New results about De Morgan triples

Ivan Mezzomo and Benjamín Callejas Bedregal

Department of Exact and Natural Sciences - DCEN
Rural Federal University of SemiArid - UFERSA,
Mossoró, Rio Grande do Norte, Brazil
imezzomo@ufersa.edu.br
Department of Informatics and Applied Mathematics - DIMAp
Federal University of Rio Grande do Norte - UFRN
Natal, Rio Grande do Norte, Brazil
bedregal@dimap.ufrn.br

Abstract. In this paper we consider the notions of t-norm, t-conorm, fuzzy negation and De Morgan triples and prove some new results about them. We define t-(co)norm S_N (T_N) N-dual of S (T) and prove that T_N is a t-conorm and S_N is a t-norm. We prove if (T, S, N) is a De Morgan triple, then (T, S, N^{-1}) is a De Morgan triple, and if N is strict, then $(S_N, T_N, N) = (T, S, N)$. Also, if (T, S, N) is a De Morgan triple and ρ an automorphism, then (T^ρ, S^ρ, N^ρ) is a De Morgan triple.

Keywords: t-norm, t-conorm, fuzzy negation, De Morgan triples, automorphism.

1 Introduction

There are several ways to extend the propositional classical connectives for a set $[0, 1]$, but not always these extentions preserve the properties of the classical conectives. Triangular norms (t-norms) and triangular conorms (t-conorms) were first studied by Menger [16] and, Scheweizer and Sklar [19] in probabilistic metric spaces in which triangular inequalities were extended using t-norms and t-conorms theory.

The name triangular norm refers to the fact that in the framework of probabilistic metric spaces t-norms are used to generalize triangle inequality of ordinary metric spaces and t-conorms are dual to t-norms under the order-reversing operation. The defining conditions of the t-norm are exactly those of the partially ordered Abelian monoid on the real unit interval [0, 1].

T-norms are used to represent logical conjunction in fuzzy logic and intersepction in fuzzy set theory, whereas t-conorms are used to represent logical disjunction in fuzzy logic and union in fuzzy set theory. A t-norm (t-conorm) is a binary operation defined in $[0, 1]$, which is commutative, associative, nondecreasing and with neutral element 1 (0).

In 19th century, De Morgan introduced the De Morgan's laws which in propositional logic and boolean algebra is a pair of transformation rules that are both valid rules of inference. This rules allow the expression of conjunctions and disjunctions purely in terms of each other via negation. In formal language, these rules can be expressed as "the negation of a conjunction is the disjunction of the negations" and "the negation of a disjunction is the conjunction of the negations". De Morgan's laws are also apply in the more general context of Boolean algebra and, in particular, in the Boolean algebra of set theory. In fuzzy logic, the triples formed by a t-norm, t-conorm and standard complement is called De Morgan triples if it fulfills De Morgan laws. De Morgan triples were first introduced by Zadeh in 1965.

In this contribuition, our aim is prove some new important results about t-norm and t-conorm theory or that they are not readily found in the literature. In this paper, we define t-(co)norm T_N (S_N) N-dual of T (S) and prove that T_N is a t-conorm and S_N is a t-norm. We demonstrate that if (T, S, N) is a De Morgan triple, then (T, S, N^{-1}) is a De Morgan triple, and if N is strict, then $(S_N, T_N, N) = (T, S, N)$. Also, if (T, S, N) is a De Morgan triple and ρ an automorphism, then (T^ρ, S^ρ, N^ρ) is a De Morgan triple and we prove the same for supremum and infimum of the t-norm and t-conorm.

2 Preliminaries

In this section, we will briefly review some basic concepts which are necessary for the development of this paper. The definitions and additional results can be found in [1], [2], [4], [5], [7], [8],[10], [12], [14].

2.1 t-norms, t-conorms and fuzzy negations

Definition 1. *A function* $T : [0, 1]^2 \to [0, 1]$ *is a t-norm if, for all* $x, y, z \in [0, 1]$, *the following axioms are satisfied:*

1. *Symmetric:* $T(x, y) = T(y, x)$;
2. *Associative:* $T(x, T(y, z)) = T(T(x, y), z)$;
3. *Monotonic: If* $x \leq y$, *then* $T(x, z) \leq T(y, z)$;
4. *One identity:* $T(x, 1) = x$.

A t-norm T is called positive if satifies the condiction: $T(x, y) = 0$ iff $x = 0$ or $y = 0$.

Example 1. Some examples of t-norms:

1. **Gödel t-norm:** $T_G(x, y) = \min(x, y)$;
2. **Product t-norm:** $T_P(x, y) = x.y$;
3. **Lukasiewicz t-norm:** $T_L(x, y) = \max(0, x + y - 1)$;
4. **Drastic t-norm:**

$$T_D(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, 1[^2; \\ \min(x, y) & \text{otherwise.} \end{cases}$$

■

Definition 2. *A function* $S : [0,1]^2 \to [0,1]$ *is a t-conorm if, for all* $x, y, z \in [0,1]$*, the following axioms are satisfied:*

1. *Symmetric:* $S(x,y) = S(y,x)$*;*
2. *Associative:* $S(x, S(y,z)) = S(S(x,y), z)$*;*
3. *Monotonic: If* $x \leq y$*, then* $S(z,x) \leq S(z,y)$*;*
4. *Zero identity:* $S(x,0) = x$*.*

A t-conorm S is called positive if satifies the condiction: $S(x,y) = 1$ iff $x = 1$ or $y = 1$.

Example 2. Some examples of t-conorms:

1. Gödel t-conorm: $S_G(x,y) = \max(x,y)$;
2. Probabilistic sum: $S_P(x,y) = x + y - x.y$;
3. Lukasiewicz t-conorm: $S_L(x,y) = \min(x+y, 1)$;
4. Drastic sum:

$$S_D(x,y) = \begin{cases} 1 & \text{if } (x,y) \in]0,1]^2; \\ \max(x,y) & \text{otherwise.} \end{cases}$$

■

A function $N : [0,1] \to [0,1]$ is a fuzzy negation if

N1: $N(0) = 1$ and $N(1) = 0$;
N2: Decreasing: If $x \leq y$, then $N(x) \geq N(y)$, for all $x, y \in [0,1]$.

A fuzzy negations N is strict if it is continuous and strictly decreasing, i.e., $N(x) < N(y)$ when $y < x$. A fuzzy negations N that satisfying the condition N3 is called strong

N3: Involutive: $N(N(x)) = x$ for each $x \in [0,1]$.

A fuzzy negation is called crisp if satisfies N4

N4: Crisp: For all $N(x) \in \{0,1\}$ iff $x = 0$ or $x = 1$.

Example 3. Some examples of fuzzy negations:

1. Standard negation: $N_S(x) = 1 - x$;
2. Strict non-strong negation: $N_{S^2}(x) = 1 - x^2$;
3. Bottom negation:

$$N_\perp(x) = \begin{cases} 0 \text{ if } x > 0 \\ 1 \text{ if } x = 0 \end{cases}$$

4. Top negation:

$$N_\top(x) = \begin{cases} 0 \text{ if } x = 1 \\ 1 \text{ if } x < 1 \end{cases}$$

In [11], Dimuro *et.al.* define N_α and N^α as

$$N_\alpha(x) = \begin{cases} 0 \text{ if } x > \alpha \\ 1 \text{ if } x \leq \alpha \end{cases} \tag{1}$$

$$N^\alpha(x) = \begin{cases} 0 \text{ if } x \geq \alpha \\ 1 \text{ if } x < \alpha \end{cases} \tag{2}$$

According to [11], a fuzzy negation N is crisp iff there exists a $\alpha \in [0,1]$ such that $N = N_\alpha$ or $N = N^\alpha$.

Note that:

1. If N is strong then it has an inverse N^- which also is a strict fuzzy negation;
2. If N is strong then N is strict.

Definition 3. *[3, Definitions 2.3.8 and 2.3.14] Let T be a t-norm, S be a t-conorm and N be a fuzzy negation. The pair (T,N) satisfies the law of contradiction if*

$$T(x, N(x)) = 0,\ \forall\ x \in [0,1] \tag{LC}$$

The pair (S,N) satisfies the law of excluded middle if

$$S(x, N(x)) = 1,\ \forall\ x \in [0,1] \tag{LEM}$$

The supremum and infimum of the t-norms T_1 and T_2 is defined as

$$T_1 \wedge T_2(x,y) = \min\{T_1(x,y), T_2(x,y)\}$$
$$T_1 \vee T_2(x,y) = \max\{T_1(x,y), T_2(x,y)\}.$$

Similarly, we define the supremum and infimum of the t-conorms S_1 and S_2.

Proposition 1. *Let T_1 and T_2 be t-norms. Then, $T_1 \wedge T_2$ and $T_1 \vee T_2$ are symmetric, increasing and have 1 as neutral element.*

Proof. Let T_1 and T_2 be t-norms. Then, for all $x, y, z \in [0,1]$,

1. Symmetry:

$$\begin{aligned} T_1 \wedge T_2(x,y) &= \min\{T_1(x,y), T_2(x,y)\} \\ &= \min\{T_1(y,x), T_2(y,x)\} \\ &= T_1 \wedge T_2(y,x). \end{aligned}$$

2. Monotonicity: If $x \leq y$ then, since T is a t-norm, $T_1(x,z) \leq T_1(y,z)$ and $T_2(x,z) \leq T_2(y,z)$. Thus, $\min\{T_1(x,z), T_2(x,z)\} \leq \min\{T_1(y,z), T_2(y,z)\}$. Therefore, $T_1 \wedge T_2(x,z) \leq T_1 \wedge T_2(y,z)$.

3. Border Condition: For all $x \in [0, 1]$, we have that

$$\begin{aligned} T_1 \wedge T_2(x, 1) &= \min\{T_1(x, 1), T_2(x, 1)\} \\ &= \min\{x, x\} \\ &= x. \end{aligned}$$

Analogously we prove that $T_1 \vee T_2$ is symmetric, increasing and have 1 as neutral element. ∎

Notice that, not always $T_1 \wedge T_2$ and $T_1 \vee T_2$ are associative and therefore, t-norms.

Proposition 2. *Let S_1 and S_2 be t-conorms. Then, $S_1 \wedge S_2$ and $S_1 \vee S_2$ are symmetric, increasing and have 1 as neutral element.*

Proof. Analogous from Proposition 1. ∎

Definition 4. *Let T be a t-norm, S be a t-conorm and N be a strict fuzzy negation. T_N is the N-dual of T if, for all $x, y \in [0, 1]$, $T_N(x, y) = N^{-1}(T(N(x), N(y)))$. Similarly, S_N is the N-dual of S if, for all $x, y \in [0, 1]$, $S_N(x, y) = N^{-1}(S(N(x), N(y)))$.*

Proposition 3. *Let T be a t-norm, S be a t-conorm and N be a fuzzy negation. Then, T_N is a t-conorm and S_N is a t-norm.*

Proof. Let T be a t-norm and N be a fuzzy negation. Then, for all $x, y, z \in [0, 1]$,

1. Symmetry:

$$\begin{aligned} T_N(x, y) &= N^{-1}(T(N(x), N(y))) \\ &= N^{-1}(T(N(y), N(x))) \\ &= T_N(y, x). \end{aligned}$$

2. Associativity:

$$\begin{aligned} T_N(x, T_N(y, z)) &= N^{-1}(T(N(x), N(N^{-1}(T(N(y)))))) \\ &= N^{-1}(T(N(x), T(N(y)))) \\ &= N^{-1}(T(T(N(x), N(y)))) \\ &= N^{-1}(T(N(N^{-1}(T(N(x), N(y)))))) \\ &= T_N(T_N(x, y), z) \end{aligned}$$

3. Monotonicity: If $y \leq z$, then $N(z) \leq N(y)$. Since T is a t-norm, then $T(N(x), N(z)) \leq T(N(x), N(y))$. Thus, $N^{-1}(T(N(x), N(y))) \leq N^{-1}(T(N(x), N(z)))$. Therefore, $T_N(x, y) \leq T_N(x, z)$.

4. Border Condition: For all $x \in [0,1]$, we have that

$$\begin{aligned}T_N(x,0) &= N^{-1}(T(N(x),N(0)))\\ &= N^{-1}(T(N(x),1))\\ &= N^{-1}(N(x))\\ &= x.\end{aligned}$$

Therefore, T_N is a t-conorm. Analogously we prove that S_N is a t-norm. ■

Definition 5. *A function $\rho : [0,1] \to [0,1]$ is an automorphism if it is bijective and increasing, i.e., for each $x, y \in [0,1]$, if $x \leq y$, then $\rho(x) \leq \rho(y)$.*

According with [8, Definition 0], a function $\rho\colon U \to U$ is an **automorphism** if it is continuous, strictly increasing and verifies the boundary conditions $\rho(0)=0$ and $\rho(1)=1$, i.e., if it is an increasing bijection on U, meaning that for each $x, y \in [0,1]$, if $x \leq y$, then $\rho(x) \leq \rho(y)$. The set of all automorphisms on $[0,1]$ will be denoted by $Aut([0,1])$.

Automorphisms are closed under composition if $\rho, \rho' \in Aut([0,1])$, then $\rho \circ \rho' \in Aut([0,1])$, where $\rho \circ \rho'(x) = \rho(\rho'(x))$. In addition, the inverse ρ^{-1} of an order automorphism ρ is also an order automorphism.

The function T^ρ (S^ρ), called as the **ρ-conjugate** of a t-(co)norm T (S), is obtained by action of $\rho \in Aut([0,1])$ on T (S) and defined in the following:

$$T^\rho(x,y) = \rho^{-1}(T(\rho(x),\rho(y))), \tag{3}$$

$$S^\rho(x,y) = \rho^{-1}(S(\rho(x),\rho(y))),\ \forall\ x,y \in [0,1]. \tag{4}$$

2.2 De Morgan triples

According to [14], the triple (T, S, N) where T is a t-norm, S is a t-conorm and N a fuzzy negation is called De Morgan triples if satisfies the following conditions:

$$\begin{aligned}T(x,y) &= N(S(N(x),N(y)));\\ S(x,y) &= N(T(N(x),N(y))),\end{aligned}$$

which naturally imply that N is a strong fuzzy negation.

There are several different notions of De Morgan triples as we can see in [6], [9], [13], [15], [17], [18]. In this paper we will use the definition of De Morgan triples laws as done in [10], [15], [18], [20] which not implies in involution of the fuzzy negation. Thus,

Definition 6. *Let T is a t-norm, S a t-conorm, N a fuzzy negation. Then (T, S, N) is a De Morgan triple if, for each $x, y \in [0,1]$,*

$$N(T(x,y)) = S(N(x),N(y)); \tag{5}$$

$$N(S(x,y)) = T(N(x),N(y)). \tag{6}$$

Example 4. (T_G, S_G, N_S), $(T_P, S_P, N_\perp)$ and (T_L, S_L, N_S) are examples of De Morgan triples. ■

Definition 7. *(T, S, N) is a semi De Morgan triple if satify the Eq. (5) for each $x \in [0,1]$ or (6) for each $x \in [0,1]$.*

3 New results

In this section, we will prove some propositions using definitions introduced on the previous section.

Proposition 4. *Let T be a t-norm, S be a t-conorm and $\alpha \in [0,1]$. If, $T(x,y) > \alpha \Leftrightarrow x, y > \alpha$ and $S(x,y) \leq \alpha \Leftrightarrow x, y \leq \alpha$, then (T, S, N_α) is a De Morgan triple.*

Proof. Suppose that $x, y > \alpha$, then $T(x,y) > \alpha$ and $N_\alpha(T(x,y)) = 0$. On the other hand, $S(N_\alpha(x), N_\alpha(y)) = S(0,0) = 0$;

If $x \leq \alpha$, then $T(x,y) \leq \alpha$ and therefore, $N_\alpha(T(x,y)) = 1$. On the other hand, $S(N_\alpha(x), N_\alpha(y)) = S(1, N_\alpha(y)) = 1$;

If $y \leq \alpha$, the proof is analogous. Therefore, $N_\alpha(T(x,y)) = S(N_\alpha(x), N_\alpha(y))$.

Now, we will prove that $N_\alpha(S(x,y)) = T(N_\alpha(x), N_\alpha(y))$. Suppose that $x, y \leq \alpha$, then $S(x,y) \leq \alpha$ and $N_\alpha(S(x,y)) = 1$. On the other hand, $T(N_\alpha(x), N_\alpha(y)) = T(1,1) = 1$;

If $x > \alpha$, then $S(x,y) > \alpha$ and therefore, $N_\alpha(S(x,y)) = 0$. On the other hand, $T(N_\alpha(x), N_\alpha(y)) = T(0, N_\alpha(y)) = 0$;

If $y > \alpha$, the proof is analogous. Therefore, $N_\alpha(S(x,y)) = T(N_\alpha(x), N_\alpha(y))$.

Therefore, (T, S, N_α) is a De Morgan triple. ■

Corollary 1. *Let T be a positive t-norm and S be a positive t-conorm. Then, $(T, S, N_\perp)$ is a De Morgan triple.*

Proof. Analogous from Proposition 4. ■

Proposition 5. *Let T be a t-norm, S be a t-conorm and $\alpha \in [0,1]$. If, $T(x,y) \geq \alpha \Leftrightarrow x, y \geq \alpha$ and $S(x,y) < \alpha \Leftrightarrow x, y < \alpha$, then (T, S, N^α) is a De Morgan triple.*

Proof. Analogous from Proposition 4. ■

Corollary 2. *Let T be a positive t-norm and S be a positive t-conorm. Then, $(T, S, N_\top)$ is a De Morgan triple.*

Proof. Analogous from Proposition 4. ■

Proposition 6. *Let (T, S, N) be a De Morgan triple. If N is strict then (T, S, N^{-1}) is also De Morgan triple.*

Proof. Let $x, y \in [0, 1]$, then

$$\begin{aligned} N^{-1}(T(x,y)) &= N^{-1}(T(N(N^{-1}(x)), N(N^{-1}(y)))) \\ &= N^{-1}(N(S(N^{-1}(x), N^{-1}(y)))) \\ &= S(N^{-1}(x), N^{-1}(y)). \end{aligned}$$

Analogously we prove that $N^{-1}(S(x,y)) = T(N^{-1}(x), N^{-1}(y))$. ∎

The following lemmas will give us important results for to prove the propositions envolving contradiction law and law of excluded middle.

Lemma 1. *Let T be a t-norm and N be a strict fuzzy negation. (T, N) satisfies (LC) iff (T, N^{-1}) satisfies (LC).*

Proof. Since N is strict then it has an inverse strict fuzzy negation N^{-1}.
($\Rightarrow$) Since (T, N) satisfies (LC), then $T(N(x), N^{-1}(N(x))) = 0$, for all $x \in [0, 1]$.
So, for all $x \in [0, 1]$, $T(x, N(x)) = T(N(x), x) = T(N(x), N^{-1}(N(x))) = 0$.
($\Leftarrow$) Analogous. ∎

Lemma 2. *Let S be a t-conorm and N be a strict fuzzy negation. (S, N) satisfies (LEM) iff (S, N^{-1}) satisfies (LEM).*

Proof. Analogous from Lemma 1. ∎

Proposition 7. *Let (T, S, N) be a semi De Morgan triple with respect the Eq. (5) and N be a strict fuzzy negation. If (T, N) satisfies (LC) then (S, N) satisfies (LEM).*

Proof. Let $x \in [0, 1]$. Since (T, N) satisfies (LC), then $T(x, N(x)) = 0$. So, $N^{-1}(T(x, N(x))) = 1$, for all $x \in [0, 1]$. By Eq. (5) and Proposition 6, $S(N^{-1}(x), N^{-1}(N(x))) = 1$ and therefore, $S(N^{-1}(x), x) = 1$, for all $x \in [0, 1]$. Thus, by Lemma 2, $S(N(x), x) = 1$. Therefore, (S, N) satisfies (LEM). ∎

Proposition 8. *Let (T, S, N) be a semi De Morgan triple with respect the Eq. (6) and N be a strict fuzzy negation. If (S, N) satisfies (LEM) then (T, N) satisfies (LC).*

Proof. Let $x \in [0, 1]$. Since (S, N) satisfies (LEM), then $S(N(x), x) = 1$ and then $N^{-1}(S(N(x), x)) = 0$, for all $x \in [0, 1]$. So, by Eq. (6) and Proposition 6, $T(N^{-1}(N(x)), N^{-1}(x)) = 0$ and hence, $T(x, N^{-1}(x)) = 0$, for all $x \in [0, 1]$. Thus, by Lemma 1, $T(x, N(x)) = 0$. Therefore, (T, N) satisfies (LC). ∎

Corollary 3. *Let (T, S, N) be a De Morgan triple and N be a strict fuzzy negation. Then, (T, N) satisfies (LC) iff (S, N) satisfies (LEM).*

Proof. Straighforward from Propositions 7 and 8. ■

Proposition 9. *Let (T, S, N) be a De Morgan triple. If N is strict then $(S_N, T_N, N) = (T, S, N)$.*

Proof. Since N is strict, then $S_N(x, y) = N^{-1}(S(N(x), N(y))) = N^{-1}(N(T(x, y))) = T(x, y)$. Analogously we prove that $T_N(x, y) = S(x, y)$. ■

Proposition 10. *Let (T, S, N) be a De Morgan triple. If N is strict, then $(S_N, T_{N^{-1}}, N)$ and $(S_{N^{-1}}, T_N, N)$ are semi De Morgan triple.*

Proof. Let N be a strict fuzzy negation and $x, y \in [0, 1]$. Then

$$\begin{aligned} N(S_N(x, y)) &= N(N^{-1}(S(N(x), N(y)))) \\ &= S(N(x), N(y)) \\ &= N(T(x, y)) \\ &= N(T(N^{-1}(N(x)), N^{-1}(N(y)))) \\ &= T_{N^{-1}}(N(x), N(y)). \end{aligned}$$

Thus, $(S_N, T_{N^{-1}}, N)$ satisfies the Eq. (6) and therefore, it is a semi De Morgan triple. Analogously, we prove that $(S_{N^{-1}}, T_N, N)$ satisfies Eq. (5). ■

Now, using the notion of automorphism ρ from t-norms, t-conorms and fuzzy negation, we show that the triple (T^ρ, S^ρ, N^ρ) is a De Morgan triple.

Proposition 11. *Let (T, S, N) be a De Morgan triple and ρ be an automorphism. Then, (T^ρ, S^ρ, N^ρ) is a De Morgan triple.*

Proof. Let $x, y \in [0, 1]$, then

$$\begin{aligned} N^\rho(T^\rho(x, y)) &= N^\rho(\rho^{-1}(T(\rho(x), \rho(y)))) \\ &= \rho^{-1}(N(\rho(\rho^{-1}(T(\rho(x), \rho(y)))))) \\ &= \rho^{-1}(N(T(\rho(x), \rho(y)))) \\ &= \rho^{-1}(S(N(\rho(x)), N(\rho(y)))) \\ &= \rho^{-1}(S(\rho(\rho^{-1}(N(\rho(x)))), \rho(\rho^{-1}(N(\rho(y)))))) \\ &= S^\rho(\rho^{-1}(N(\rho(x))), \rho^{-1}(N(\rho(y)))) \\ &= S^\rho(N^\rho(x), N^\rho(y)). \end{aligned}$$

Analogously we prove that $N^\rho(S^\rho(x, y)) = T^\rho(N^\rho(x), N^\rho(y))$. ■

Proposition 12. *Let* (T_1, S_1, N) *and* (T_2, S_2, N) *be De Morgan triples. Then,* $T_1 \leq T_2$ *iff* $S_1 \geq S_2$.

Proof. $(\Rightarrow)$ Let $T_1 \leq T_2$ and $x, y \in [0, 1]$. Then, $N(S_1(x, y)) = T_1(N(x), N(y)) \leq T_2(N(x), N(y)) = N(S_2(x, y))$. Therefore, $S_1(x, y) \geq S_2(x, y)$.

$(\Leftarrow)$ Analogous. ■

Proposition 13. *Let* (T_1, S_1, N) *and* (T_2, S_2, N) *be De Morgan triples. If* $T_1 \wedge T_2$ *and* $S_1 \vee S_2$ *are t-norm and t-conorm, respectively, then* $(T_1 \wedge T_2, S_1 \vee S_2, N)$ *is a De Morgan triple. Dually, if* $T_1 \vee T_2$ *and* $S_1 \wedge S_2$ *are t-norm, t-conorm, respectively, then* $(T_1 \vee T_2, S_1 \wedge S_2, N)$ *is a De Morgan triple.*

Proof. Let $x, y \in [0, 1]$. Then,

$$\begin{aligned}
N(T_1 \wedge T_2(x, y)) &= N(\min\{T_1(x, y), T_2(x, y)\}) \\
&= \max\{N(T_1(x, y)), N(T_2(x, y))\} \\
&= \max\{S_1(N(x), N(y)), S_2(N(x), N(y))\} \\
&= S_1 \vee S_2(N(x), N(y)).
\end{aligned}$$

Analogously we prove that $N(S_1 \vee S_2(x, y)) = T_1 \wedge T_2(N(x), N(y))$. Therefore, $(T_1 \wedge T_2, S_1 \vee S_2, N)$ is a De Morgan triple.

Dually, we prove that $(T_1 \vee T_2, S_1 \wedge S_2, N)$ is a De Morgan triple. ■

4 Conclusion

In this paper we consider the notions of t-norm, t-conorm, fuzzy negation and De Morgan triples and prove some new results about them as, if (T, S, N) is a De Morgan triple, then (T, S, N^{-1}) is a De Morgan triple, and if N is strict, then $(S_N, T_N, N) = (T, S, N)$. Also, if (T, S, N) is a De Morgan triple and ρ an automorphism, then (T^ρ, S^ρ, N^ρ) is a De Morgan triple.

As further work, we will prove other results as the ordinal sum of De Morgan triples is De Morgan triple.

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