$ [27]. As observed in [14], AMMs can also be defined as follows for an arbitrary complete lattice-ordered group $\mathbb{F}$.

Let $X \in \mathbb{F}^{n\times k}$ be the matrix whose ξ-th column is $\mathbf{x}^\xi$ for $\xi = 1, \dots, k$. The AMM $\mathcal{M}_{XX}$ is the mapping $\mathbb{F}^n \to \mathbb{F}^n$ determined by the equation

$$\mathcal{M}_{XX}(\mathbf{x}) = M_{XX} \boxwedge \mathbf{x} \;\forall \mathbf{x} \in \mathbb{F}^n, \tag{10}$$

where the synaptic weight matrix M_{XX} is given by $M_{XX} = X \boxvee X^*$. The dual AMM model $\mathcal{W}_{XX} : \mathbb{F}^n \to \mathbb{F}^n$ is determined by the equation

$$\mathcal{W}_{XX}(\mathbf{x}) = W_{XX} \boxvee \mathbf{x} \;\forall \mathbf{x} \in \mathbb{F}^n, \tag{11}$$

where the synaptic weight matrix W_{XX} is given by $W_{XX} = X \boxwedge X^*$. If $\mathbb{G}$ denotes the completion of $\mathbb{F}$, then $\mathbb{G}$ represents a complete l-group extension and Equations 10 and 11 can also be applied to $\mathbf{x} \in \mathbb{G}^n$. As shown in [7], the respective extended mappings $\mathbb{G}^n \to \mathbb{G}^n$ represent elementary operations of MM in complete lattices - in this case from the complete lattice $\mathbb{G}^n$ to the complete lattice $\mathbb{G}^n$ - and this is the reason why the AMs $\mathcal{M}_{XX}$ and $\mathcal{W}_{XX}$ are called "morphological".

Recently, Sussner and Medeiros introduced a AMM model in complete semilattices. Let us briefly review this model. As before, we consider patterns $\mathbf{x}^1, \dots, \mathbf{x}^k \in \mathbb{F}^n$. Let $\rho : \mathbb{F}^n \to \mathbb{F}^n$ be an arbitrary function. Given an arbitrary element $\mathbf{x}$ of $\mathbb{F}^n$, $\rho(\mathbf{x})$ will play the role of a reference element and therefore we may refer to ρ as a "reference function". Let $X_\rho^\pm \in \mathbb{F}^{n\times 2k}$ be the matrix whose ξth column is $(\mathbf{x}^\xi - \rho(\mathbf{x}^\xi))^+$ and $(\xi + k)$th column is given by $(\mathbf{x}^\xi - \rho(\mathbf{x}^\xi))^-$ for all $\xi = 1, \dots, k$. In addition, let M_{XX}^ρ denotes the matrix $M_{X_\rho^\pm X_\rho^\pm} \in \mathbb{F}^{n\times n}$. The following equation yields an auto-associative memory $\mathcal{M}_\rho : \mathbb{F}^n \to \mathbb{F}^n$ [14]:

$$\mathcal{M}_\rho(\mathbf{x}) = M_{XX}^\rho \boxwedge (\mathbf{x} - \rho(\mathbf{x}))^+ - M_{XX}^\rho \boxwedge (\mathbf{x} - \rho(\mathbf{x}))^- + \rho(\mathbf{x}) \,\forall\, \mathbf{x} \in \mathbb{F}^n. \tag{12}$$

Theorem 1. *Let $\mathcal{M}_\rho : \mathbb{F}^n \to \mathbb{F}^n$ be defined as in Equation* 12*. The function $\mathcal{M}_\rho$ represents an associative memory model, that is guaranteed to yield perfect recall for an arbitrary set of patterns $\left\{\mathbf{x}^1, \dots, \mathbf{x}^k\right\} \subset \mathbb{F}^n$ for an arbitrary number of patterns $k \in \mathbb{N}$. Formally we have:*

$$\mathcal{M}_\rho(\mathbf{x}^\xi) = \mathbf{x}^\xi \,\forall\, \xi = 1, \dots, k\,, \tag{13}$$

For an arbitrary input pattern $\mathbf{x} \in \mathbb{F}^n$, the output pattern $\mathcal{M}_\rho(\mathbf{x}) \in \mathbb{F}^n$ satisfies

$$\rho(\mathbf{x}) \preceq_{\rho(\mathbf{x})} \mathcal{M}_\rho(\mathbf{x}) \preceq_{\rho(\mathbf{x})} \mathbf{x}. \tag{14}$$

If $\rho(\mathbf{x}) = \mathbf{r} \in \mathbb{F}^n$ for all $\mathbf{x} \in \mathbb{F}^n$, then $\mathcal{M}_\rho$ represents an erosion from the cisl $\mathbb{F}_0^n$ to the cisl $\mathbb{F}_\rho^n$.

4 Simulations in Gray-Scale Images Reconstruction

In this section we perform some experiments using the ten images that are displayed in 1. These images have size 64×64 and 256 gray levels. In the simulations concerning FAM models, we converted these images into ten fuzzy images by normalizing the respective pixel values within the range $[0, 1]$. For each of these images, we generated a vector $\mathbf{x}^\xi$ of length n = 4096.

Recall that $\mathcal{M}_\rho$ depends on the choice of reference vectors which can be accomplished by means of a reference function $\rho : \mathbb{F}^n \to \mathbb{F}^n$. Here, $\mathbb{F}$ can be assumed to be $\mathbb{Z}$ or $\mathbb{R}$. In a previous papers, the median filter was chosen as a reference function.

In this paper, the reference functions are chosen to be the adaptive median (AMF) and Wiener filters [22, 24]. The adaptive median and Wiener filters have outperformed than median filter with respect to image noise reduction in a number of simulations.

Using a range of window sizes varying between an initial size 3×3 and a maximal window size, the adaptive median filter (AMF) represents an iterative procedure in which some pixel locations are flagged as noisy and in which the values of these presumably noisy pixels are replaced by the ones obtained from the median filter while other pixels are left unaltered. The AMF ensures that most of the pixels that are corrupted by impulsive noise, in particular salt and pepper noise, are detected even at a high noise level provided that the maximal window size is large enough.

Wiener filtering (after N. Wiener, who first proposed the method in 1942) is one of the earliest and best approaches to linear image restoration [24]. The Wiener filter is optimal in terms of the mean square error. In other words, let $x(n)$ is a wide sense stationary (WSS) random process. Suppose that we want to determine the unit sample response or frequency response of the LTI (linear time invariance) system such that the filter output $\hat{f}(n)$ is the minimum-mean square error (MMSE) estimate of some "target" process $f(n)$ that is jointly WSS with $x(n)$ by carrying out the following minimization:

$$\hat{h}(n) = argmin_{f(n)} E\left[|e(n)|^2\right] = argmin_{f(n)} E\left[\left|f(n) - \hat{f}(n)\right|^2\right]. \tag{15}$$

Here $E[\cdot]$ is the expectation operator and $e(n)$ is the estimation error. The resulting filter $\hat{h}(n)$ is called the Wiener filter for estimation of $f(n)$ from $x(n)$. This approach often produces better results than linear filtering. The adaptive filter is more selective than a comparable linear filter, preserving edges and other high frequency parts of an image. In addition, there are no design task; the Wiener function handles all preliminary computations and implements the filter for an input image, Wiener, however, does require more computation time than linear filtering. The Wiener function works best when the noise is constant power ("white") additive noise, such as Gaussian white noise.

Specifically, we employed $\alpha, \omega : \mathbb{R}^n \to \mathbb{R}^n$, where α and ω correspond respectively to the adaptive median with maximum window size 39×39 and the Wiener filters.

As expected, both $\mathcal{M}_\alpha$ and $\mathcal{M}_\omega$ succeeded in achieving perfect recall of the original patterns $\mathbf{x}^1, \ldots, \mathbf{x}^{10}$ in accordance with Theorem 1. In order to verify the tolerance of $\mathcal{M}_\alpha$ and $\mathcal{M}_\omega$ models with respect to noise, we corrupted $\mathbf{x}^1, \ldots, \mathbf{x}^{10}$ by introducing the following types of noise:

1. Salt and pepper noise with density 0.50;
2. Gaussian noise with mean 0 and variance 0.15;

We conducted this experiment 100 times for each type of noise and each original pattern $\mathbf{x}^\xi$, where $\xi = 1, \ldots, 10$. Fig. 2. provides a visual interpretation. The first column displays the original images. The second column displays the corrupted versions $\widetilde{\mathbf{x}}^1$, and

Fig. 1. Original gray-scale images that were used in constructing SLAMs Models.

$\widetilde{\mathbf{x}}^2$ of $\mathbf{x}^1$, and $\mathbf{x}^2$, respectively. The third column shows the images retrieved by the filters α (top) and ω (bottom). The last image in the top row shows $\mathcal{M}_\alpha(\widetilde{\mathbf{x}}^1)$ and the last image in the bottom row shows $\mathcal{M}_\omega(\widetilde{\mathbf{x}}^2)$. A close visual inspection reveals that $\mathcal{M}_\alpha(\widetilde{\mathbf{x}}^1)$ and $\mathcal{M}_\omega(\widetilde{\mathbf{x}}^2)$ are slightly more similar to $\mathbf{x}^1$ and $\mathbf{x}^2$ than $\alpha(\widetilde{\mathbf{x}}^1)$ and $\omega(\widetilde{\mathbf{x}}^2)$.

Table 1 lists the *normalized mean square errors* (NMSEs) produced by the adaptive median and Wiener filters as well as the SLAM models $\mathcal{M}_\alpha$ and $\mathcal{M}_\omega$ for each type of noise. Note that, for each type of noise, $\mathcal{M}_\alpha$ and $\mathcal{M}_\omega$ outperformed the filters α and ω, respectively, in terms of the NMSE. The associative memories $\mathcal{M}_\alpha$ and $\mathcal{M}_\omega$ yielded the best results in this experiment with respect to reducing the amounts of salt and pepper noise and Gaussian noise, respectively.

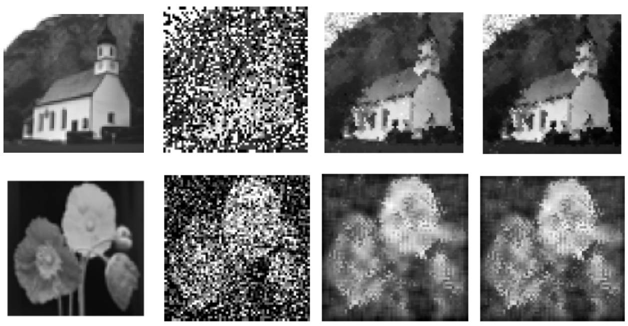

Fig. 2. The first row depicts the original image $\mathbf{x}^1$, a noisy image $\widetilde{\mathbf{x}}^1$, i.e., $\mathbf{x}^1$ corrupted by salt and pepper noise, $\alpha(\widetilde{\mathbf{x}}^1)$ and $\mathcal{M}_\alpha(\widetilde{\mathbf{x}}^1)$, i.e the outputs produced by the adaptive median filter and by $\mathcal{M}_\alpha$. The second row depicts the original image $\mathbf{x}^2$, a noisy image $\widetilde{\mathbf{x}}^2$, i.e., $\mathbf{x}^2$ corrupted by Gaussian noise, The remaining images in the second rows, correspond to the outputs produced by the Wiener filter ω and $\mathcal{M}_\omega$.

NMSE (Comparison with Original Images)					
Noise	$\widetilde{\mathbf{x}}$	$\alpha(\widetilde{\mathbf{x}})$	$\mathcal{M}_\alpha(\widetilde{\mathbf{x}})$	$\omega(\widetilde{\mathbf{x}})$	$\mathcal{M}_\omega(\widetilde{\mathbf{x}})$
Salt and Pepper	0.7535	0.1408	**0.1359**	0.3371	0.3330
Gaussian	0.5786	0.4268	0.4241	0.2544	**0.2529**

Table 1. NMSEs of the corrupted images as well as the errors produced by the adaptive median and Wiener filters the corresponding memory models.

Furthermore, we performed experiments using the morphological associative memory $W_{XX}+\nu$) [27] and the optimal linear associative memory (OLAM) proposed by Kohonen and Ruohonen [32]. The synaptic weight matrix W of the OLAM model is given by $W = XX^+$, where X^+ denotes the pseudo-inverse of the matrix $X = [\mathbf{x}^1 \ldots \mathbf{x}^k] \in \mathbb{R}^{n\times k}$ such that $X^+X = I$, which means that $\mathbf{x}^\xi$ for $1 \leq \xi \leq k$ are linear independent vectors. In addition, we compared the results produced by $\mathcal{M}_\alpha(\widetilde{\mathbf{x}}^1)$ and $\mathcal{M}_\omega(\widetilde{\mathbf{x}}^1)$ with the ones produced by other distributed associative memories, namely Kosko's max-min [33] and Junbo's fuzzy autoassociative memories (FAMs) [34] as well as the complex-valued Hopfield net of Tanaka et al. [13].

The outcome of this experiment is visualized in Fig. 3. Table 2 lists the resulting mean NMSEs produced by $\mathcal{M}_\alpha$, $\mathcal{M}_\omega$, $\mathcal{M}_\rho$ (where ρ denotes the median filter), the MAM $W_{XX} + \nu$ [14], the OLAM, the complex-valued Hopfield net, and the aforementioned FAM models. Table 2 reveals that the SLAM model $\mathcal{M}_\alpha$, that uses the AMF as a reference function, exhibits the highest tolerance with respect to salt and pepper noise. The OLAM, followed by the SLAM model $\mathcal{M}_\omega$, achieved the highest error correction capability with respect to Gaussian noise of mean 0 and variance 0.15. Unlike the AMF, the Wiener filter does not include a noise detection phase. Therefore a SLAM model that uses an Gaussian noise filter having a noise detection phase [35] may be able to produce better results than $\mathcal{M}_\omega$. Finally, note that for images corrupted by an unknown type of noise, a universal filter [36, 37] may be used as a reference function of a SLAM model.

5 Conclusion

In this paper, we reviewed MM as well as associative memories complete semi-lattices (SLAMs). We argued that in applications concerning the recall of noisy images, the adaptive median and Wiener filters are more suited than the conventional median filter to serve as reference functions for SLAMs. We conducted a series of experiments concerning the restoration of gray-scale images that were corrupted by salt-and-pepper as well as Gaussian noise. In these experiments, the SLAM models based on the adaptive median and Wiener filters outperformed the SLAM model based on the median filter and the fuzzy and gray-scale morphological associative memories that we tested. Moreover, the SLAM models $\mathcal{M}_\alpha$ and $\mathcal{M}_\omega$ exhibited better results that the adaptive median and Wiener filters alone. In future research, we intend to make further steps towards solving the problem of choosing a reference function for a SLAM model.

Associative Memory	Salt & Pepper $d = 0.5$	Gaussian $\sigma^2 = 0.15$
$\mathcal{M}_\alpha$	**0.1359**	0.4241
$\mathcal{M}_\omega$	0.3330	0.2529
$\mathcal{M}_\rho$	0.3734	0.3201
$W_{XX} + \nu$	0.4537	0.4126
OLAM	0.2342	**0.1201**
Complex Hopfield	0.3252	0.6870
Kosko's FAM	0.8230	0.8228
Junbo's FAM	0.7466	0.6049

Table 2. NMSEs produced by associative memory models in applications to patterns that were corrupted by introducing salt and pepper or Gaussian noise.

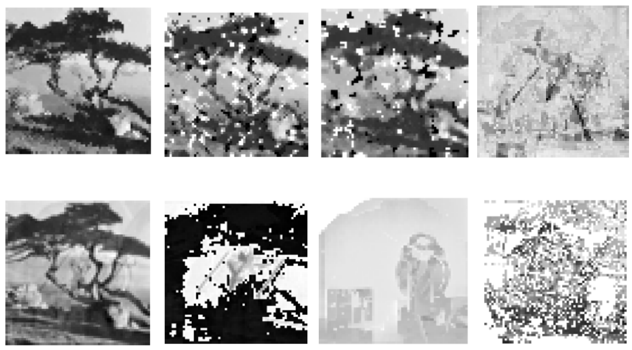

Fig. 3. Consider the tree image corrupted by salt and pepper noise with density 0.5. The images in first row show the outputs of $\mathcal{M}_\alpha$, $\mathcal{M}_\omega$, $\mathcal{M}_\rho$, and $W_{XX} + \nu$. The bottom row shows the outputs produced by the OLAM, the complex-valued Hopfield net, Kosko's FAM, and Junbo's FAM.

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