A theorem to construct fuzzy subsethood measures

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Abstract. Inspired in the mostly known axiomatizations of fuzzy subsethood measures, we proposed in a previous paper a new axiomatization for fuzzy subsethood measures in such a way that we could construct subsethood grades aggregating implication operators. We continue that study in this paper presenting the approach of constructing fuzzy subsethood measures by means of aggregating implication operators. We present a new theorem to construct them.

Keywords: Fuzzy subsethood measures; Implication operator; Aggregation functions

1 Introduction

The idea of measuring up to what extent a given fuzzy set is included into another was firstly given by Zadeh [34] and has led to various axiomatizations over the years. In general, these axiomatizations propose to indicate the degree of which a fuzzy set A is included in B, called an inclusion degree or a subsethood measure.

In the literature, fuzzy subsethood measures between fuzzy sets have been used in different applications, namely in: mathematical morphology [20,29], clustering [11,33], fuzzy relational databases [19], intelligent systems [22], fuzzy decision [18], image processing [5,10], formal concept lattice analysis [12], etc.

Such measures arise from the partial order relation given by Zadeh for fuzzy sets, i.e., given $A, B \in F(X)$ with $X = \{x_1, \cdots, x_n\}$: $A \subseteq B$ if and only if $A(x) \leq B(x)$ for all $x \in X$, where $F(X)$ is the set of all fuzzy subsets on a universe X.

Observe that if only a single $x \in X$ does not satisfy $A(x) \leq B(x)$, then $A \nsubseteq B$. Clearly, this definition is inherently crisp (given such a harsh condition) and has been criticized since its introduction. In 1980, Bandler and Kohout [3] took into account this fact and proposed the following expression to measure the subsethood grade of a set A in a set B: $\sigma(A, B) = \inf_{x \in X} \mathcal{J}(A(x), B(x))$, where $\mathcal{J} : [0,1]^2 \to [0,1]$ is such that $\mathcal{J}(0,0) = \mathcal{J}(0,1) = \mathcal{J}(1,1) = 1$ and $\mathcal{J}(1,0) = 0$.

Bandler and Kohout's proposal has led many authors to consider functions of the type $\sigma : F(X) \times F(X) \to [0,1]$ such that $\sigma(A, B)$ quantifies up to what extent a set

A is included in a set B. The conditions (axioms) which are requested to σ will depend on the application.

We have found at least six different axiomatizations in the literature. Historically, the most relevant ones are: Kitainik [16], Sinha and Dougherty [28], Young [33], Fan [11], Bustince [8] and Zhang [35].

In [26], we proposed a new axiomatization for fuzzy subsethood measures in such a way that we could construct subsethood grades aggregating implication operators. We continue that study and propose here a theorem to construct fuzzy subsethood measures on X from $\mathcal{A}$ fixed and functions I generated from functions of the type $[0,1]^2 \to \mathbb{R}^+$.

The structure of this paper is as follows. In the next section we present some basic concepts used along the text. Next we recall our axiomatization and then we present our contribution, i.e., the construction of subsethood measure by means of aggregating implications operators. We finish with some conclusions, acknowledgments and references.

2 Preliminaries

In this section we review some concepts and definitions that will be used throughout this paper. First, we should state that we only deal with finite referential set $X = \{x_1, \ldots, x_n\}$ and will denote real variables in the unit interval by x.

Given two fuzzy sets $A, B \in F(X)$ we write $A \leq B$ whenever $A(x) \leq B(x)$ for every $x \in X$.

Definition 1. *A fuzzy negation is a decreasing function $N : [0,1] \to [0,1]$ such that $N(0) = 1$ and $N(1) = 0$. A negation N is strong if $N(N(x)) = x$ for all $x \in [0,1]$.*

A function $\mathcal{A} : [0,1]^n \to [0,1]$ with $n \geq 2$ is an aggregation function if it is increasing and satisfies the boundary conditions $\mathcal{A}(0, \cdots, 0) = 0$ and $\mathcal{A}(1, \cdots, 1) = 1$ [4,15]. Nevertheless, in this paper we will follow [9] and consider a definition that is more restrictive, as follows:

Definition 2. *An (n-ary) aggregation function is a function $\mathcal{A} : [0,1]^n \to [0,1]$ such that:*

(A1) $\mathcal{A}(x_1, \ldots, x_n) = 0$ *if and only if* $x_1 = \cdots = x_n = 0$*;*
(A2) $\mathcal{A}(x_1, \ldots, x_n) = 1$ *if and only if* $x_1 = \cdots = x_n = 1$*;*
(A3) $\mathcal{A}$ *is increasing.*

An aggregation function $\mathcal{A}$ is symmetric if its output does not depend on the order in which the inputs are considered. That is, if $\mathcal{A}(x_1, \cdots, x_n) = \mathcal{A}(x_{p(1)}, \cdots, x_{p(n)})$ for every permutation p of $\{1, \cdots, n\}$. Finally, an aggregation function $\mathcal{A}$ is idempotent if $\mathcal{A}(x, \cdots, x) = x$ for all $x \in X$.

We now recall the notion of (fuzzy) implication function. An implication function (in the sense of Fodor and Roubens [13,14], see [1,2,6,21]) is a mapping $I\colon [0,1]^2 \to [0,1]$ such that, for every $x, y, z \in [0,1]$:

(I1) If $x \leq z$ then $I(x,y) \geq I(z,y)$;

(I2) If $y \leq z$ then $I(x, y) \leq I(x, z)$;
(I3) $I(0, x) = 1$;
(I4) $I(x, 1) = 1$;
(I5) $I(1, 0) = 0$.

Other properties can be demanded to implication functions, mostly depending on the application (see [2]). A non-exhaustive list includes the following

(I6) $I(1, x) = x$;
(I7) $I(x, I(y, z)) = I(y, I(x, z))$;
(I8) $I(x, y) = 1$ if and only if $x \leq y$;
(I9) $I(x, 0) = N(x)$ is a strong negation;
(I10) $I(x, y) \geq y$;
(I11) $I(x, y) = I(N(y), N(x))$ for a given strong negation N.

The relations that exist between all these properties have been studied in different works, for instance in [6,27].

3 On fuzzy subsethood measures

In [26], we presented the following definition for fuzzy subsethood measure.

Definition 3. *A function $\sigma : F(X) \times F(X) \to [0, 1]$ is called a fuzzy subsethood measure, if σ satisfies the following properties:*

(a) $\sigma(A, B) = 1$ if and only if $A \leq B$;
(b) $\sigma(A, B) = 0$ if and only if $A(x) = 1$ and $B(x) = 0$ for every $x \in X$;
(c) If $A \leq B$, then $\sigma(A, C) \geq \sigma(B, C)$ and $\sigma(C, A) \leq \sigma(C, B)$.

Example 1. 1. Goguen's subsethood degree,

$$\sigma_G(A, B) = \frac{1}{n} \sum_{i=1}^{n} \wedge(1, 1 - A(x_i) + B(x_i)) \tag{1}$$

2.

$$\sigma(A, B) = \frac{1}{n} \sum_{i=1}^{n} \begin{cases} 1 & \text{if } A(x_i) \leq B(x_i), \\ \dfrac{\vee(1 - A(x_i), B(x_i))}{2} & \text{if } A(x_i) > B(x_i). \end{cases}$$

3.

$$\sigma(A, B) = \begin{cases} 1 & \text{if } A \leq B, \\ 0 & \text{if } A = 1 \text{ and } B = 0, \\ \dfrac{1}{2} & \text{otherwise.} \end{cases}$$

In Figure 1 [26] it can be seen the relationships between our proposal of fuzzy subsethood measure (σ) as in Definition 3 and the subsethood measures defined according to Fan (σ_*), Young (σ_Y), Zhang (σ_Z) and Bustince (σ_{DI}).

Notice that despite our measure (σ) is the most restrictive one in the literature, we obtained a simpler axiomatization (by only demanding three axioms) without losing its

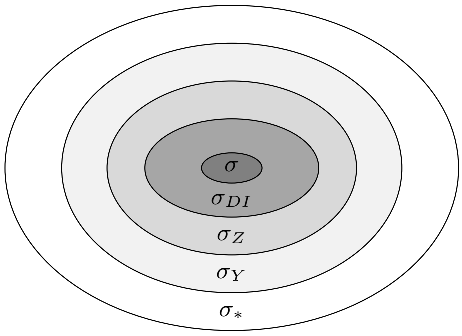

Fig. 1. Relationships between σ and other axiomatizations.

strength, as for instance, we have that Goguen's measure (Eq. 1) is a specific example of our proposal.

In [26] we also considered a method following the works by Sanchez [25], Bandler and Kohout [3], L. Kitainik [16,17], E. Ruspini [24] and Willmott [30,31,32], which consisted of building fuzzy subsethood measures by aggregating implication operators-like operators, as in the following equation: $\sigma(A, B) = \mathop{\mathcal{A}}_{i=1}^{n} (I(A(x_i), B(x_i)))$.

Proposition 1. *[26] Let* $\sigma : F(X) \times F(X) \to [0, 1]$ *be given by:* $\sigma(A, B) = \mathop{\mathcal{A}}_{i=1}^{n} (I(A(x_i), B(x_i)))$*, where* $\mathcal{A} : [0, 1]^n \to [0, 1]$ *is an aggregation function and* I *is a function of* $[0, 1]^2 \to [0, 1]$ *that satisfies* $I1$*,* $I2$*,* $I8$ *and Eq. (2) below.*

$$I(x, y) = 0 \text{ if and only if } x = 1 \text{ and } y = 0 \text{ (see [7]).} \tag{2}$$

Then σ *is a fuzzy subsethood measure on* X*.*

In the two following corollaries we show fuzzy subsethood measures constructed from functions $\mathcal{A}$ and I that satisfy the conditions of Proposition 1 and we use for $\mathcal{A}$, the constructions studied in [26].

Corollary 1. *Let* $g, h : [0, 1]^n \to [0, 1]$ *be such that*

$i)$ $g(x_1, \cdots, x_n) = 0$ *if and only if* $x_i = 0$ *for all* $i \in \{1, \cdots, n\}$*;*
$ii)$ $h(x_1, \cdots, x_n) = 0$ *if and only if* $x_i = 1$ *for all* $i \in \{1, \cdots, n\}$*;*
$iii)$ g *is non decreasing and* h *is non increasing;*
$iv)$ g *and* h *are symmetric.*

and let $I : [0, 1]^2 \to [0, 1]$ *be such that it satisfies* $I1$*,* $I2$*,* $I8$ *and Eq. (2).*
In these conditions

$$\sigma(A, B) = \frac{g(I(A(x_1), B(x_1)), \cdots, I(A(x_n), B(x_n)))}{g(I(A(x_1), B(x_1)), \cdots,) + h(I(A(x_1), B(x_1)), \cdots,)}$$

is a fuzzy subsethood measure on X*.*

Proof. By Proposition 1 in [9] we have: $\mathcal{A} : [1, 0]^n \to [0, 1]$ satisfies $A1 - A3$ if and only if $\mathcal{A}(x_1, \cdots, x_n) = \frac{g(x_1,\cdots,x_n)}{g(x_1,\cdots,x_n)+h(x_1,\cdots,x_n)}$, where $g, h : [0, 1]^n \to [0, 1]$ satisfy $i) - iv)$.

Taking into account that the expression of the statement is obtained by replacing $\mathcal{A}$ with the value above; that is,

$$\sigma(A, B) = \mathop{\mathcal{A}}_{i=1}^{n}(I(A(x_i), B(x_i))) = \frac{g(I(A(x_1), B(x_1)), \cdots, I(A(x_n), B(x_n)))}{g(I(A(x_1), B(x_1)), \cdots,) + h(I(A(x_1), B(x_1)), \cdots,)}$$

it results that $\mathcal{A}$ and I satisfy the conditions of Proposition 1. ■

Example 2. In the following examples we take $I(x, y) = \wedge(1, 1 - x + y)$ and the standard negation. In the first column we write the quotient $\frac{g}{h}$. In the second column we show the resulting fuzzy subsethood measure.

1)	$\dfrac{\frac{1}{n}\sum_{i=1}^{n} x_i}{\left(\frac{1}{n}\sum_{i=1}^{n}(1-x_i)^2\right)^{\frac{1}{2}}}$	$\dfrac{\frac{1}{n}\sum_{i=1}^{n} \wedge(1, 1 - A(x_i) + B(x_i))}{\frac{1}{n}\sum_{i=1}^{n} \wedge(1, 1 - A(x_i) + B(x_i)) + \left(\frac{1}{n}\sum_{i=1}^{n}(\vee(0, A(x_i) - B(x_i)))^2\right)^{\frac{1}{2}}}$
2)	$\dfrac{\lambda \bigwedge_{i=1}^{n} x_i + (1-\lambda)\bigvee_{i=1}^{n} x_i}{1 - (\lambda \bigwedge_{i=1}^{n} x_i + (1-\lambda)\bigvee_{i=1}^{n} x_i)}$ with $\lambda \in (0, 1)$	$\lambda \bigwedge_{i=1}^{n} \wedge(1, 1 - A(x_i) + B(x_i)) +$ $(1-\lambda)\bigvee_{i=1}^{n} \wedge(1, 1 - A(x_i) + B(x_i))$

Corollary 2. *In the same conditions as in the corollary above, if N is a strong negation, $g(x_1, \cdots, x_n) \geq h(N(x_1), \cdots, N(x_n))$ and I satisfies $I10$, then*

$$\sigma(A, B) = \frac{g(I(A(x_1), B(x_1)), \cdots, I(A(x_n), B(x_n)))}{g(I(A(x_1), B(x_1)), \cdots,) + h(I(A(x_1), B(x_1)), \cdots,)}$$

is a fuzzy subsethood measure on X that satisfies Axiom 10 [1] *of Sinha and Dougherty [28].*

Proof. Bearing in mind Lemma 1 in [9]; that is, $\mathcal{A} : [0, 1]^n \to [0, 1]$ satisfies $A1 - A3$ and $\mathcal{A}(x_1, \cdots, x_n) \geq 1 - \mathcal{A}(N(x_1), \cdots, N(x_n))$ if and only if

$$\mathcal{A}(x_1, \cdots, x_n) = \frac{g(x_1, \cdots, x_n)}{g(x_1, \cdots, x_n) + h(x_1, \cdots, x_n)}$$

where $g, h : [0, 1]^n \to [0, 1]$ satisfy $i) - iv)$ and also satisfy:

$v)$ $g(x_1, \cdots, x_n) \geq h(N(x_1), \cdots, N(x_n))$.

[1] Sinha and Dougherty considered twelve axioms, among which Axioms 6 is $(SD6)$ $\sigma_{SD}(A, B) = \sigma_{SD}(B_N, A_N)$ and Axiom 10 is $(SD10)$ $\sigma_{SD}(A, B) + \sigma_{SD}(A_N, B_N) \geq 1$.

Taking into account that $\mathcal{A}$ in these conditions satisfies $A1 - A3$ and the conditions of the item i) of Corollary 1 and I satisfies $I10$, then by the same Corollary 1 we have that

$$\sigma(A,B) = \underset{i=1}{\overset{n}{\mathcal{A}}}(I(A(x_i),B(x_i))) = \frac{g(I(A(x_1),B(x_1)),\cdots,I(A(x_n),B(x_n)))}{g(I(A(x_1),B(x_1)),\cdots,)+h(I(A(x_1),B(x_1)),\cdots,)}$$

is a fuzzy subsethood measure on X that satisfies Axiom 10. ∎

Example 3. In the following examples we take $I(x,y) = \wedge(1, 1-x+y)$ and the standard negation. In the first column we write the quotient $\frac{g}{h}$. In the second column we show the resulting fuzzy subsethood measure.

1)	$\dfrac{\bigvee_{i=1}^{n} x_i}{\bigvee_{i=1}^{n} (1-x_i)}$	$\dfrac{\bigvee_{i=1}^{n} \wedge(1, 1-A(x_i)+B(x_i))}{\bigvee_{i=1}^{n} \wedge(1, 1-A(x_i)+B(x_i)) + \bigvee_{i=1}^{n} \vee(0, A(x_i)-B(x_i))}$
2)	$\dfrac{\bigvee_{i=1}^{n} x_i}{\left(\frac{1}{n}\sum_{i=1}^{n}(1-x_i)^2\right)^{\frac{1}{2}}}$	$\dfrac{\bigvee_{i=1}^{n} \wedge(1, 1-A(x_i)+B(x_i))}{\bigvee_{i=1}^{n} \wedge(1, 1-A(x_i)+B(x_i)) + \left(\frac{1}{n}\sum_{i=1}^{n}(\vee(0, A(x_i)-B(x_i)))^2\right)^{\frac{1}{2}}}$

3.1 Construction of fuzzy subsethood measures on X from $\mathcal{A}$ fixed and functions I generated from functions of $[0,1]^2$ in $\mathbb{R}^+$

In [26] we analyzed some properties that the fuzzy subsethood measure may have depending on the aggregation functions considered. The theorem below is one of them,

Theorem 1. *Let us consider $\sigma : F(X) \times F(X) \to [0,1]$, given by: $\sigma(A,B) = \underset{i=1}{\overset{n}{\mathcal{A}}}(I(A(x_i),B(x_i)))$, for all $A, B \in F(X)$, where $\mathcal{A} : [0,1]^n \to [0,1]$ is an idempotent aggregation function and I is a function $I : [0,1]^2 \to [0,1]$. Then, σ is a fuzzy subsethood measure on X if and only if I satisfies $I1$, $I2$, $I8$ and Eq.(2)*

Proof. See [26] ∎

Our main contribution in this work is presented in the following theorem where we show the way of constructing fuzzy subsethood measures using functions $\mathcal{A}$, functions of the type $[0,1]^2 \to [0,1]$ and functions of $[0,1]^2 \to \mathbb{R}^+$.

Theorem 2. *Let N be a strong negation and let $\sigma : F(X) \times F(X) \to [0,1]$, be given by: $\sigma(A,B) = \underset{i=1}{\overset{n}{\mathcal{A}}}(I(A(x_i),B(x_i)))$ for all $A, B \in F(X)$, where $\mathcal{A} : [0,1]^n \to [0,1]$ is a function that satisfies A1, A2, A3, is idempotent such that $\mathcal{A}(x_1,\cdots,x_n) \geq \frac{1}{n}\sum_{i=1}^{n} x_i$ and I is a function of the type $[0,1]^2 \to [0,1]$. In these conditions the following items are equivalent:*

$i)$ *σ is a fuzzy subsethood measure on X that satisfies Sinha and Dougherty [28] axioms $SD6$ and $SD10$.*

$ii)$ *I satisfies $I1$, $I8$, $I11$ and*

$$\begin{cases} I(x,y)=0 \text{ if and only if } x=1 \text{ and } y=0 \\ I(x,y)+I(x,N(y))\geq 1. \end{cases}$$

$iii)$ $\sigma(A,B)=\overset{n}{\underset{i=1}{\mathcal{A}}}\left(\frac{G(N(A(x_i)),B(x_i))}{G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i))}\right)$,

where the functions $G:[0,1]^2\to[0,1]$ and $H:[0,1]^2\to\mathbb{R}^+$ are such that:

(a') $G(x,y)\geq H(x,N(y))$ for all $x,y\in[0,1]$;

(b') $G(x,y)=0$ if and only if $x=y=0$;

(c') $H(x,y)=0$ if and only if $x\geq N(y)$;

(d') G is non decreasing in both arguments and H is non increasing in both arguments;

(e') G and H are symmetric;

(f') $H(0,0)=1$.

Proof. i)⇒ii) Similar to the one done in the necessary condition of Theorem 1 and in the first part of the theorem above. Let us see that I satisfies $I(p,q)+I(p,N(q))\geq 1$ for all $p,q\in[0,1]$.

Let $p,q\in[0,1]$, let us take the sets $A=\{\langle x_i,A(x_i)=p\rangle\colon x_i\in X\}$ and $B=\{\langle x_i,B(x_i)=q\rangle\colon x_i\in X\}$.

Bearing in mind that $\mathcal{A}$ is idempotent and σ satisfies $SD10$ we have

$$I(p,q)+I(p,N(q))=\overset{n}{\underset{i=1}{\mathcal{A}}}(I(p,q))+\overset{n}{\underset{i=1}{\mathcal{A}}}(I(p,N(q)))=$$
$$\overset{n}{\underset{i=1}{\mathcal{A}}}(I(A(x_i),B(x_i)))+\overset{n}{\underset{i=1}{\mathcal{A}}}(I(A(x_i),N(B(x_i))))=\sigma(A,B)+\sigma(A,B_N)\geq 1.$$

ii)⇒iii) Taking into account that $\sigma(A,B)=\overset{n}{\underset{i=1}{\mathcal{A}}}(I(A(x_i),B(x_i)))$ we have

$$\sigma(A,B)=\overset{n}{\underset{i=1}{\mathcal{A}}}\left(\frac{G(N(A(x_i)),B(x_i))}{G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i))}\right)$$

where functions $G:[0,1]\to[0,1]$ and $H:[0,1]\to\mathbb{R}^+$ are such that (a'), (b'), (c'), (d'), (e') and (f') hold.

iii)⇒i) (a) If $\sigma(A,B)=1=\overset{n}{\underset{i=1}{\mathcal{A}}}\left(\frac{G(N(A(x_i)),B(x_i))}{G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i))}\right)$, then by $A2$ we have $\frac{G(N(A(x_i)),B(x_i))}{G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i))}=1$ for all $i\in\{1,\cdots,n\}$. That is,

$$G(N(A(x_i)),B(x_i))=G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i)),$$

then

$$H(N(A(x_i)),B(x_i))=0.$$

Taking into account that H satisfies (c') we have $N(A(x_i))\geq N(B(x_i))$; that is $A(x_i)\leq B(x_i)$.

If $A(x_i)\leq B(x_i)$ for all $i\in\{1,\cdots,n\}$, then $N(A(x_i))\geq N(B(x_i))$. By (c') we have $H(N(A(x_i)),B(x_i))=0$. Therefore

$$\frac{G(N(A(x_i)),B(x_i))}{G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i))}=1.$$

Bearing in mind that $\mathcal{A}$ satisfies $A2$ we have that

$$\sigma(A,B)=\mathop{\mathcal{A}}_{i=1}^{n}\left(\frac{G(N(A(x_i)),B(x_i))}{G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i))}\right)=\mathcal{A}(1,\cdots,1)=1.$$

(b) If $\sigma(A,B)=0=\mathop{\mathcal{A}}_{i=1}^{n}\left(\frac{G(N(A(x_i)),B(x_i))}{G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i))}\right)$.

By $A1$ we have that $\frac{G(N(A(x_i)),B(x_i))}{G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i))}=0$ for all $i\in\{1,\cdots,n\}$. Since

$$G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i))\neq 0,$$

then $G(N(A(x_i)),B(x_i))=0$. By (b') we have $N(A(x_i))=B(x_i)=0$; that is, $A(x_i)=1$ and $B(x_i)=0$ for all $i\in\{1,\cdots,n\}$.

If $A(x_i)=1$ and $B(x_i)=0$ for all $i\in\{1,\cdots,n\}$, then $N(A(x_i))=B(x_i)=0$, therefore by (b') $G(N(A(x_i)),B(x_i))=0$. Then

$$\frac{G(N(A(x_i)),B(x_i))}{G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i))}=0$$

for all $i\in\{1,\cdots,n\}$. Since $\mathcal{A}$ satisfies $A1$ we have

$$\sigma(A,B)=\mathop{\mathcal{A}}_{i=1}^{n}\left(\frac{G(N(A(x_i)),B(x_i))}{G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i))}\right)=\mathcal{A}(0,\cdots,0)=0.$$

(c) If $A\leq B$, then $A(x_i)\leq B(x_i)$; that is $N(A(x_i))\geq N(B(x_i))$ for all $i\in\{1,\cdots,n\}$. By (d') G is non decreasing in both arguments and H is no increasing in both arguments, therefore

$$G(N(A(x_i)),C(x_i))\geq G(N(B(x_i)),C(x_i)) \text{ and } H(N(B(x_i)),C(x_i))\geq H(N(A(x_i)),C(x_i)).$$

Therefore,

$G(N(A(x_i)),C(x_i))H(N(B(x_i)),C(x_i))\geq G(N(B(x_i)),C(x_i))H(N(A(x_i)),C(x_i))$; that is,

$$\begin{aligned}&G(N(A(x_i)),C(x_i))G(N(B(x_i)),C(x_i))+G(N(A(x_i)),C(x_i))H(N(B(x_i)),C(x_i))\geq\\&G(N(A(x_i)),C(x_i))G(N(B(x_i)),C(x_i))+G(N(B(x_i)),C(x_i))H(N(A(x_i)),C(x_i)).\end{aligned}$$

Then

$$\begin{aligned}&G(N(A(x_i)),C(x_i))[G(N(B(x_i)),C(x_i))+H(N(B(x_i)),C(x_i))]\geq\\&G(N(B(x_i)),C(x_i))[G(N(A(x_i)),C(x_i))+H(N(A(x_i)),C(x_i))]\end{aligned}$$

therefore,

$$\frac{G(N(A(x_i)),C(x_i))}{G(N(A(x_i)),C(x_i))+H(N(A(x_i)),C(x_i))}\geq\frac{G(N(B(x_i)),C(x_i))}{G(N(B(x_i)),C(x_i))+H(N(B(x_i)),C(x_i))}$$

that is, bearing in mind that $\mathcal{A}$ satisfies $A3$ we have that $\sigma(A,C)\geq\sigma(B,C)$.

If $A\leq B$, then $A(x_i)\leq B(x_i)$. As G and H fulfill (d') we have that

$$G(N(C(x_i)),A(x_i))\leq G(N(C(x_i)),B(x_i)) \text{ and } H(N(C(x_i)),B(x_i))\leq H(N(C(x_i)),A(x_i)).$$

Therefore,

$$G(N(C(x_i)),A(x_i))H(N(C(x_i)),B(x_i))\leq G(N(C(x_i)),B(x_i))H(N(C(x_i)),A(x_i));$$

that is,

$$G(N(C(x_i)), A(x_i))G(N(C(x_i)), B(x_i)) + G(N(C(x_i)), A(x_i))H(N(C(x_i)), B(x_i)) \leq$$
$$G(N(C(x_i)), A(x_i))G(N(C(x_i)), B(x_i)) + G(N(C(x_i)), B(x_i))H(N(C(x_i)), A(x_i)).$$

then

$$G(N(C(x_i)), A(x_i))[G(N(C(x_i)), B(x_i)) + H(N(C(x_i)), B(x_i))] \leq$$
$$G(N(C(x_i)), B(x_i))[G(N(C(x_i)), A(x_i)) + H(N(C(x_i)), A(x_i))]$$

therefore

$$\frac{G(N(C(x_i)), A(x_i))}{G(N(C(x_i)), A(x_i)) + H(N(C(x_i)), A(x_i))} \leq \frac{G(N(C(x_i)), B(x_i))}{G(N(C(x_i)), B(x_i)) + H(N(C(x_i)), B(x_i))};$$

that is $\sigma(C, A) \leq \sigma(C, B)$.

Since G and H fulfill (e') and N is a strong negation we have that

$$\sigma(B_N, A_N) = \overset{n}{\underset{i=1}{\mathcal{A}}} \left(\frac{G(B(x_i), N(A(x_i)))}{G(B(x_i), N(A(x_i))) + H(B(x_i), N(A(x_i)))} \right) =$$
$$\overset{n}{\underset{i=1}{\mathcal{A}}} \left(\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))} \right) = \sigma(A, B).$$

We know by (a') that $G(A(x_i), B(x_i)) \geq H(A(x_i), N(B(x_i)))$ for all $i \in \{1, \cdots, n\}$, then bearing in mind that N is involutive we have $G(N(A(x_i)), B(x_i)) \geq H(N(A(x_i)), N(B(x_i)))$ and $G(N(A(x_i)), N(B(x_i))) \geq H(N(A(x_i)), B(x_i))$, therefore

$$G(N(A(x_i)), B(x_i))G(N(A(x_i)), N(B(x_i))) \geq H(N(A(x_i)), B(x_i))H(N(A(x_i)), N(B(x_i))),$$

then

$$G(N(A(x_i)), B(x_i))G(N(A(x_i)), N(B(x_i))) + G(N(A(x_i)), B(x_i))H(N(A(x_i)), N(B(x_i))) \geq$$
$$G(N(A(x_i)), B(x_i))H(N(A(x_i)), N(B(x_i))) + H(N(A(x_i)), B(x_i))H(N(A(x_i)), N(B(x_i))),$$

that is:

$$G(N(A(x_i)), B(x_i))[G(N(A(x_i)), N(B(x_i))) + H(N(A(x_i)), N(B(x_i)))] \geq$$
$$H(N(A(x_i)), N(B(x_i)))[G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))].$$

By (b') and (f') we have that $G(N(A(x_i)), N(B(x_i))) + H(N(A(x_i)), N(B(x_i))) \neq 0$ and $G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i)) \neq 0$. Therefore

$$\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))} \geq$$
$$\frac{H(N(A(x_i)), N(B(x_i)))}{H(N(A(x_i)), N(B(x_i))) + G(N(A(x_i)), N(B(x_i)))} =$$
$$1 - \frac{G(N(A(x_i)), N(B(x_i)))}{G(N(A(x_i)), N(B(x_i))) + H(N(A(x_i)), N(B(x_i)))},$$

that is:

$$\frac{G(N(A(x_i)), B(x_i))}{G(N(A(x_i)), B(x_i)) + H(N(A(x_i)), B(x_i))} +$$
$$\frac{G(N(A(x_i)), N(B(x_i)))}{G(N(A(x_i)), N(B(x_i))) + H(N(A(x_i)), N(B(x_i)))} \geq 1.$$

Therefore, since $\mathcal{A}(x_1,\cdots,x_n) \geq \frac{1}{n}\sum\limits_{i=1}^{n} x_i$, we have

$$\begin{aligned}\sigma(A,B)+\sigma(A,B_N) = & \mathop{\mathcal{A}}_{i=1}^{n}\left(\frac{G(N(A(x_i)),B(x_i))}{G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i))}\right)+\\ & \mathop{\mathcal{A}}_{i=1}^{n}\left(\frac{G(N(A(x_i)),N(B(x_i)))}{G(N(A(x_i)),N(B(x_i)))+H(N(A(x_i)),N(B(x_i)))}\right)\geq\\ & \frac{1}{n}\sum_{i=1}^{n}\left(\frac{G(N(A(x_i)),B(x_i))}{G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i))}\right)+\\ & \frac{1}{n}\sum_{i=1}^{n}\left(\frac{G(N(A(x_i)),N(B(x_i)))}{G(N(A(x_i)),N(B(x_i)))+H(N(A(x_i)),N(B(x_i)))}\right)=\\ & \frac{1}{n}\sum_{i=1}^{n}\frac{G(N(A(x_i)),B(x_i))}{G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i))}+\\ & \frac{G(N(A(x_i)),N(B(x_i)))}{G(N(A(x_i)),N(B(x_i)))+H(N(A(x_i)),N(B(x_i)))}\geq 1. \quad \blacksquare\end{aligned}$$

Example 4. Let us take:

$$G(x,y)=\begin{cases}1 \;\; if \;\; x\geq 1-y\\ \vee(x,y) \;\; if \;\; x<1-y\end{cases} \qquad H(x,y)=\begin{cases}0 \;\; if \;\; x\geq 1-y\\ \wedge(1-x,1-y) \;\; if \;\; x<1-y.\end{cases}$$

In these conditions,

	$\mathcal{A} =$	$\sigma(A,B) \quad =$
1)	$\frac{1}{n}\sum\limits_{i=1}^{n} x_i$	$\frac{1}{n}\sum\limits_{i=1}^{n}\begin{cases}1 & \text{if } A(x_i)\leq B(x_i)\\ \vee(1-A(x_i),B(x_i)) & \text{if } A(x_i)>B(x_i)\end{cases}$

Corollary 3. *In the same conditions as in the theorem above, if* $f:[0,1]\to\mathbb{R}$ *is a continuous, strictly increasing and convex function, then*

$$\sigma(A,B)=f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{G(N(A(x_i)),B(x_i))}{G(N(A(x_i)),B(x_i))+H(N(A(x_i)),B(x_i))}\right)\right)$$

is a fuzzy subsethood measure on X *that satisfies axioms* $SD6$ *and* $SD10$ *of Sinha and Dougherty.*

Proof. We only need to take into account the theorem above, and item (a) in section 6.1 of [9] by which if f is convex we have $\mathcal{A}_k(x_1,\cdots,x_n) > \frac{1}{n}\sum_{i=1}^{n} x_i$. And Theorem 5 in [9], by which: $\mathcal{A}(x_1,\cdots,x_n)=f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)$ $\blacksquare$

Example 5. Construction of fuzzy subsethood measures on X from functions of $[0,1]$ in $\mathbb{R}$ and functions I generated from functions of $[0,1]^2$ in $\mathbb{R}^+$.

	$f \quad =$	$\sigma(A,B) \quad =$
1)	x^{λ} $\lambda>1$	$\left(\frac{1}{n}\sum\limits_{i=1}^{n}\left(\begin{cases}1 & \text{if } A(x_i)\leq B(x_i)\\ \vee(1-A(x_i),B(x_i)) & \text{if } A(x_i)>B(x_i)\end{cases}\right)^{\lambda}\right)^{\frac{1}{\lambda}}$

4 Conclusions

In this paper, we have introduced a theorem by which we could construct fuzzy subethood measures using functions with certain properties. The importance of using different ways to construct fuzzy subsethood measures comes from the fact that they can be used to obtain other measures namely, entropies, similarity measures, penalty functions, etc. [26]. Besides, we use functions I with properties very similar to those of implication operators in order to construct fuzzy subsethood measures. Thus we can benefit from all the studies regarding such operators [2,6,23].

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