Prediction of the Economically Active Population Index Using Interval-Valued Fuzzy Morphological Associative Memories

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Abstract. The last few decades have witnessed rapid progress in the field of type-2 fuzzy systems and in particular interval type-2 fuzzy systems. Encouraged by this progress, we present in this paper some theoretical foundations and applications of interval-valued fuzzy morphological associative memories (IV-FMAMs) as a rule-based system. We perform simulations concerning the application of IV-FMAMs to the prediction of the monthly rate of participation of certain age groups in the work force of the metropolitan area of São Paulo. The performance in terms of mean squared prediction errors of the IV-FMAM approach is then compared with the one of a conventional type-2 fuzzy inference system.

1 Introduction

Type-2 fuzzy sets, and in particular interval type-2 fuzzy sets have found a wide variety of applications in engineering, control, computing with words, and approximate reasoning [8,16,17]. Although interval-valued fuzzy sets (IV fuzzy sets) are related to interval type-2 fuzzy sets [2], these concepts are different. In fact, interval-valued fuzzy sets correspond to closed interval type-2 fuzzy sets whose secondary membership functions are characteristic functions of closed intervals [14]. To our knowledge, all interval type-2 fuzzy systems that have appeared in the literature and are used in practice represent closed interval type-2 fuzzy systems.

Like general type-2 fuzzy sets and (closed) interval type-2 fuzzy sets, interval-valued fuzzy sets can be employed to model the inherent uncertainties regarding fuzzy set membership functions. An approach for dealing with interval-valued fuzzy data is given by IV-FMAMs [24], a recent extension of fuzzy morphological associative memories. The latter can be used to implement fuzzy rule-based systems, in particular for applications in time-series prediction [22,27]. In this paper, we apply the proposed IV-FMAM model to the problem of predicting the monthly percentages of participation of certain age groups in the work force of the metropolitan area of São Paulo [4]. Solving this prediction problem may offer decision support for government, industry and trade unions.

The paper is organized as follows. Section 2 provides a brief review of pertinent notions on lattice theory, $\mathbb{L}$-fuzzy logical operators and $\mathbb{L}$-fuzzy mathematical morphology. In Section 3, we present the interval-fuzzy morphological associative memories (IV-FMAMs), that yield associations between IV fuzzy sets. Section 4 describes an application of the IV-FMAMs as rule-based systems to a social index time-series prediction. The results are compared with the ones obtained using a Mamdani(-Assilian) type-2 fuzzy system [13], followed by some concluding remarks.

2 Theoretical background

2.1 Some Relevant Concepts of Lattice Theory and Mathematical Morphology

A complete lattice $\mathbb{L}$ is a partially ordered set such that every subset $X \in \mathbb{L}$ has an infimum $\bigwedge X$ and a supremum $\bigvee X$ in $\mathbb{L}$. Recall that a *partial order* is a reflexive, antisymmetric, and transitive binary relation "$\leq$" [7]. The unit interval $[0,1]$ with the usual (total) ordering yields an example of a complete lattice. Another example is given by $\mathbb{I} = \{u = [\underline{u}, \overline{u}] \subseteq [0,1]\}$ if we consider the following partial order:

$$u \leq v \ \Leftrightarrow\ \underline{u} \leq \underline{v} \text{ and } \overline{u} \leq \overline{v}\,. \tag{1}$$

A component-wise partial order can also be defined on $\mathbb{L}^n$ by setting

$$(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n) \Leftrightarrow a_i \leq b_i,\ i = 1, \ldots, n. \tag{2}$$

If $\mathbb{L}$ is a (complete) lattice, then $\mathbb{L}^n$ is also a (complete) lattice. Similarly, we have that if $\mathbb{L}$ is a (complete) lattice then the class of functions $X \to \mathbb{L}$, $\mathbb{L}^X$, is also a (complete) lattice. Here, the partial order on $\mathbb{L}^X$ is given as follows for all $f, g \in \mathbb{L}^X$:

$$f \leq g \Leftrightarrow f(x) \leq g(x)\ \forall x \in X. \tag{3}$$

Let $\mathbb{L}$ be a complete lattice and $X \neq \emptyset$. The partial order on $\mathbb{L}^X$ of Equation 3 induces a partial order on $\mathcal{F}_{\mathbb{L}}(X)$, the class of $\mathbb{L}$-fuzzy sets over the universe X. Recall that an $\mathbb{L}$-fuzzy set A consists of a universe X together with a membership function $\mu_A : X \to \mathbb{L}$ [6]. For $\mathbb{L}$-fuzzy sets A and B, we have $A \leq B$ if and only if $\mu_A \leq \mu_B$. The class of fuzzy sets over the universe X,$\mathcal{F}(X)$, and the class of interval-valued fuzzy sets over the universe X,$\mathcal{F}_{\mathbb{I}}(X)$, represent particular classes of $\mathbb{L}$-fuzzy sets for particular choices of $\mathbb{L}$.

It is well established that complete lattices form an appropriate framework for *mathematical morphology* (MM) [19,26]. Although MM was originally conceived as a theory for image and signal processing based on geometrical and topological concepts, its operators naturally adhere to the lattice theory as their algebraic framework [10,20].

In this framework, the basic operators of MM are the (algebraic) *erosion*, $\varepsilon : \mathbb{L} \to \mathbb{M}$ and the (algebraic) *dilation*, $\delta : \mathbb{L} \to \mathbb{M}$, that satisfies, respectively, the equations below for all $M \subseteq \mathbb{L}$:

$$\varepsilon(\bigwedge M) = \bigwedge \varepsilon(M) \text{ and } \delta(\bigvee M) = \bigvee \delta(M). \tag{4}$$

The concept of an *adjunction*, that is also known as a monotone or isotone Galois connection [1] plays a very prominent role in MM [10].

Definition 1. *A pair (ε, δ) consisting of mappings $\varepsilon : \mathbb{M} \to \mathbb{L}$ and $\delta : \mathbb{L} \to \mathbb{M}$ is called an* adjunction *between $\mathbb{M}$ and $\mathbb{L}$ if and only if for all $x \in \mathbb{L}$ and all $y \in \mathbb{M}$:*

$$\delta(x) \leq y \Leftrightarrow x \leq \varepsilon(y). \tag{5}$$

If the pair (ε, δ) is an adjunction then ε is an erosion and δ a dilation [11].

2.2 $\mathbb{L}$-Fuzzy Operators

The purpose of this section is to recall the definitions of some morphological operators and matrix products on the complete lattice $\mathcal{F}_{\mathbb{L}}(X)$ [24]. Let us begin by the definition of $\mathbb{L}$-fuzzy conjunctions and implications [5]:

Definition 2. *Let $\mathbb{L}$ be a complete lattice.*

- *A* conjunction *on $\mathbb{L}$ or* $\mathbb{L}$-fuzzy conjunction *is defined as an increasing mapping $\mathcal{C} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ that satisfies $C(0_{\mathbb{L}}, 0_{\mathbb{L}}) = \mathcal{C}(0_{\mathbb{L}}, 1_{\mathbb{L}}) = \mathcal{C}(1_{\mathbb{L}}, 0_{\mathbb{L}}) = 0_{\mathbb{L}}$ and $\mathcal{C}(1_{\mathbb{L}}, 1_{\mathbb{L}}) = 1_{\mathbb{L}}$. In particular, a commutative and associative $\mathbb{L}$-fuzzy conjunction $T : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ that satisfies $T(x, 1_{\mathbb{L}}) = x$ for every $x \in \mathbb{L}$ is called* triangular norm *or simply* t-norm *on $\mathbb{L}$.*
- *An operator $\mathcal{I} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ that is decreasing in the first argument and that is increasing in the second argument is called an* implication *on $\mathbb{L}$ or* $\mathbb{L}$-fuzzy implication *if the equations $\mathcal{I}(0_{\mathbb{L}}, 0_{\mathbb{L}}) = \mathcal{I}(0_{\mathbb{L}}, 1_{\mathbb{L}}) = \mathcal{I}(1_{\mathbb{L}}, 1_{\mathbb{L}}) = 1_{\mathbb{L}}$ and $\mathcal{I}(1_{\mathbb{L}}, 0_{\mathbb{L}}) = 0_{\mathbb{L}}$ are satisfied.*
- *In the special case where $\mathbb{L} = \mathbb{I}$, we speak of* interval-valued fuzzy (IV fuzzy) operators*, in particular of* IV fuzzy conjunctions, t-norms, and implications.

Several types of matrix products arise from $\mathbb{L}$-fuzzy operators including the following ones [24]:

Definition 3. *Given an $\mathbb{L}$-fuzzy conjunction $\mathcal{C}$ and an $\mathbb{L}$-fuzzy implication $\mathcal{I}$, the* sup-$\mathcal{C}$ product *of $A \in \mathbb{L}^{m \times k}$ and $B \in \mathbb{L}^{k \times n}$, denoted by $E = A \circ_{\mathcal{C}} B$, and the* inf-$\mathcal{I}$ product*, denoted by $G = A \circledast B$ are defined, respectively, for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, as follows:*

$$e_{ij} = \bigvee_{\xi=1}^{k} \mathcal{C}(a_{i\xi}, b_{\xi j}), \text{ and } g_{ij} = \bigwedge_{\xi=1}^{k} \mathcal{I}(b_{\xi j}, a_{i\xi}) \ . \tag{6}$$

The following proposition holds for every complete lattice $\mathbb{L}$:

Proposition 1 *The operator $\delta_W : \mathbb{L}^n \to \mathbb{L}^m$ that are given by*

$$\delta_W(\mathbf{x}) = W \circ \mathbf{x}, \ \forall \mathbf{x} \in \mathbb{L}^n. \tag{7}$$

represents a dilation for every $W \in \mathbb{L}^{m\times n}$ from the complete lattice $\mathbb{L}^n$ to the complete lattice $\mathbb{L}^m$ if and only if $\mathcal{C}(w, \cdot) : \mathbb{L} \to \mathbb{L}$ represents a dilation for every $w \in \mathbb{L}$.

As we shall point out in the next section for the interval-valued case, the output of an interval-valued fuzzy associative memory can be modeled by means of Equation 7. Furthermore, the model given by Equation 7 will be called *morphological*. The next section focuses on interval-valued fuzzy morphological associative memories.

3 Interval-Valued FMAMs

In this section we will present an associative memory model for interval-valued fuzzy sets. Consider initially X and Y arbitrary universes and $\left\{\left(\mathbf{p}^\xi, \mathbf{q}^\xi\right) : \xi \in \mathcal{K}\right\} \subseteq \mathcal{F}_{\mathbb{I}}(X) \times \mathcal{F}_{\mathbb{I}}(Y)$ a set of pairs or associations, the set $\left\{\left(\mathbf{p}^\xi, \mathbf{q}^\xi\right) : \xi \in \mathcal{K}\right\}$ is called the *fundamental memory set* and its elements are called *fundamental memories.*

An *IV fuzzy associative memory* (*IV*-FAM) is an input-output system given by a mapping $\mathcal{W} : \mathcal{F}_{\mathbb{I}}(X) \to \mathcal{F}_{\mathbb{I}}(Y)$ that should ideally satisfy the following conditions [9]:

1. $\mathcal{W}(\mathbf{p}^\xi) = \mathbf{q}^\xi$ for all $\xi \in \mathcal{K}$;
2. $\mathcal{W}(\tilde{\mathbf{p}}^\xi) = \mathbf{q}^\xi$ for $\tilde{\mathbf{p}}^\xi \approx \mathbf{p}^\xi$.

Since our objective is to employ IV-FAMs in order to implement rule-based systems, it would be more adequate to replace items 1 and 2 with the following condition:

$$\tilde{\mathbf{p}}^\xi \approx \mathbf{p}^\xi \Rightarrow \mathcal{W}(\tilde{\mathbf{p}}^\xi) \approx \mathbf{q}^\xi\,. \tag{8}$$

For simplicity, we concentrate on the case where X, Y, and $\mathcal{K}$ are finite. Let $|X| = n$, $|Y| = m$, and $|\mathcal{K}| = k$. Thus, we view an IV-FAM as a mapping $\mathcal{W} : \mathbb{I}^n \to \mathbb{I}^m$. We say that $\mathcal{W}$ represents a *sup-$\mathcal{C}$ IV-FAM* if $\mathcal{W}(\mathbf{x}) = \delta_W(\mathbf{x}) = W \circ_{\mathcal{C}} \mathbf{x}, \ \forall \mathbf{x} \in \mathbb{I}^n$, for some $W \in \mathbb{I}^{m\times n}$.

In this paper we focus on the construction of IV-FMAMs based on representable IV fuzzy conjunctions and their adjoint implications [5,23]. We will now present a recipe for constructing IV fuzzy conjunctions and their adjoint implications, beginning by the definition of a representable IV fuzzy conjunction [5]:

Proposition 2 *If C is a fuzzy conjunction, then the operator $\mathcal{C}^r_C$, that is defined as follows for all $u = [\underline{u}, \overline{u}], v = [\underline{v}, \overline{v}] \in \mathbb{I}$, yields an IV fuzzy conjunction.*

$$\mathcal{C}^r_C(u, v) = [C(\underline{u}, \underline{v}), C(\overline{u}, \overline{v})]. \tag{9}$$

Definition 4. *The IV fuzzy conjunction $\mathcal{C}^r_C$ of Equation 9 is referred to as the* representable conjunction with representative C.

Let $\mathcal{C}_C^r$ be a representable interval-valued fuzzy conjunction with representative C and I the adjoint fuzzy implication of C. The adjoint IV fuzzy implication of $\mathcal{C}_C^r$ can be obtained by means of Equation 10, as follows [23]:

$$\mathcal{I}_I^n(u,v) = [I(\underline{u},\underline{v}) \wedge I(\overline{u},\overline{v}), I(\overline{u},\overline{v})]. \tag{10}$$

An example of dilative fuzzy conjunction is the cross-ratio uninorm [30], denoted using the symbol C_F. The formulas for C_F and its adjoint implication I_F can be found below:

$$C_F(x,y) = \begin{cases} 1, & \text{if } (x,y) = (0,1) \text{ or } (1,0) \\ \frac{xy}{(1-y)(1-x)+xy}, & \text{otherwise.} \end{cases} \tag{11}$$

$$I_F(x,y) = \begin{cases} 1, & \text{if } (x,y) = (0,0) \text{ or } (1,1) \\ \frac{(1-x)y}{y(1-x)+x(1-y)}, & \text{otherwise.} \end{cases} \tag{12}$$

Observe that the representable IV fuzzy conjunction $\mathcal{C}_{C_F}^r$ and its adjoint implication $\mathcal{I}_{I_F}^n$, given by means of Equation 10, represents a pair consisting of an erosion and a dilation. Hence, an IV-FAM based on $\mathcal{C}_{C_F}^r$, that will be denoted here by $\mathcal{W}_F^r$, will be morphological.

In this research paper, we will employ sup-$\mathcal{C}_C^r$ matrix products in the recall phases of IV-FAMs, in particular sup-$\mathcal{C}_C^r$ matrix products that give rise to dilations and, therefore, IV-FMAMs.

Let $P = [\mathbf{p}^1, ..., \mathbf{p}^k] \in \mathbb{I}^{n\times k}$, $Q = [\mathbf{q}^1, ..., \mathbf{q}^k] \in \mathbb{I}^{m\times k}$ be matrices which columns are formed by the pairs $(\mathbf{p}^j, \mathbf{q}^j)$ of fundamental memories and let $\mathcal{C}_C^r$ be an IV fuzzy conjunction. Consider the problem of determining the weight matrix W of a sup-$\mathcal{C}_C^r$ IV-FMAM given by the dilation δ_W. As an extension of the *fuzzy learning by adjunction* [22] for sup-C FMAMs to *IV fuzzy learning by adjunction (IV-FLA)* for sup-$\mathcal{C}_C^r$ FMAMs, we propose to construct its weight matrix $W \in \mathbb{I}^{n\times k}$ as follows:

$$W = Q \circledast_{n,I} P^t, \tag{13}$$

where the IV fuzzy implication used in the Inf-$\mathcal{I}$ product is the adjoint pair of $\mathcal{C}_C^r$. Finally, upon presentation of an input pattern $\mathbf{x} \in \mathbb{I}^n$, the output pattern $\mathbf{y} \in \mathbb{I}^m$ of the corresponding IV-FMAM can be calculated by the sup-$\mathcal{C}_C^r$ product of W and $\mathbf{x}$. In the following section, we employ the IV-FMAM $\mathcal{W}_F^r$ in order to implement a IV fuzzy inference system for a time series prediction problem.

4 An IV-FMAM Approach Towards Time Series Prediction

4.1 Experimental Setup for the IV-FMAM Approach

In this section, we present the experimental setup regarding the use of sup-$\mathcal{C}_C^r$ IV-FMAMs in a time-series forecasting problem, namely the Economically Active

Population Index (PEA), a socio-economic index used by Brazilian governmental and non-governmental sectors for strategic decision making [4]. Specifically, a sup-$\mathcal{C}_F^r$ IV-FMAM was employed to model a rule-based system consisting of k rules, each one having a IV fuzzy antecedents and b IV fuzzy consequents. To this end, the crisp training data, that are contained along with the testing data in a finite universe $\mathcal{U} \times \mathcal{V} \subseteq \mathbb{R}^a \times \mathbb{R}^b$, were clustered by means of the *IV fuzzy c-means clustering technique (IV-FCM)* [12]. More precisely, given a number k of clusters and a "fuzzifier parameter $\mathbf{m} = [m_1, m_2]$, IV fuzzy c-means clustering produces the cluster centers $c_{\mathbf{p}}^{\gamma} \in \mathbb{R}^a$ and $c_{\mathbf{q}}^{\xi} \in \mathbb{R}^b$ with respective component-wise standard deviations $\sigma_{\mathbf{p}} \in \mathbb{R}^a$ and $\sigma_{\mathbf{q}} \in \mathbb{R}^b$ of IV fuzzy Gaussian membership functions $\mathbf{p}^\xi$ and $\mathbf{q}^\xi$ for $\xi = 1, \ldots, k$.

The antecedents $\mathbf{p}^\xi$ and the consequents $\mathbf{q}^\xi$ are respectively contained in $\mathcal{F}_{\mathbb{I}}(\mathcal{U})$ and $\mathcal{F}_{\mathbb{I}}(\mathcal{V})$, where $\mathcal{U} = \{\mathbf{u}^1, \ldots, \mathbf{u}^m\}$ and $\mathcal{V} = \{\mathbf{v}^1, \ldots, \mathbf{v}^n\}$. Their components can be calculated as follows:

$$p_j^\xi = exp\left[-\frac{1}{2}\sum_{l=1}^{a}\left|\frac{(\mathbf{u}^j)_l - (\mathbf{c}_{\mathbf{p}}^\xi)_l}{(\sigma_{\mathbf{p}})_l}\right|^{\frac{2}{m_2-1}}, -\frac{1}{2}\sum_{l=1}^{a}\left|\frac{(\mathbf{u}^j)_l - (\mathbf{c}_{\mathbf{p}}^\xi)_l}{(\sigma_{\mathbf{p}})_l}\right|^{\frac{2}{m_1-1}}\right], \quad (14)$$

$$q_i^\xi = exp\left[-\frac{1}{2}\sum_{l=1}^{b}\left|\frac{(\mathbf{v}^i)_l - (\mathbf{c}_{\mathbf{q}}^\xi)_l}{(\sigma_{\mathbf{q}})_l}\right|^{\frac{2}{m_2-1}}, -\frac{1}{2}\sum_{l=1}^{b}\left|\frac{(\mathbf{v}^i)_l - (\mathbf{c}_{\mathbf{q}}^\xi)_l}{(\sigma_{\mathbf{q}})_l}\right|^{\frac{2}{m_1-1}}\right]. \quad (15)$$

In principle, the pairs $(\mathbf{p}^\xi, \mathbf{q}^\xi)$ corresponding to the training data can be used to compute the weight matrix $W \in \mathbb{I}^{m\times n}$ by means of Equation 13.

Training:

1. Apply the IV-FCM clustering technique with k centers and fuzzifier parameter $\mathbf{m}$ to the training data in $\mathcal{U} \times \mathcal{V} \subseteq \mathbb{R}^a \times \mathbb{R}^b$ in order to obtain k centers $c_{\mathbf{p}}^{\gamma} \in \mathbb{R}^a$ and $c_{\mathbf{q}}^{\xi} \in \mathbb{R}^b$ with respective component-wise standard deviations $\sigma_{\mathbf{p}} \in \mathbb{R}^a$ and $\sigma_{\mathbf{q}} \in \mathbb{R}^b$;
2. Create the discrete intervalar gaussian antecedents $\mathbf{p}^\xi$ in $\mathcal{F}_{\mathbb{I}}(\mathcal{U})$ and consequents $\mathbf{q}^\xi$ in $\mathcal{F}_{\mathbb{I}}(\mathcal{V})$ using Equations 14 and 15;
3. Compute the weight matrix $W \in \mathbb{I}^{m\times n}$ by means of the inf-I product $W = (\mathbf{q}^\xi)^t \circledast \mathbf{p}^\xi$.

The test is performed as follows. Given the weight matrix W, for each $\mathbf{p}$ in the test data set, do:

1. Compute the sup-C product $\mathbf{q} = W \circ \mathbf{p}$ to obtain an interval-valued fuzzy vector;
2. Average $\mathbf{q}$ component-wise to type-reduce it (Nie-Tan type-reduction method [18]);
3. Defuzzify the type-reduced vector using the *centroid* defuzzification method to obtain a real valued output.

However, an explicit construction of W is not required since the representation as an IV fuzzy set of the antecedent part of a crisp test datum $\mathbf{u}^d$ in $\mathcal{U}$ yields an input pattern $\mathbf{p} \in \mathcal{F}_{\mathbb{I}}(\mathcal{U})$ of the form $\mathbf{p} = [0_{\mathbb{I}}, \ldots, 0_{\mathbb{I}}, e_{\mathbb{I}}, 0_{\mathbb{I}}, \ldots, 0_{\mathbb{I}}]^T$. More precisely, we have $p_j = \begin{cases} e_{\mathbb{I}}, & \text{if } j = d \\ 0_{\mathbb{I}}, & \text{otherwise.} \end{cases}$. Here, the symbol $e_{\mathbb{I}}$ denotes the identity element $[0.5, 0.5]$ of the IV fuzzy conjunction $\mathcal{C}_F^r$. For $\mathbf{p} \in \mathcal{F}_{\mathcal{I}}(\mathcal{U}) \simeq \mathbb{I}^m$ of this form, $\mathbf{q} = W \circ_{r,F} \mathbf{p}$ has the following components for $i = 1, \ldots, m$:

$$q_i = W \circ_{r,F} p_i = \bigvee_{j=1}^{n} \mathcal{C}_F^r(w_{ij}, p_j) = w_{id} = \bigwedge_{\xi=1}^{k} \mathcal{I}_F^n(p_j^\xi, q_i^\xi) \,. \tag{16}$$

We obtain a final, crisp prediction value from $\mathbf{q}$ after an application of type-reduction and defuzzification. For computational reasons, we employed the *Nie-Tan* method, which is nearly as fast as a conventional fuzzy centroid defuzzification [28]. The Nie-Tan method consists of a type-reduction procedure followed by a *centroid defuzzification* of the resulting type-1 fuzzy set.

4.2 Prediction of a Brazilian Socioeconomic Index

The Economically Active Population Index (PEA) refers to the percentage of economically active persons, i.e., people who are currently employed or actively looking for a job, within certain age groups. The data for the computation of the PEA index are collected by DIEESE, the "Inter-Union Department of Statistics and Socio-Economic Studies", a creation of the Brazilian trade union movement. DIEESE was founded in 1955 to develop research on which workers' demands could be based [4].

The PEA index, since its conception, has provided guidance - not only to some governmental sectors such as the social security office but also to trade unions and private corporations since this index yields valuable information concerning the current state of the workforce. This information can also be used as a decision making tool for politicians to avoid future social and economical problems. The PEA index is computed monthly as part of some other more comprehensive indices, such as the Monthly Employment Survey Index (PME/IBGE) and the Employment and Unemployment ResearchIndex (PED/DIEESE), in the metropolitan area of São Paulo and other major Brazilian metropolitan areas.

Here, we employed the methodology described in order to forecast the PEA index of the metropolitan area of São Paulo from January 1985 to December 2012. The population under consideration comprises approximately 17 million inhabitants and is divided into the following age brackets: 10-15, 16-24, 25-39, 40-49, 50-59 and 60+. The PEA index values are given by the percentage of the working age population that includes every person older than nine.

Note that the PEA index includes 10 to 15 year old economically active children that are part, albeit illegally, of Brazil's working population. Thus, the PEA index serves to aide governmental and non-governmental organizations to understand and remedy the child labor problem. Since the insertion of the 10-15 years age group in the PEA index, initiatives such as the creation of the Statute of the Child and the Adolescent ("Estatuto da Criança e do Adolescente") and several non-governmental campaigns helped to decrease the amount of child labor from more than 20% in 1985 to 4.5% at the end of 2012.

We chose to use one predictor model for each month and for age-group, in view of seasonal differences in the PEA values. We standardized the original data to lie within the range $[-5, 5]$ by subtracting the mean and dividing by the standard deviation. Let $s_\gamma \in \mathbb{R}$ be samples of the seasonal time series. The goal is to estimate the value of s_q from a subset of the past values $\{s_1, s_2, ..., s_{q-1}\}$. In particular, we simply used the last three monthly PEA indices $\{s_{\gamma-3}, s_{\gamma-2}, s_{\gamma-1}\}$ to predict the next index s_γ.

Given a fuzzifier parameter $\mathbf{m}$ and a number of clusters k, IV-FCM clustering produced cluster centers $c^\xi_{\mathbf{p}} \in \mathbb{R}^3$ and $c^\xi_{\mathbf{q}} \in \mathbb{R}$ with respective component-wise standard deviations $\sigma_{\mathbf{p}} \in \mathbb{R}^3$ and $\sigma_{\mathbf{q}} \in \mathbb{R}$. We considered finite universes of discourse $\mathcal{U} = \{\mathbf{u}^1, \ldots, \mathbf{u}^m\}$ and $\mathcal{V} = \{\mathbf{v}^1, \ldots, \mathbf{v}^n\}$ comprising $m = 50^3$ and $n = 50$ equally spaced points in $[-5, 5]^3$ and $[-5, 5]$, respectively. The values $c^\xi_{\mathbf{p}}, \sigma_{\mathbf{p}}, c^\xi_{\mathbf{q}}$, and $\sigma_{\mathbf{q}}$ can be used to compute the entries $p^\xi_j = [\underline{p^\xi_j}, \overline{p^\xi_j}]$ and $q^\xi_i = [\underline{q^\xi_i}, \overline{q^\xi_i}]$ of the finite Gaussian IV fuzzy sets $\mathbf{p}^\xi \in \mathcal{F}_{\mathbb{I}}(\mathcal{U}) \simeq \mathbb{I}^n$ and $\mathbf{q}^\xi \in \mathcal{F}_{\mathbb{I}}(\mathcal{V}) \simeq \mathbb{I}^m$ via Equations 14 to 15.

The performance of the IV-FMAM approach suggested in this paper depends on the choices of the number of clusters and a fuzzifier parameter $\mathbf{m} = [m_1, m_2]$ as inputs to the IV-FCM clustering algorithm. We employed a fixed number of clusters $k = 10$ for all monthly models. We considered three different options of $\mathbf{m}$, namely $[2.0 - \alpha, 2.0 - \alpha]$ for $\alpha = 0, 0.1, 0.2$. Note that $\mathbf{m} = [2.0, 2.0]$ yields fuzzy Gaussian membership functions $\mathbf{p}^\xi \in \mathcal{F}(\mathcal{U})$ and $\mathbf{q}^\xi \in \mathcal{F}(\mathcal{V})$ and therefore $\mathcal{W}^r_C$ corresponds to $\mathcal{W}_C$ for every commutative and dilative fuzzy conjunction C. Since $\mathcal{W}_F$ achieved the best validation performance in previous experiments concerning time-series prediction [27], we only performed simulations using $\mathcal{W}^r_F$.

The fuzzifier parameter $\mathbf{m}$ was selected by means of leave-one-out cross-validation on the data from January 1985 to December 2002, according to the Table 1 for each age bracket. Here, the performance was measured in terms of the mean absolute error (MAE), the root mean squared error (RMSE), the mean percentage error (MPE), and the correlation coefficient ρ (the higher the values of ρ, the better the performance).

We compared the prediction results produced by the sup-$\mathcal{C}^r_F$ IV-FMAM model with $\mathbf{m} = [1.9, 2.1]$ with the ones produced by an interval type-2 fuzzy inference system (IT-2 FIS) with the same fuzzifier parameter $\mathbf{m} = [1.9, 2.1]$ using the test data from January 2003 to December 2012. Note that the IT2-FIS with $\mathbf{m} = [2.0, 2.0]$ corresponds to a conventional type-1 fuzzy inference system [15]. Tables 2 and 3 display the testing errors and the correlation coefficients produced by the $\mathcal{W}^r_F$ and the interval type-2 fuzzy inference system, respectively. Note that the IV-FMAM $\mathcal{W}^r_F$ evidently outperformed the IT-2 FIS in these simulations. Figures 1 and 2 illustrate the prediction results obtained by $\mathcal{W}^r_F$ and a conventional interval type-2 fuzzy system, respectively, in comparison with the real data for the testing period.

Table 1. Leave-one-out cross-validation errors from 1985 to 2003.

Age Bracket	**m**	MAE	RMSE	MRE(%)	ρ
10-15	$[2.0, 2.0]$	1.70	2.77	11.10	0.798
	$[1.9, 2.1]$	1.70	2.75	10.97	0.802
	$[\mathbf{1.8}, \mathbf{2.2}]$	*1.68*	*2.75*	*10.91*	*0.803*
16-24	$[2.0, 2.0]$	3.17	8.25	4.79	0.231
	$[\mathbf{1.9}, \mathbf{2.1}]$	*3.04*	*8.10*	*4.74*	*0.233*
	$[1.8, 2.2]$	3.11	8.32	4.87	0.230
25-39	$[2.0, 2.0]$	3.03	7.80	4.40	0.278
	$[\mathbf{1.9}, \mathbf{2.1}]$	*3.00*	*7.76*	*4.38*	*0.283*
	$[1.8, 2.2]$	3.04	7.96	4.46	0.2801
40-49	$[2.0, 2.0]$	*2.45*	6.00	*3.69*	0.334
	$[\mathbf{1.9}, \mathbf{2.1}]$	*2.45*	*5.99*	*3.69*	*0.335*
	$[1.8, 2.2]$	2.54	6.30	3.89	0.330
50-59	$[2.0, 2.0]$	*2.44*	5.35	*4.95*	0.418
	$[\mathbf{1.9}, \mathbf{2.1}]$	2.45	*5.34*	*4.95*	*0.425*
	$[1.8, 2.2]$	2.48	5.45	5.05	0.417
60+	$[2.0, 2.0]$	*1.32*	2.53	6.87	0.144
	$[\mathbf{1.9}, \mathbf{2.1}]$	*1.32*	*2.51*	*6.86*	0.144
	$[1.8, 2.2]$	1.34	2.54	6.92	*0.148*

Table 2. Testing errors produced by the $\mathcal{W}_F^r$ for the each age group of the PEA index.

Age Bracket	MAE	RMSE	MRE(%)	ρ
10-15	0.90	0.60	8.66	0.870
16-24	0.62	0.50	0.66	0.861
25-39	0.50	0.35	0.41	0.809
40-49	0.62	0.46	0.58	0.876
50-59	0.91	0.70	1.13	0.920
60+	0.83	0.66	3.04	0.624

Table 3. Testing errors produced by the IT2 Mamdani inference system for the each age group of the PEA index.

Age Bracket	MAE	RMSE	MRE(%)	ρ
10-15	1.15	0.91	13.16	0.791
16-24	0.78	0.63	0.84	0.754
25-39	0.60	0.45	0.53	0.710
40-49	0.82	0.64	0.81	0.804
50-59	1.40	1.17	1.87	0.849
60+	0.97	0.76	3.49	0.447

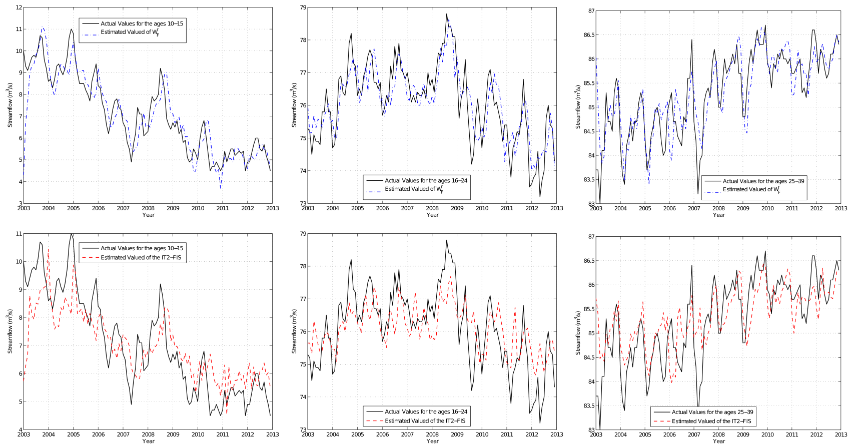

Fig. 1. Predictions obtained by $\mathcal{W}_F^r$ (top row) and the IT2-FIS (bottom row) for the age groups: 10-15, 16-24 and 25-39, from left to right.

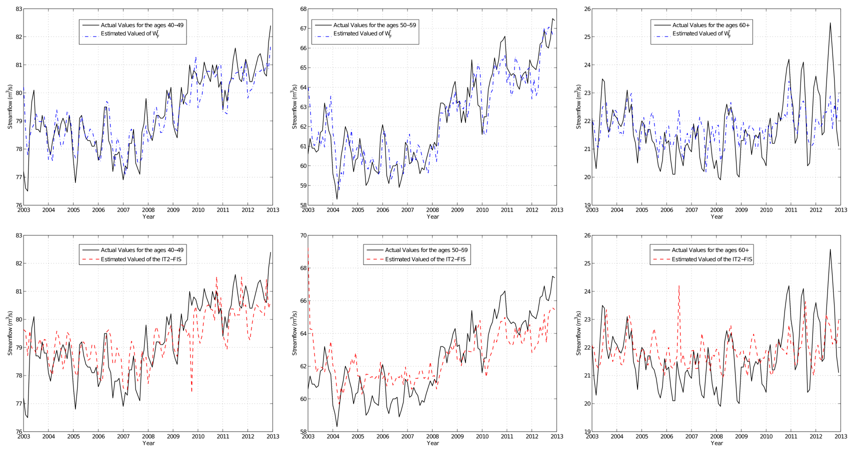

Fig. 2. Predictions obtained by $\mathcal{W}_F^r$ (top row) and the IT2-FIS (bottom row) for the age groups: 40-49, 50-59 and 60+, from left to right.

5 Concluding Remarks

Type-2 fuzzy systems have been successfully employed in a variety of applications for their capability of handling uncertainties that are intrinsic in real-world data better then traditional (type-1) fuzzy systems [3]. In particular, interval-valued FSs have found far more applications then full type-2 FS due to their simplicity and to the merely linear increase in computational complexity in comparison to type-1 fuzzy systems [13,25].

We applied in this paper our alternative approach towards an IV fuzzy system, namely the sup-$\mathcal{C}_C^r$ IV-FMAM, to the problem of forecasting the monthly rates of participation of given age groups in the work force of the metropolitan area of São Paulo. To this purpose, we generated an IV fuzzy inference system based on the aforementioned IV-FMAMs. The same methodology was employed in conjunction with an Mamdani(-Assilian) IT-2 FIS. In our simulations the sup-$\mathcal{C}_C^r$ IV-FMAM approach exhibited significantly better results than the interval type-2 fuzzy system with respect to four different performance measures.

Note that the IV-FMAM approach presented in this paper depends on the complete lattice structure of $\mathbb{I}$ and in particular on the fact that elements of $\mathbb{I}$ were partially ordered in terms of Equation 1. In the future, we intend to investigate the suitability of other partial orders in conjunction with IV-FAMs. Furthermore, we plan to develop full type-2 fuzzy associative memories (T2-FAMs) as particular cases of $\mathbb{L}$-fuzzy associative memories and use them to build full type-2 fuzzy inference systems.

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