# An invitation to Study Fuzzy and Generalized Uncertainty Optimization

## A Historical and Contemporary View

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# MOTIVATION AND OBJECTIVE

- One of the most significant features of human beings is the decision-making of everyday problems.

- The database of the practical problems, in many cases, have approximate and/or imprecise values.

- The goal of this course is to present a brief description how to use fuzzy set and possibility theories in optimization methods.

# OUTLINE

- Generalized Uncertain Mathematical Programming – the beginning

- Approaches in uncertain environment
  - costs in the objective function and/or;
  - coefficients of the set of constraints

# WHERE EVERYTHING BEGAN

Fuzzy Sets and Systems 100 Supplement (1999) 9–34 9
North-Holland

**FUZZY SETS AS A BASIS FOR A THEORY OF POSSIBILITY***

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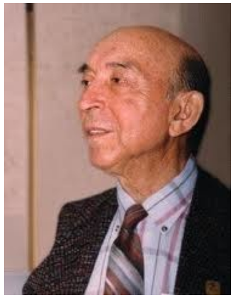
Received February 1977
Revised June 1977

The theory of possibility described in this paper is related to the theory of fuzzy sets by defining the concept of a possibility distribution as a fuzzy restriction which acts as an elastic constraint on the values that may be assigned to a variable. More specifically, if $F$ is a fuzzy subset of a universe of discourse $U = \{u\}$ which is characterized by its membership function $\mu_F$, then a proposition of the form "$X$ is $F$," where $X$ is a variable taking values in $U$, induces a possibility distribution $\Pi_X$ which



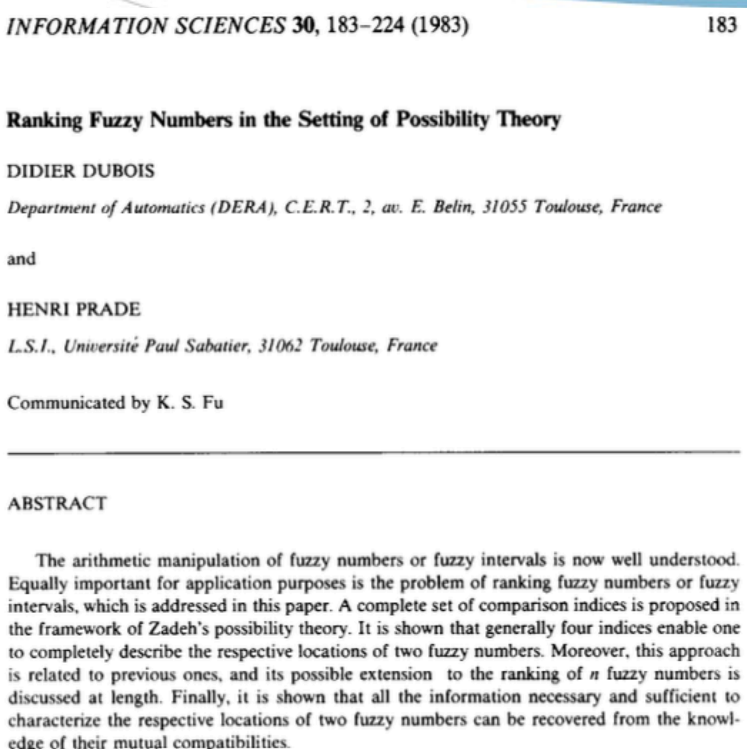


**Lotfi A. Zadeh**

**"Fuzzy Sets as a basis for a Theory of Possibility"**
***Fuzzy Sets and Systems*. 1 3–28. 1978**

# RANKING FUZZY NUMBERS IN THE SETTING OF POSSIBILITY THEORY



*INFORMATION SCIENCES* **30**, 183–224 (1983) 183

**Ranking Fuzzy Numbers in the Setting of Possibility Theory**

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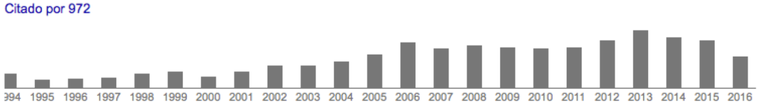
Communicated by K. S. Fu

ABSTRACT

The arithmetic manipulation of fuzzy numbers or fuzzy intervals is now well understood. Equally important for application purposes is the problem of ranking fuzzy numbers or fuzzy intervals, which is addressed in this paper. A complete set of comparison indices is proposed in the framework of Zadeh's possibility theory. It is shown that generally four indices enable one to completely describe the respective locations of two fuzzy numbers. Moreover, this approach is related to previous ones, and its possible extension to the ranking of $n$ fuzzy numbers is discussed at length. Finally, it is shown that all the information necessary and sufficient to characterize the respective locations of two fuzzy numbers can be recovered from the knowledge of their mutual compatibilities.








**Didier Dubois**

**Henri Prade**

**"Ranking Fuzzy Numbers in the Setting of Possibility Theory"**

***Information Sciences*. 30 183–224. 1983**

# RANKING FUZZY NUMBERS IN THE SETTING OF POSSIBILITY THEORY

- Let *U* be a set of elementary event and an event *A*, which is contained. The possibility measure is defined by

$$\Pi(\varnothing)=0, \qquad \Pi(U)=1,$$

$$\forall A, B \in \mathscr{P}(U), \qquad \Pi(A \cup B)=\max(\Pi(A), \Pi(B)).$$

- Given a normalized fuzzy set *F*,

$$\Pi_F(A)=\sup_{u \in A} \mu_F(u) \qquad \forall A \subseteq U$$

# RANKING FUZZY NUMBERS IN THE SETTING OF POSSIBILITY THEORY

- When both *A* and *F* are fuzzy,

$$\Pi_F(A) = \sup_u \min(\mu_F(u), \mu_A(u)).$$

- An extension is interpreted in terms of the intersection of the level cuts of *F* and *A*

$$\Pi_F(A) = \sup_{\alpha,\beta} \min\left(\alpha, \beta, \Pi_{F_\alpha}(A_\beta)\right)$$

- when $A_\beta = \{u \mid \mu_A(u) \geq \beta\}$, . $F_\alpha = \{u \mid \mu_F(u) \geq \alpha\}$

# RANKING FUZZY NUMBERS IN THE SETTING OF POSSIBILITY THEORY

- The necessity measure is a set function defined by

$$\mathcal{N}(\varnothing)=0, \qquad \mathcal{N}(U)=1,$$

$$\mathcal{N}(A\cap B)=\min(\mathcal{N}(A),\mathcal{N}(B)) \qquad \forall A,B\subseteq U$$

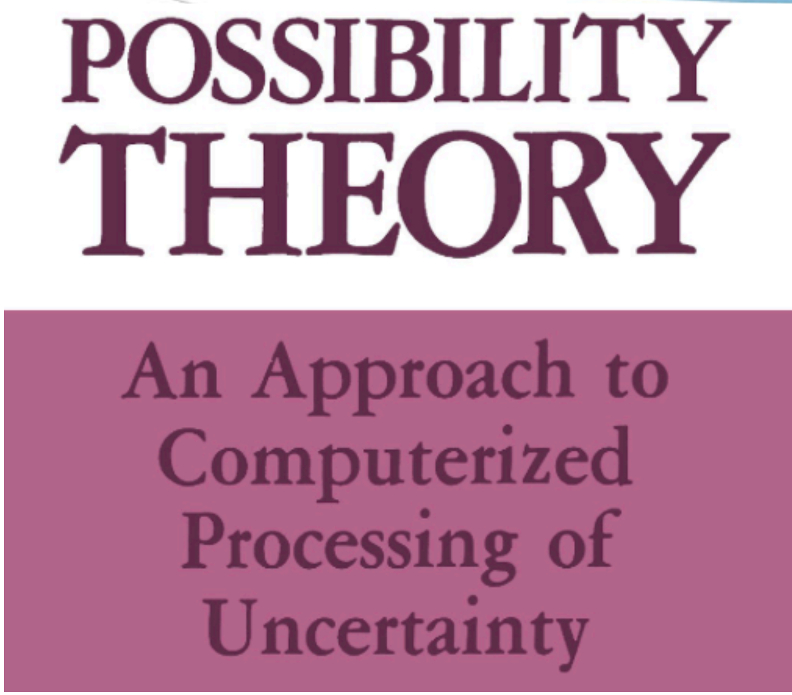
- Let the complementary set of A, and a possibility measure, the necessary measure can be defined by

$$\forall A\subseteq U, \qquad \mathcal{N}(A)\triangleq 1-\Pi(\bar{A})$$

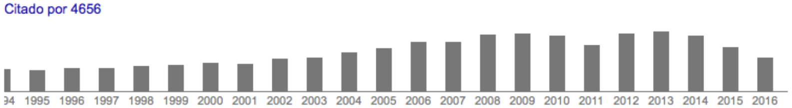
- If the possibility measure derived from a normalized membership function.

$$\forall A, \qquad \mathcal{N}_F(A)\triangleq 1-\Pi_F(\bar{A})=\inf_{u\in\bar{A}} 1-\mu_F(u)$$

# POSSIBILITY THEORY






Possibility theory: an approach to computerized processing of uncertainty *
D Dubois, H Prade - 2012


**Didier Dubois**
**Henri Prade**
**"Possibility Theory"**
***Plemun Press*, 1988.**

# FUZZY LINEAR PROGRAMMING PROBLEMS WITH FUZZY NUMBERS

Fuzzy Sets and Systems 13 (1984) 1–10 1
North-Holland

**FUZZY LINEAR PROGRAMMING PROBLEMS WITH FUZZY NUMBERS**

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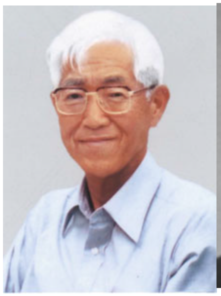
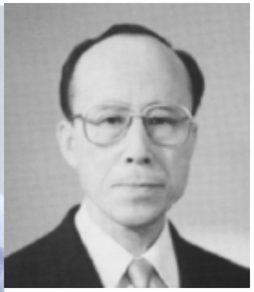
Revised July 1983

Linear programming problems with fuzzy parameters are formulated by fuzzy functions. The ambiguity considered here is not randomness, but fuzziness which is associated with the lack of a sharp transition from membership to nonmembership. Parameters on constraint and objective functions are given by fuzzy numbers. In this paper, our object is the formulation of a fuzzy linear programming problem to obtain a reasonable solution under consideration of the ambiguity of parameters. This fuzzy linear programming problem with fuzzy numbers can be regarded as a model of decision problems where human estimation is influential.

*Keywords:* Fuzzy linear programming, Fuzzy numbers, Fuzzy parameters, Fuzzy decision, Fuzzy function.







**Hideo Tanaka**
**Kiyoji Asai**

"Fuzzy Linear Programming Problems with Fuzzy Numbers"
*Fuzzy Sets and Systems*. Vol. 13, 1-10, 1984

# FUZZY LINEAR PROGRAMMING PROBLEMS WITH FUZZY NUMBERS

**Definition 1.** The fuzzy function is denoted by

$$f: X \to \mathscr{F}(Y), \quad \tilde{Y} = f(x, \tilde{A}), \tag{4}$$

where $\mathscr{F}(Y)$ is the set of all fuzzy subsets on $Y$. The fuzzy set $\tilde{Y}$ is defined by the membership function

$$\mu_Y(y) = \begin{cases} \max\limits_{\{a \mid y = f(x,a)\}} [\mu_A(a)], & \{a \mid y = f(x, a)\} \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

where $\tilde{A}$ is a fuzzy set on the product space consisting of $[a_1 \times a_2 \times \cdots \times a_n]$ whose membership function is denoted by $\mu_A(a)$.

# FUZZY LINEAR PROGRAMMING PROBLEMS WITH FUZZY NUMBERS

**Definition 2.** Fuzzy parameters are defined by such fuzzy sets as illustrated in Figure 1. These fuzzy sets can be represented as

$$\mu_{\mathbf{A}}(a) = \min_j [\mu_{\mathbf{A}_i}(a_j)], \tag{6}$$

$$\mu_{\mathbf{A}_i}(a_j) = \begin{cases} 1 - \dfrac{|\alpha_j - a_j|}{c_j}, & \alpha_i - c_j \leqslant a_j \leqslant \alpha_i + c_j, \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

where $c_j > 0$.

# FUZZY LINEAR PROGRAMMING PROBLEMS WITH FUZZY NUMBERS

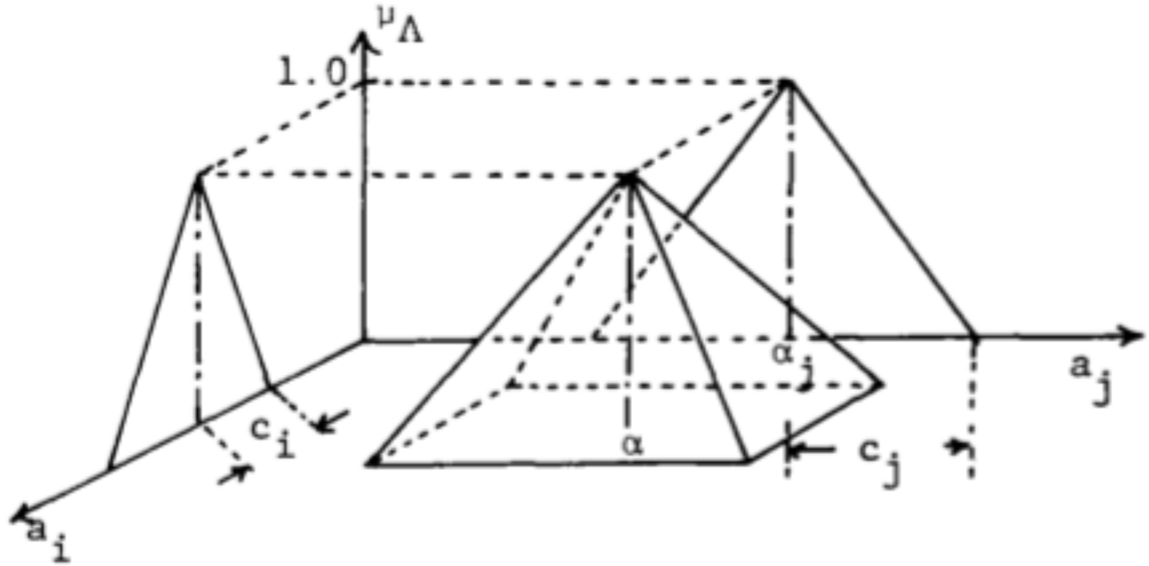



Fig. 1. Fuzzy set of the parameter 'approximately $\alpha$'.

# FUZZY LINEAR PROGRAMMING PROBLEMS WITH FUZZY NUMBERS

**Proposition 1.** *Given the fuzzy parameter $\tilde{A} = \{\alpha, c\}$, the fuzzy linear function*

$$\tilde{Y} = \tilde{A}_1 x_1 + \cdots + \tilde{A}_n x_n = \tilde{A}x \tag{10}$$

*is obtained as the following membership function (see Appendix):*

$$\mu_Y(y) = \begin{cases} 1 - \dfrac{|y - x^t\alpha|}{c^t\,|x|}, & x \neq 0, \\ 1, & x = 0,\ y = 0, \\ 0, & x = 0,\ y \neq 0, \end{cases} \tag{11}$$

*where $|x| = (|x_1|, \ldots, |x_n|)^t$ and $\mu_Y(y) = 0$ if $c^t\,|x| \leqslant |y - x^t\alpha|$.*

**Definition 3.** '$\tilde{Y}_i$ is almost positive' denoted by $\tilde{Y}_i \geqslant 0$ is defined by

$$\tilde{Y}_i \geqslant 0 \ \Leftrightarrow\ \mu_{Y_i}(0) \leqslant 1 - h,\ x^t\alpha_i \geqslant 0, \tag{16}$$

where $h$ stands for the degree of $\tilde{Y}_i \geqslant 0$ and the larger $h$ is, the stronger the meaning of 'almost positive' is (see Figure 2).

# FUZZY MATHEMATICAL PROGRAMMING THE BEGINNING

Fuzzy Sets and Systems 24 (1987) 363–375
North-Holland



**FUZZY DATA ANALYSIS BY POSSIBILISTIC LINEAR MODELS**

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Received November 1985
Revised June 1986

Since fuzzy data can be regarded as distribution of possibility, fuzzy data analysis by possibilistic linear models is proposed in this paper. Possibilistic linear systems are defined by the extension principle. Fuzzy parameter estimations are discussed in possibilistic linear systems and possibilistic linear models are employed for fuzzy data analysis with non-fuzzy inputs and fuzzy outputs defined by fuzzy numbers. The estimated possibilistic linear system can be obtained by solving a linear programming problem. This approach can be regarded as fuzzy interval analysis.

*Keywords:* Possibilistic linear systems, Fuzzy data analysis, Fuzzy parameter estimation, Fuzzy interval analysis, Fuzzy linear regression models.



**Hideo Tanaka**

"Fuzzy Data Analysis by Possibilistic Linear Models" *Fuzzy Sets and Systems*. Vol. 24, 363-375, 1987

# FUZZY MATHEMATICAL PROGRAMMING THE BEGINNING

**Definition 2.2.** Given a function $y = f(x_1, x_2)$, the fuzzy output $Y = f(A, B)$ is defined by the following membership function:

$$\mu_Y(y) = \sup_{y=f(x_1,x_2)} \mu_A(x_1) \wedge \mu_B(x_2). \tag{2.6}$$

**Definition 2.3.** The inclusion of fuzzy numbers with degree $0 \leqslant h < 1$ denoted $A \supsetneq_h B$ is defined by $[A]_h \supseteq [B]_h$.

It follows from Definition 2.3 that $[A_h] \supseteq [B]_h$ is equivalent to

$$\alpha_1 \leqslant \alpha_2 + |L^{-1}(h)|\,(c_1 - c_2), \qquad \alpha_1 \geqslant \alpha_2 - |L^{-1}(h)|\,(c_1 - c_2) \tag{2.7}$$

where $\mu_A(x) = L((x - \alpha_1)/c_1)$ and $\mu_B(x) = L((x - \alpha_2)/c_2)$.

# FUZZY MATHEMATICAL PROGRAMMING THE BEGINNING

Fuzzy Sets and Systems 30 (1989) 257–282
North-Holland



## FUZZY OPTIMIZATION: AN APPRAISAL

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Dedicated to Professor H.-J. Zimmermann

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Revised September 1987

This paper takes a general look at core ideas that make up the burgeoning body of Fuzzy mathematical programming emphasizing the methodological view.
Although Fuzzy mathematical programming has enjoyed a rapidly increasing acceptance within the scientific community, some technical hurdles exist to hinder a unanimity. Reasons for this as well as possible ways for improvement are also discussed.

*Keywords:* Fuzzy, Optimization, Decision-making.






**M . K. Luhandjula**

"Fuzzy Optimization: An Appraisal"
*Fuzzy Sets and Systems*
*Vol.* 30, pp. 257-282, 1989.

# FUZZY MATHEMATICAL PROGRAMMING THE BEGINNING

Fuzzy Sets and Systems 31 (1989) 329–341
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329

**SOLVING POSSIBILISTIC LINEAR PROGRAMMING PROBLEMS**

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Received July 1987
Revised January 1988

*Abstract:* We present a solution to certain possibilistic linear programming problems which have been previously introduced in [1].

*Keywords:* Fuzzy programming; possibility distributions; fuzzy numbers.

Solving possibilistic linear programming problems - ScienceDirect
www.sciencedirect.com/science/article/pii/0165011489902042 ▾ Traduzir esta página
de JJ Buckley - 1989 - Citado por 159 - Artigos relacionados
21 de mai de 2003 - In this paper we present an alternate method to **solve** certain **possibilistic linear programming problems**. First, in this section, we will briefly ...

**J. J. Buckley**

**"Solving Possibilistic Linear Programming Problems"**
***Fuzzy Sets and Systems***
***Vol.* 31, pp. 329-341, 1989.**

# LINEAR PROGRAMMING WITH FUZZY OBJECTIVES

Fuzzy Sets and Systems 29 (1989) 31–48 31
North-Holland

## LINEAR PROGRAMMING WITH FUZZY OBJECTIVES

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Received November 1985
Revised February 1987

This paper presents a new method for solving linear programming problems with fuzzy parameters in the objective function. To determine a compromise solution the authors do not reduce the set of infinitely many objective functions to a single 'compromise objective function' as done in classical deterministic or stochastic procedures. Here the problem is only reduced to a few extreme objective functions. The information contained in the membership functions can be used to any extent by a method called '$\alpha$-level related pair formation'. Moreover, if different level sets are considered the stability of the optimal solution can be tested.

*Keywords:* Fuzzy linear programming, Decision making, Multicriteria analysis.

Linear programming with fuzzy objectives - ScienceDirect
www.sciencedirect.com/science/article/pii/0165011489901346 ▾ Traduzir esta página
de H Rommelfanger - 1989 - Citado por 285 - Artigos relacionados
20 de mai de 2003 - This paper presents a new method for solving **linear programming** problems with **fuzzy** parameters in the **objective** function. To determine a ...

H. Rommelfanger
R. Hanuscheck
J. Wolf

"Linear Programming with Fuzzy Objectives"
*Fuzzy Sets and Systems*
Vol 29, pp. 31-48, 1989.

# POSSIBILISTIC LINEAR PROGRAMMING A BRIEF REVIEW …

ELSEVIER

Fuzzy Sets and Systems 111 (2000) 3–28

FUZZY sets and systems

www.elsevier.com/locate/fss

## Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem

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Received October 1998

**Abstract**

In this paper, we review some fuzzy linear programming methods and techniques from a practical point of view. In the first part, the general history and the approach of fuzzy mathematical programming are introduced. Using a numerical example, some models of fuzzy linear programming are described. In the second part of the paper, fuzzy mathematical programming approaches are compared to stochastic programming ones. The advantages and disadvantages of fuzzy mathematical programming approaches are exemplified in the setting of an optimal portfolio selection problem. Finally, some newly developed ideas and techniques in fuzzy mathematical programming are briefly reviewed. 

*Keywords:* Fuzzy mathematical programming; Fuzzy constraint; Fuzzy goal; Possibility measure; Necessity measure; Simplex method; Stochastic programming; Portfolio selection

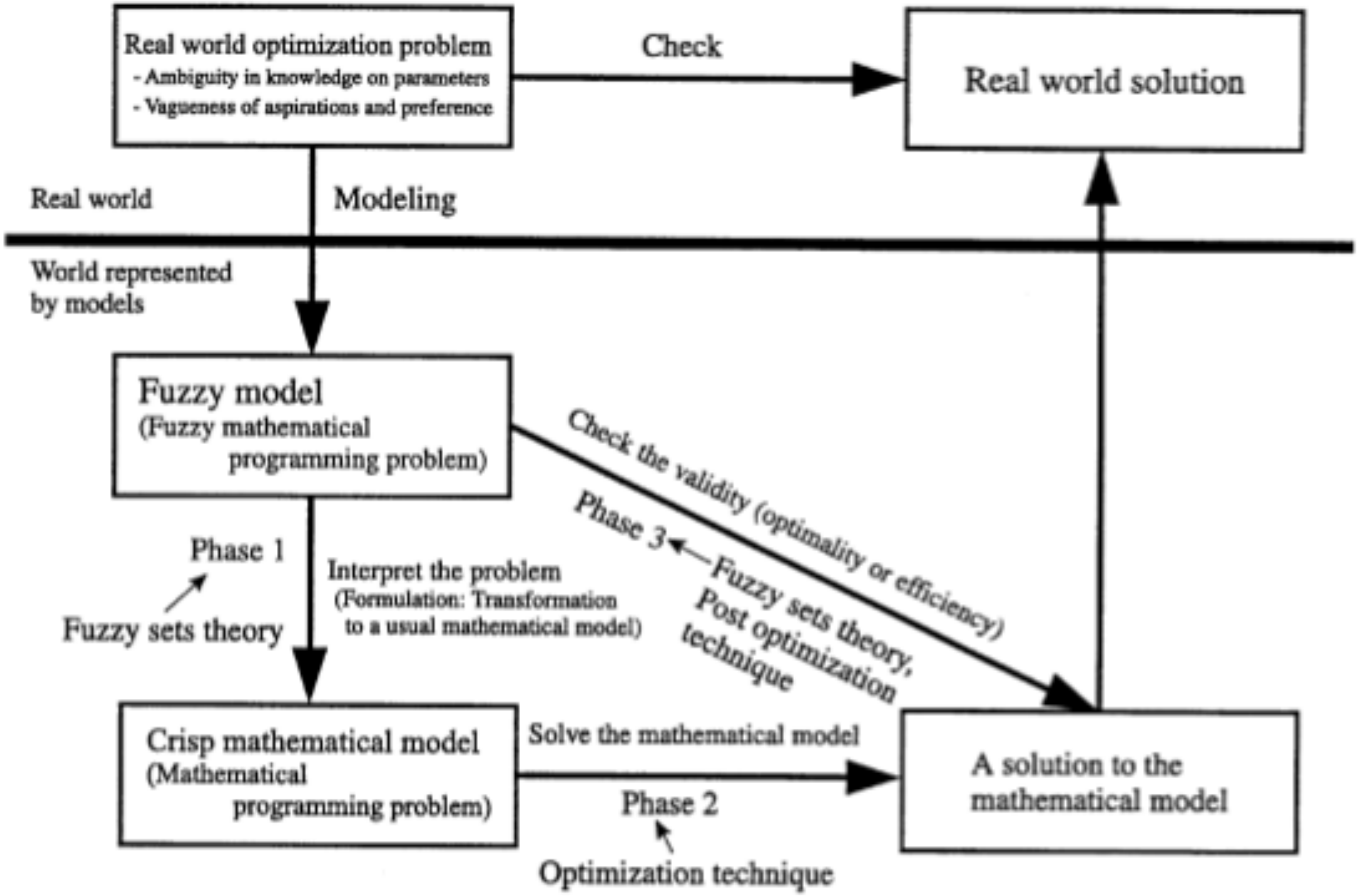



Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem
M Inuiguchi, J Ramık - Fuzzy sets and systems, 2000


**M. Inuiguchi**
**J. Ramik**

**"Possilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem."**
***Fuzzy Sets and Systems***
**Vol 111, pp. 3-28, 2000.**

# POSSIBILISTIC LINEAR PROGRAMMING A BRIEF REVIEW …



# RELATING DIFFERENT APPROACHES

Fuzzy Sets and Systems 37 (1990) 33–42 33
North-Holland

**RELATING DIFFERENT APPROACHES TO SOLVE LINEAR PROGRAMMING PROBLEMS WITH IMPRECISE COSTS**

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Received February 1988
Revised April 1989

*Abstract:* Linear programming problems involving coefficients in the objective function with some lack of precision are usual. To solve them several approaches have been proposed. In this paper the different methods studied to solve the problem are briefly described. Next, it is shown that the different solutions obtained from them may be obtained by solving a multiobjective linear programming problem.

*Keywords:* Mathematical programming; fuzzy linear programming; fuzzy objectives.



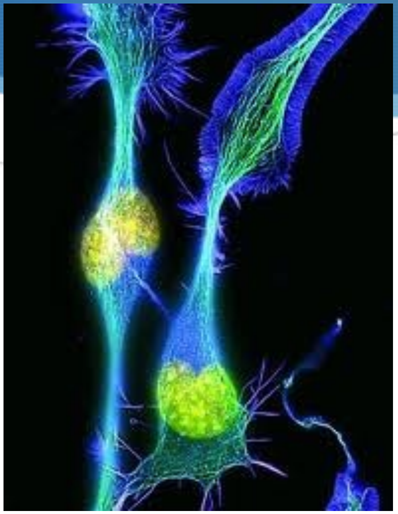





**Miguel Delgado**
**J.L. Verdegay**
**Amparo Vila**

**"Relating Different Approaches to Solve Linear Programming Problems with Imprecise Costs."**
***Fuzzy Sets and Systems*, Vol. 37, pp 33-42, 1990.**

# FUZZY MATHEMATICAL PROGRAMMING



# FUZZY QUADRATIC PROGRAMMING

Fuzzy Optim Decis Making (2011) 10:11–30
DOI 10.1007/s10700-010-9092-z

**The use of possibility theory in the definition of fuzzy Pareto-optimality**

**Ricardo C. Silva · Akebo Yamakami**

Published online: 1 December 2010


**Abstract** Pareto-optimality conditions are crucial when dealing with classic multi-objective optimization problems. Extensions of these conditions to the fuzzy domain have been discussed and addressed in recent literature. This work presents a novel approach based on the definition of a fuzzily ordered set with a view to generating the necessary conditions for the Pareto-optimality of candidate solutions in the fuzzy domain. Making use of the conditions generated, one can characterize fuzzy efficient solutions by means of carefully chosen mono-objective problems and Karush-Kuhn-Tucker conditions to fuzzy non-dominated solutions. The uncertainties are inserted into the formulation of the studied fuzzy multi-objective optimization problem by means of fuzzy coefficients in the objective function. Some numerical examples are analytically solved to illustrate the efficiency of the proposed approach.

**Keywords** Fuzzy logic · Possibility theory · Multi-objective optimization · Fuzzy Pareto-optimality conditions · Fuzzy optimization

$$\widetilde{\min}\ F(\tilde{\mathbf{a}};\mathbf{x}) \quad \text{s.t. } \mathbf{x} \in \tilde{\Omega}$$

$$Poss\{\tilde{a}_1 R^f \tilde{a}_2\} = \sup_{u,v \in U;\, u \leq v} \min(\mu_{\tilde{a}_1}(u), \mu_{\tilde{a}_2}(v))$$

# FUZZY QUADRATIC PROGRAMMING

**Definition 3** *(Ordered fuzzily set)* A fuzzy subset $A \subset \mathbb{F}(\mathbb{R})$ is fuzzily ordered with respect to the possibility measure if each element in $A$ satisfies the following basic properties:

1. $Poss[\tilde{\mathbf{a}}_1 \leq \tilde{\mathbf{a}}_1] = 1$;
2. $Poss[\tilde{\mathbf{a}}_1 \leq \tilde{\mathbf{a}}_2] \geq \alpha_1$ and $Poss[\tilde{\mathbf{a}}_2 \leq \tilde{\mathbf{a}}_3] \geq \alpha_2 \Rightarrow Poss[\tilde{\mathbf{a}}_1 \leq \tilde{\mathbf{a}}_3] \geq \min\{\alpha_1, \alpha_2\}$;
3. $Poss[\tilde{\mathbf{a}}_1 \leq \tilde{\mathbf{a}}_2] \geq \alpha_1$ and $Poss[\tilde{\mathbf{a}}_2 \leq \tilde{\mathbf{a}}_1] \geq \alpha_2 \Rightarrow Poss[\tilde{\mathbf{a}}_1 = \tilde{\mathbf{a}}_2] \geq \min\{\alpha_1, \alpha_2\}$;

$\forall\, \tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \tilde{\mathbf{a}}_3 \in A$ and $\forall\, \alpha_1, \alpha_2 \in [0, 1]$.

$$\begin{aligned}
\Omega_{<}(\mathbf{x}^0; \alpha) &\triangleq \{\mathbf{x} \in \mathbb{R}^n : Poss[F(\tilde{\mathbf{a}}; \mathbf{x}) \leq F(\tilde{\mathbf{a}}; \mathbf{x}^0)] \geq \alpha \text{ and} \\
&\qquad Poss[F(\tilde{\mathbf{a}}; \mathbf{x}) = F(\tilde{\mathbf{a}}; \mathbf{x}^0)] < 1\} \\
\Omega_{\geq}(\mathbf{x}^0; \alpha) &\triangleq \{\mathbf{x} \in \mathbb{R}^n : Poss[F(\tilde{\mathbf{a}}; \mathbf{x}) \geq F(\tilde{\mathbf{a}}; \mathbf{x}^0)] \geq \alpha\} \\
\Omega_{\sim}(\mathbf{x}^0; \alpha) &\triangleq \{\mathbf{x} \in \mathbb{R}^n : \max\{Poss[F(\tilde{\mathbf{a}}; \mathbf{x}) \leq F(\tilde{\mathbf{a}}; \mathbf{x}^0)], \\
&\qquad Poss[F(\tilde{\mathbf{a}}; \mathbf{x}) \geq F(\tilde{\mathbf{a}}; \mathbf{x}^0)]\} \leq \alpha\}
\end{aligned}$$

# FUZZY QUADRATIC PROGRAMMING

**Definition 5** *(Fuzzy Pareto-optimal solution)* $\mathbf{x}^* \in \Omega$ is said be a fuzzy Pareto-optimal solution if there exists no other $x \in \Omega$ such that $Poss[f_i(\tilde{\mathbf{a}}_i; \mathbf{x}) \leq f_i(\tilde{\mathbf{a}}_i; \mathbf{x}^*)] \geq \alpha_i,\ \forall i$ and $Poss[f_j(\tilde{\mathbf{a}}_j; \mathbf{x}) = f_j(\tilde{\mathbf{a}}_j; \mathbf{x}^*)] < 1$ for at least one $j$, where $\alpha_i \in [0, 1],\ \forall i$.

**Definition 6** *(Fuzzy local Pareto-optimal solution)* $\mathbf{x}^* \in \Omega$ is said to be a fuzzy local Pareto-optimal solution if there is a real number $\delta \geq 0$ such that there exists no other $x \in \Omega \cap \mathcal{N}(x^*, \delta)$ such that $Poss[f_i(\tilde{\mathbf{a}}_i; \mathbf{x}) \leq f_i(\tilde{\mathbf{a}}_i; \mathbf{x}^*)] \geq \alpha_i,\ \forall i$ and $Poss[f_j(\tilde{\mathbf{a}}_j; \mathbf{x}) = f_j(\tilde{\mathbf{a}}_j; \mathbf{x}^*)] < 1$ in at least one $j$, where $\alpha_i \in [0, 1],\ \forall i$.

# FUZZY QUADRATIC PROGRAMMING

Fuzzy Optim Decis Making (2013) 12:231–248
DOI 10.1007/s10700-013-9153-1

## Fuzzy costs in quadratic programming problems

Ricardo C. Silva · Carlos Cruz · José L. Verdegay

$$\begin{aligned} \min\ & \tilde{\mathbf{c}}^t\mathbf{x} + \tfrac{1}{2}\mathbf{x}^t\tilde{\mathbf{Q}}\mathbf{x} \\ \text{s.t.}\ & \mathbf{A}\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge 0 \end{aligned}$$

Published online: 16 March 2013


**Abstract** Although quadratic programming problems are a special class of nonlinear programming, they can also be seen as general linear programming problems. These quadratic problems are of the utmost importance in an increasing variety of practical fields. As, in addition, ambiguity and vagueness are natural and ever-present in real-life situations requiring operative solutions, it makes perfect sense to address them using fuzzy concepts formulated as quadratic programming problems with uncertainty, i.e., as Fuzzy Quadratic Programming problems. This work proposes two novel fuzzy-sets-based methods to solve a particular class of Fuzzy Quadratic Programming problems which have vagueness coefficients in the objective function. Moreover, two other linear approaches are extended to solve the quadratic case. Finally, it is shown that the solutions reached from the extended approaches may be obtained from two proposed parametric multiobjective approaches.

$$\mu_j, \mu_{ij} : \mathbb{R} \to [0, 1], \quad i, j \in \mathbb{J} = \{1, 2, \ldots, n\}$$

**Keywords** Fuzzy set · Decision making · Fuzzy mathematical optimization · Quadratic programming · Efficient solutions

# FUZZY QUADRATIC PROGRAMMING

$$\mu_j(y) = \begin{cases} 0 & \text{if } c_j^U \leq y \text{ or } y \leq c_j^L \\ h_j(y) & \text{if } c_j^L \leq y \leq c_j \\ g_j(y) & \text{if } c_j \leq y \leq c_j^U \end{cases} \qquad j \in \mathbb{J}$$

$$\mu_{ij}(y) = \begin{cases} 0 & \text{if } q_{ij}^U \leq y \text{ or } y \leq q_{ij}^L \\ h_{ij}(y) & \text{if } q_{ij}^L \leq y \leq q_{ij} \\ g_{ij}(y) & \text{if } q_{ij} \leq y \leq q_{ij}^U \end{cases} \qquad i, j \in \mathbb{J}$$

Then by considering the $\alpha$-cut of every cost, $\alpha \in [0, 1]$,

$$\forall x \in \mathbb{R}, \quad \mu_j(x) \geq \alpha \Leftrightarrow h_j^{-1}(\alpha) \leq x \leq g_j^{-1}(\alpha), \quad j \in \mathbb{J}$$

$$\forall x \in \mathbb{R}, \quad \mu_{ij}(x) \geq \alpha \Leftrightarrow h_{ij}^{-1}(\alpha) \leq x \leq g_{ij}^{-1}(\alpha), \quad i, j \in \mathbb{J}$$

# FUZZY QUADRATIC PROGRAMMING

$$\begin{aligned}
\min\ & \Big[(\mathbf{c}^1)^t\mathbf{x} + \tfrac{1}{2}\mathbf{x}^t\mathbf{Q}^1\mathbf{x}, (\mathbf{c}^2)^t\mathbf{x} + \tfrac{1}{2}\mathbf{x}^t\mathbf{Q}^1\mathbf{x}, \ldots, (\mathbf{c}^{2^n})^t\mathbf{x} + \tfrac{1}{2}\mathbf{x}^t\mathbf{Q}^1\mathbf{x}, \\
& (\mathbf{c}^1)^t\mathbf{x} + \tfrac{1}{2}\mathbf{x}^t\mathbf{Q}^2\mathbf{x}, \ldots, (\mathbf{c}^{2^n})^t\mathbf{x} + \tfrac{1}{2}\mathbf{x}^t\mathbf{Q}^2\mathbf{x}, \ldots, (\mathbf{c}^{2^n})^t\mathbf{x} + \tfrac{1}{2}\mathbf{x}^t\mathbf{Q}^{2^{\left(\frac{n^2+n}{2}\right)}}\mathbf{x}\Big] \\
\text{s.t.}\ & \mathbf{A}\mathbf{x} \le \mathbf{b}, \quad \mathbf{x} \ge \mathbf{0}, \\
& \mathbf{c}^k, \quad \mathbf{Q}^p \in \mathbb{E}(\alpha), \quad \alpha \in [0,1], k = 1, 2, \ldots, 2^n \text{ and } p = 1, 2, \ldots, 2^{\left(\frac{n^2+n}{2}\right)},
\end{aligned}$$

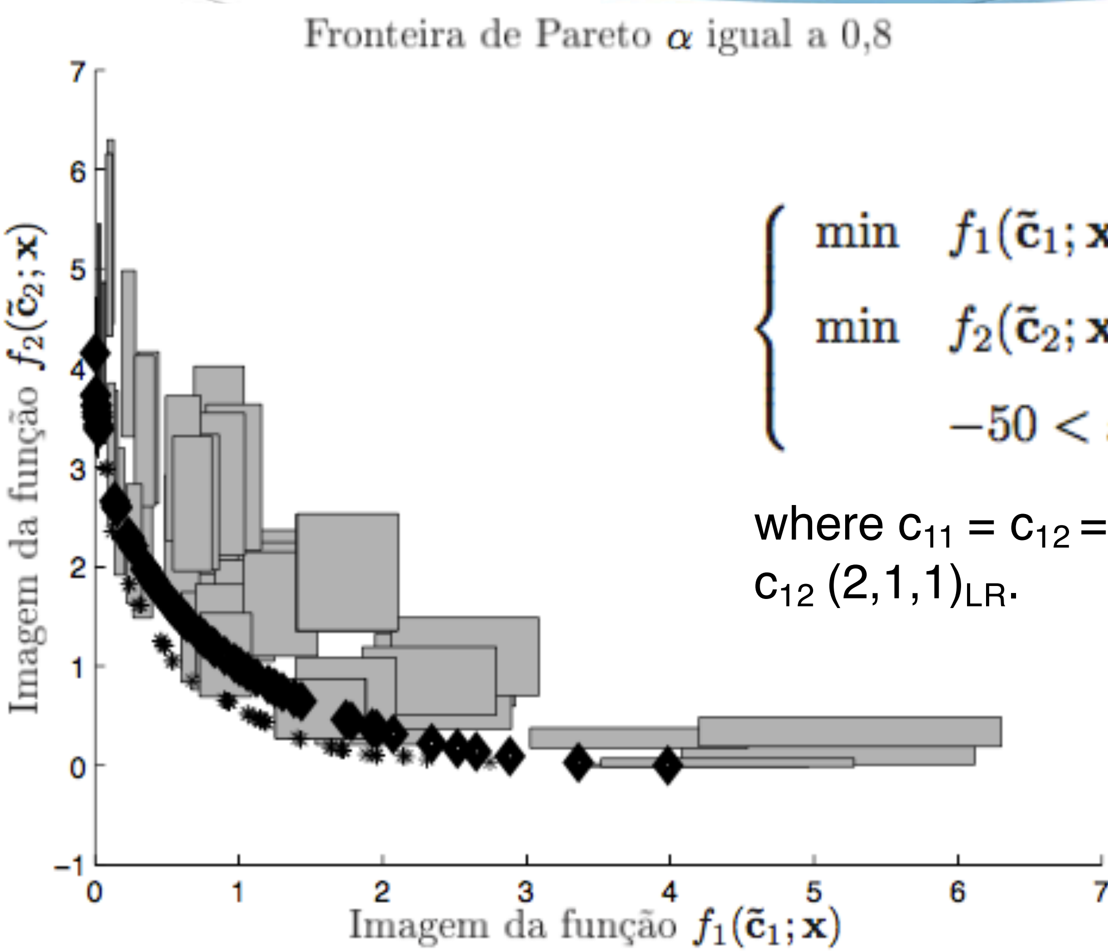
That is, $\forall k = 1, 2, \ldots, 2^n$ and $\forall p = 1, 2, \ldots, 2^{\left(\frac{n^2+n}{2}\right)}$, like is (14) and (17)

$$\mathbf{c}^k = (c_1^k, c_2^k, \ldots, c_n^k) \in \mathbb{E}(\alpha) \Leftrightarrow c_j^k = h_j^{-1}(\alpha) \text{ or } g_j^{-1}(\alpha), \quad \forall j \in \mathbb{J}$$

and

$$\mathbf{Q}^p = (q_{11}^p, \ldots, q_{1n}^p, \ldots, q_{nn}^p) \in \mathbb{E}(\alpha) \Leftrightarrow q_{ij}^k = h_{ij}^{-1}(\alpha) \text{ or } g_{ij}^{-1}(\alpha), \quad \forall i, j \in \mathbb{J}$$

# NUMERICAL EXAMPLES
## UNCONSTRAINTS PROBLEMS

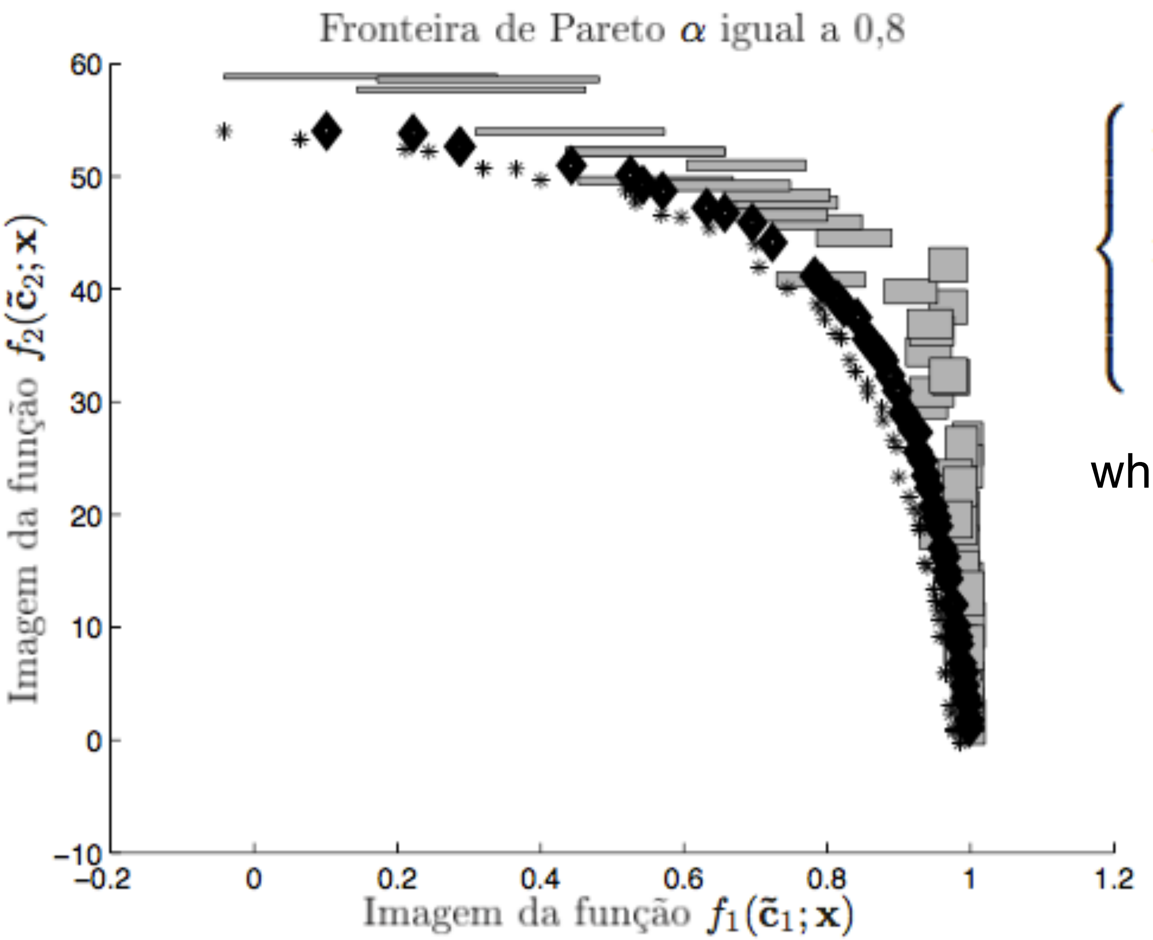



$$\begin{cases} \min & f_1(\tilde{\mathbf{c}}_1;\mathbf{x}) = \tilde{c}_{11}x_1^2 + \tilde{c}_{12}x_2^2 \\ \min & f_2(\tilde{\mathbf{c}}_2;\mathbf{x}) = (x_1 - \tilde{c}_{21})^2 + \tilde{c}_{22}x_2^2 \\ & -50 < x_1, x_2 < 50 \end{cases}$$

where $c_{11} = c_{12} = c_{22} = (1,1,1)_{LR}$ and $c_{12}$ $(2,1,1)_{LR}$.

# NUMERICAL EXAMPLES

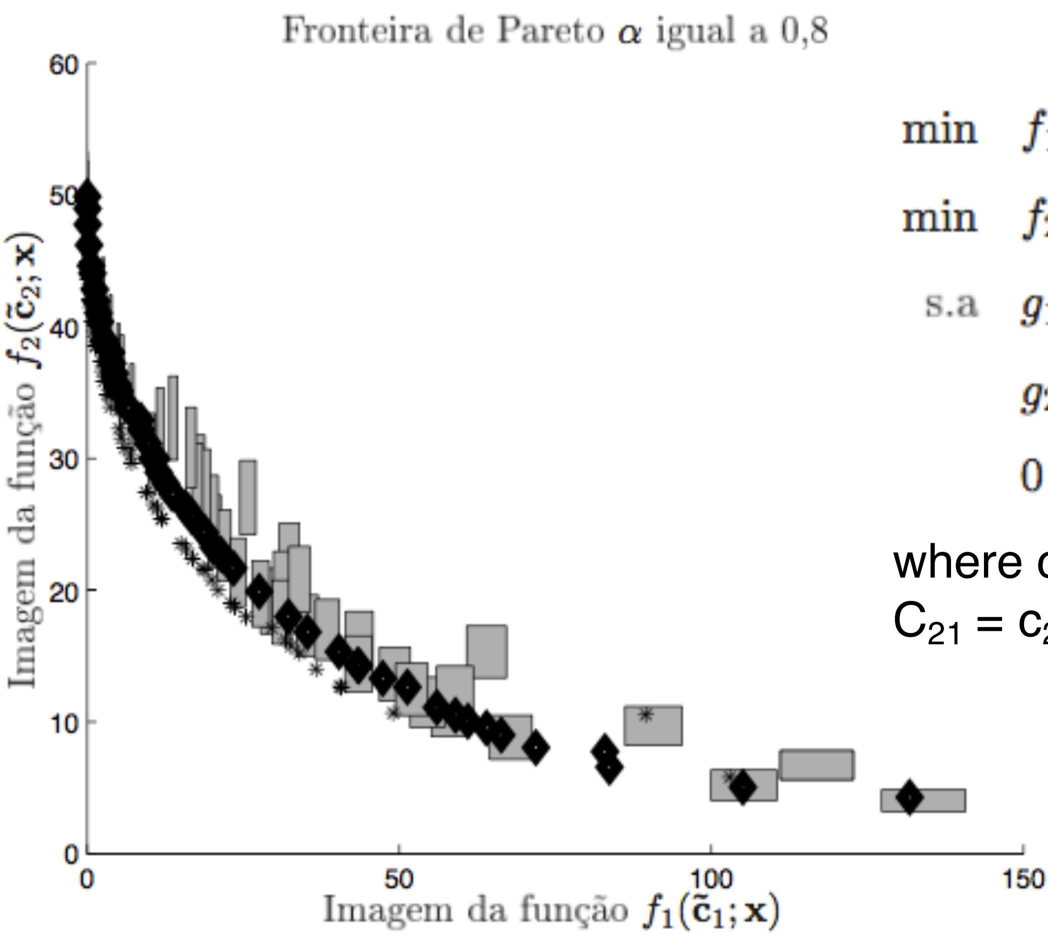
## UNCONSTRAINTS PROBLEMS




$$\begin{cases} \min & f_1(\tilde{\mathbf{c}}_1;\mathbf{x}) = 1.1 - \tilde{c}_1 x_1 \\ \min & f_2(\tilde{\mathbf{c}}_2;\mathbf{x}) = 60 - \dfrac{\tilde{c}_2 + x_3}{x_1} \\ & 0.1 < x_1 < 1, \quad 0 < x_2 < 5 \end{cases}$$

where $c_1 = c_2 = (1,1,1)_{LR}$.

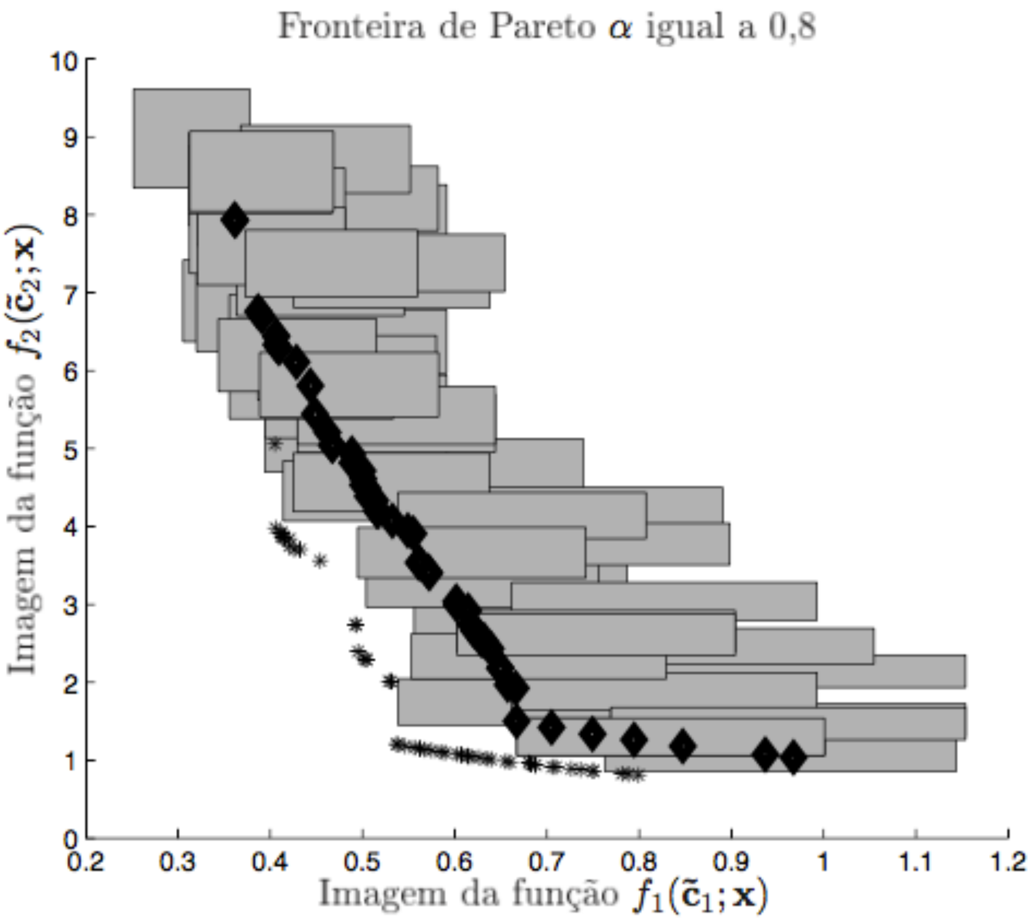
# NUMERICAL EXAMPLES
## CONSTRAINTS PROBLEMS




$$\begin{aligned} \min \quad & f_1(\tilde{\mathbf{c}}_1;\mathbf{x}) = \tilde{c}_{11}x_1^2 + \tilde{c}_{12}x_2^2 \\ \min \quad & f_2(\tilde{\mathbf{c}}_2;\mathbf{x}) = (x_1 - \tilde{c}_{21})^2 + (x_2 - \tilde{c}_{22})^2 \\ \text{s.a} \quad & g_1(\mathbf{x}) = (x_1 - 5)^2 + x_2^2 \leq 25 \\ & g_2(\mathbf{x}) = (x_1 - 8)^2 + (x_2 + 3)^2 \geq 7.7 \\ & 0 \leq x_1 \leq 5, \quad 0 \leq x_2 \leq 3 \end{aligned}$$

where $c_{11} = c_{12} = (4,1,1)_{LR}$ and $C_{21} = c_{22} = (5,1,1)_{LR}$.

# NUMERICAL EXAMPLES
## CONSTRAINTS PROBLEMS




$$
\begin{aligned}
\min \quad & f_1(\tilde{\mathbf{c}}_1;\mathbf{x}) = \tilde{c}_1 x_1 \\
\min \quad & f_2(\tilde{\mathbf{c}}_2;\mathbf{x}) = \frac{\tilde{c}_2 + x_1}{x_2} \\
\text{s.a} \quad & g_1(\mathbf{x}) = x_2 + 9x_1 \geq 6 \\
& g_2(\mathbf{x}) = -x_2 + 9x_1 \geq 1 \\
& 0.1 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 5
\end{aligned}
$$

where $c_1 = c_2 = (1,1,1)_{LR}$.

# NUMERICAL EXAMPLE

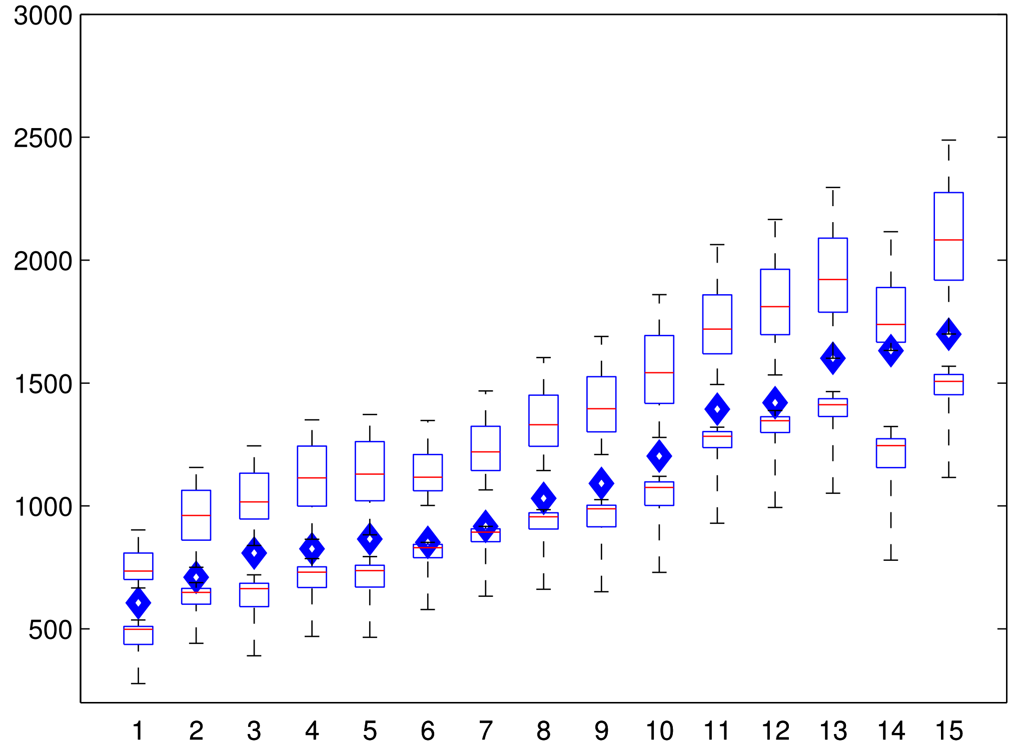
## PRICE MECHANISM OF PREFABRICATED HOUSES

| NO. | $y_j$ | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|---|
| 1 | 606 | 1 | 38.09 | 36.43 |
| 2 | 710 | 1 | 62.10 | 26.50 |
| 3 | 808 | 1 | 63.76 | 44.71 |
| 4 | 826 | 1 | 74.52 | 38.09 |
| 5 | 865 | 1 | 75.38 | 41.10 |
| 6 | 852 | 2 | 52.99 | 26.49 |
| 7 | 917 | 2 | 62.93 | 26.49 |
| 8 | 1031 | 2 | 72.04 | 33.12 |
| 9 | 1092 | 2 | 76.12 | 43.06 |
| 10 | 1203 | 2 | 90.26 | 42.64 |
| 11 | 1394 | 3 | 85.70 | 31.33 |
| 12 | 1420 | 3 | 92.27 | 27.64 |
| 13 | 1601 | 3 | 105.98 | 27.64 |
| 14 | 1632 | 3 | 79.25 | 66.81 |
| 15 | 1699 | 3 | 120.50 | 32.25 |

Table 2: Data related to prefabricate houses.

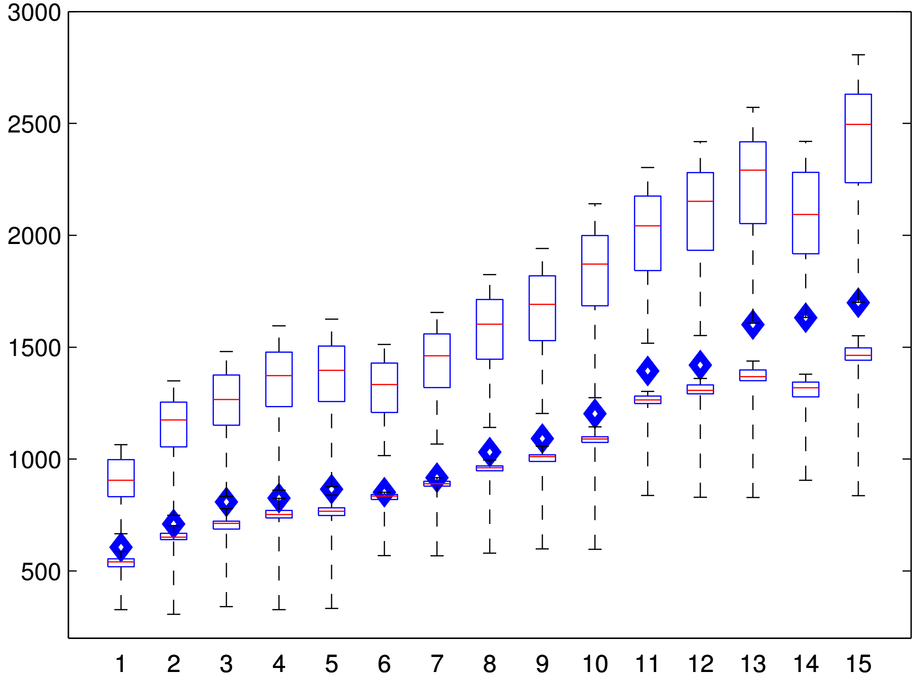
# NUMERICAL EXAMPLE

## PRICE MECHANISM OF PREFABRICATED HOUSES



# NUMERICAL EXAMPLE

## PRICE MECHANISM OF PREFABRICATED HOUSES



*In memorian to the professors*
*Bellman, Tanaka, Asai, and Chanas*

# Thanks for your attention!

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