On the solution of bounded and unbounded mixed complementarity problems

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Abstract

A reformulation of the bounded mixed complementarity problem is introduced. It is proved that the level sets of the objective function are bounded and, under reasonable assumptions, stationary points coincide with solutions of the original variational inequality problem. Therefore, standard minimization algorithms applied to the new reformulation must succeed. This result is applied to the compactification of unbounded mixed complementarity problems.

Keywords. Mixed complementarity problem, variational inequalities, box constrained minimization, reformulation.

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1 Introduction

The *mixed complementarity problem* (MCP) is the finite dimensional variational inequality problem (VIP) on a generalized $n-$dimensional box. See [7, 9, 10], among others. Given $F : I\!R^n \to I\!R^n$, $F = (f_1, \ldots, f_n)$ one wants to find $x \in \Omega$ such that

$$\langle F(x), z - x \rangle \geq 0 \ \ \forall \ z \in \Omega, \tag{1}$$

where

$$\Omega = \{x \in I\!R^n \ \mid \ a \leq x \leq b\} \tag{2}$$

and $-\infty \leq a_i < b_i \leq \infty$ for all $i = 1, \ldots, n$. (Throughout this paper, as usually, $-\infty \leq x_i$ must be read $-\infty < x_i$ and $x_i \leq \infty$ must be read $x_i < \infty$.) Solving (1) is equivalent to finding $x \in \Omega$ such that

$$f_i(x) \geq 0 \quad \text{if} \quad x_i = a_i, \tag{3}$$

$$f_i(x) \leq 0 \quad \text{if} \quad x_i = b_i \tag{4}$$

and

$$f_i(x) = 0 \quad \text{if} \quad a_i < x_i < b_i. \tag{5}$$

The classical *nonlinear complementarity problem* (NCP) is the particular case of the MCP where $\Omega = \{x \in I\!R^n \ \mid \ x \geq 0\}$. The problem of solving a nonlinear system of equations $F(x) = 0$ is also a mixed complementarity problem with $\Omega = I\!R^n$.

We say that the MCP is *bounded* if $-\infty < a_i$ and $b_i < \infty$ for all $i = 1, \ldots, n$.

Many times one needs to solve an unbounded MCP, but only solutions belonging to a smaller set are desired. For example, the original MCP can be a nonlinear system coming from a real life application where only strictly positive solutions have physical meaning. In these cases, one is tempted to solve the MCP corresponding to the smaller set. Unfortunately, the new (smaller) MCP might have solutions on the new boundary that are not solutions of the original problem. In Section 2 of this paper we give sufficient conditions under which it can be guaranteed that the solutions of the smaller MCP are also solutions of the original one.

When the conditions of Section 2 are satisfied, the MCP can be reduced to a bounded mixed complementarity problem. This fact justifies the analysis of Section 3, where the bounded MCP is considered. In this section, we introduce a smooth reformulation of the bounded MCP. The reformulation of complementarity and variational inequality problems as optimization problems with simple contraints (or without constraints at all) has attracted the attention of many researchers in the last few years.

See [1, 2, 3, 8, 12, 13, 14, 16, 17, 19, 20, 21, 22, 23] and many others. Reformulations are equivalent to the original problems in the sense that *global* minimizers are solutions of the VIP if the objective function value is null. If, at a global minimizer of the reformulation, this value is greater than zero, the original problem is not solvable. General algorithms for minimization can be used for solving the reformulation. Many times, these algorithms succeed finding a global minimizer, therefore, many problems can be solved using reformulations even under difficult conditions related to the original problem. As the reformulation of the bounded MCP introduced in [8], the reformulation introduced here uses $2n$ auxiliary unbounded variables. In spite of this unboundedness, we will prove that the objective function has bounded level sets, so that standard minimization algorithms must find, at least, stationary points. In Section 3 we also prove sufficient conditions under which stationary points of the minimization problem are solutions of the bounded MCP.

Consequences of the results given in Sections 2 and 3 are analyzed in Section 4. If the sufficient conditions of both sections hold, the MCP can be reduced to a bounded MCP, this MCP can be reduced to a minimization problem with bounded level sets and the stationary points of the optimization problem are solutions of the MCP. Conclusions of this work are stated in Section 5.

2 Solutions of the MCP in a restricted box

Suppose that we have the mixed complementarity problem (3)-(5) but we only want solutions belonging to a smaller box $\Omega_{small} \subset \Omega$. The question addressed in this section is: would it be a good strategy to solve the MCP defined by F and Ω_{small}? In general, the answer is negative. Take, for example, $\Omega = I\!R$, $F(x) \equiv 1 - x$, $\Omega_{small} = \{x \in I\!R \mid x \geq 0\}$. One of the solutions of the MCP defined by Ω_{small} is $x = 0$, which is not a solution of the original MCP.

Let us define

$$\Omega_{small} = \{x \in I\!R^n \mid a^{small} \leq x \leq b^{small}\},$$

where for all $i = 1, \ldots, n$,

$$-\infty \leq a_i \leq a_i^{small} < b_i^{small} \leq b_i \leq \infty.$$

The MCP defined by F and Ω will be called MCP(big) in this section. Analogously, the MCP defined by F and Ω_{small} will be called MCP(small). Clearly, any solution of MCP(big) belonging to Ω_{small} must be a solution of

MCP(small). We are going to prove that, under the following Assumptions 1 and 2, the reciprocal is also true.

The first assumption concerns the behavior of F on Ω_{small}. We will prove later that this assumption generalizes the property of monotonicity.

Assumption 1. *We assume that, for all $x, y \in \Omega_{small}$,*

$$[f_i(x) - f_i(y)](x_i - y_i) \leq 0 \;\; \forall \;\; i = 1, \ldots n$$

implies that

$$[f_i(x) - f_i(y)](x_i - y_i) = 0 \;\; \forall \;\; i = 1, \ldots n.$$

The second assumption says that there exists a solution of the MCP(big) that belongs to Ω_{small}. Moreover, the constraints of Ω_{small} that are not constraints of Ω are not active at this solution. A typical case where this situation occurs is when one wants to compactify the domain Ω adding artificial constraints $\|x\|_\infty \leq L$ and we know that $\|x\|_\infty < L$ at some physical solution.

Assumption 2. *There exists $\bar{x} \in \Omega_{small}$ such that $\bar{x}$ is a solution of MCP(big) and, for all $i = 1, \ldots, n$,*

(a) If $a_i < a_i^{small}$ then $\bar{x}_i > a_i^{small}$;

(b) If $b_i > b_i^{small}$ then $\bar{x}_i < b_i^{small}$.

In the following two lemmas, we prove that Assumption 1 is satisfied if F is monotone or if F is an affine function and its Jacobian is column-sufficient.

Lemma 1. *If F is monotone on Ω_{small}, then it satisfies Assumption 1.*

Proof. Let us proceed by contradiction. If $[f_i(x) - f_i(y)](x_i - y_i) \leq 0$ for all $i = 1, \ldots, n$ and there exists $j \in \{1, \ldots, n\}$ such that $[f_j(x) - f_j(y)](x_j - y_j) < 0$ then

$$\langle F(x) - F(y), x - y \rangle < 0.$$

So, F is not monotone. □

Recall that a matrix $M \in I\!R^{n \times n}$ is called *column-sufficient* if

$$[Mz]_i z_i \leq 0 \; \forall \; i = 1, \ldots n \text{ only if } [Mz]_i z_i = 0 \; \forall \; i = 1, \ldots n.$$

A matrix M is called *row-sufficient* if its transpose M^T is column-sufficient. Finally, M is called *sufficient* if it is column-sufficient and row-sufficient.

Lemma 2. *If $F(x) = Mx + q$ and M is column-sufficient, then F satisfies Assumption 1.*

Proof. If $[f_i(x) - f_i(y)](x_i - y_i) \le 0$ for all $i = 1, \dots, n$, then

$$[M(x - y)]_i(x_i - y_i) \le 0 \ \ \forall i = 1, \dots, n.$$

Since M is column-sufficient this implies that

$$[M(x - y)]_i(x_i - y_i) = 0 \ \ \forall i = 1, \dots, n.$$

Therefore, $[f_i(x) - f_i(y)](x_i - y_i) = 0$ for all $i = 1, \dots, n$. □

The main result of this section is given below. We prove that, under Assumptions 1 and 2, *all* the solutions of MCP(small) are solutions of MCP(big). As a consequence, if Assumption 1 is satisfied and we find a solution of MCP(small) which is not a solution of MCP(big), then the hypothesis on the existence of a solution satisfying Assumption 2 must be false.

Theorem 1. *Assume that F satisfies Assumption 1 and $\bar{x}$ satisfies Assumption 2. Then, every solution of MCP(small) is a solution of MCP(big).*

Proof. Let $x^* \in \Omega_{small}$ be a solution of MCP(small). We are going to prove first that

$$[f_i(x^*) - f_i(\bar{x})](x_i^* - \bar{x}_i) \le 0 \; \forall i = 1, \dots, n. \tag{6}$$

This is obviously true when $\bar{x}_i = x_i^*$. Let us consider the remaining possibilities. For all the deductions we will use that both x^* and $\bar{x}$ are solutions of MCP(small):

(1) $\bar{x}_i = a_i^{small}$ and $x_i^* > a_i^{small}$.
This implies that $f_i(\bar{x}) \ge 0$, $f_i(x^*) \le 0$ and $x_i^* - \bar{x}_i > 0$. So, (6) holds.

(2) $\bar{x}_i > a_i^{small}$ and $x_i^* = a_i^{small}$.
This implies that $f_i(\bar{x}) \le 0$, $f_i(x^*) \ge 0$ and $x_i^* - \bar{x}_i < 0$. So, (6) holds.

(3) $\bar{x}_i \in (a_i^{small}, b_i^{small})$ and $x_i^* \in (a_i^{small}, b_i^{small})$.
In this case, $f_i(x^*) = f_i(\bar{x}) = 0$. So, (6) holds.

(4) $\bar{x}_i = b_i^{small}$ and $x_i^* < b_i^{small}$.
This implies that $f_i(\bar{x}) \le 0$, $f_i(x^*) \ge 0$ and $x_i^* - \bar{x}_i < 0$. So, (6) holds.

(5) $\bar{x}_i < b_i^{small}$ and $x_i^* = b_i^{small}$.

This implies that $f_i(\bar{x}) \geq 0$, $f_i(x^*) \leq 0$ and $x_i^* - \bar{x}_i > 0$. So, (6) holds.

Therefore, (6) is proved. So, by Assumption 2,

$$[f_i(x^*) - f_i(\bar{x})](x_i^* - \bar{x}_i) = 0 \;\forall\; i = 1, \ldots, n. \tag{7}$$

Now, suppose that x^* *is not* a solution of MCP(big). Since x^* *is* a solution of MCP(small), there exists $j \in \{1, \ldots, n\}$ such that

$$x_j^* = a_j^{small} > a_j \text{ and } f_j(x^*) > 0 \tag{8}$$

or

$$x_j^* = b_j^{small} < b_j \text{ and } f_j(x^*) < 0. \tag{9}$$

Suppose that (8) holds. By Assumption 2, since $a_j^{small} > a_j$, we have that $\bar{x}_j > a_j^{small}$. Therefore,

$$x_j^* - \bar{x}_j < 0. \tag{10}$$

But $\bar{x}_j > a_j^{small}$ implies that $f_j(\bar{x}) \leq 0$. So, $f_j(x^*) - f_j(\bar{x}) > 0$. By (10), this contradicts (7).

Analogously, suppose that (9) holds. By Assumption 2, since $b_j^{small} < b_j$, we have that $\bar{x}_j < b_j^{small}$. Therefore,

$$x_j^* - \bar{x}_j > 0. \tag{11}$$

But $\bar{x}_j < b_j^{small}$ implies that $f_j(\bar{x}) \geq 0$. So, $f_j(x^*) - f_j(\bar{x}) < 0$. By (11), this contradicts (7).

It follows that x^* is a solution of MCP(big), as we wanted to prove. □

3 Reformulation of the bounded MCP

In this section, we consider the bounded MCP, so $-\infty < a_i < b_i < \infty$ for all $i = 1, \ldots, n$.

It is easy to see that (3)-(5) is equivalent to:

$$F(x) - u + v = 0, \tag{12}$$

$$\langle x - a, u \rangle = 0, \;\; \langle b - x, v \rangle = 0, \tag{13}$$

$$a \leq x \leq b, \;\; u \geq 0, v \geq 0. \tag{14}$$

For an arbitrary $\rho > 0$ we define the following optimization problem associated with (12)–(14):

$$\text{Minimize } \|F(x) - u + v\|_2^2 + \rho[\langle x - a, u \rangle^2 + \langle b - x, v \rangle^2]$$

$$\text{subject to} \quad a \leq x \leq b, \quad u \geq 0, v \geq 0. \tag{15}$$

Obviously, if (x, u, v) is a solution of (15) and the objective function value is zero, then x is a solution of (12)–(14).

As in [2, 3, 13, 14], the optimization problem (15) preserves the same differentiability properties of the original MCP. For example, if F has at most quadratic components, (15) consists on the minimization of a quartic with simple bounds.

Let us prove now that the level sets associated to this optimization problem are bounded. We denote, for all $x, u, v \in I\!R^n$,

$$\Phi(x, u, v) = \|F(x) - u + v\|_2^2 + \rho[\langle x - a, u\rangle^2 + \langle b - x, v\rangle^2].$$

Theorem 2. *Assume that F is continuous. Then, for all $\theta > 0$ the set $S \subset I\!R^{3n}$ defined by*

$$S = \{(x, u, v) \in I\!R^n \mid a \leq x \leq b, \quad u \geq 0, v \geq 0, \ \Phi(x, u, v) \leq \theta\}$$

is bounded.

Proof. Assume, by contradiction, that an unbounded sequence $\{(x^k, u^k, v^k)\} \subset S$ exists. Since $\{x^k\}$ is obviously bounded, it follows that, for an adequate subsequence, there exists $i \in \{1, \ldots, n\}$ such that $u_i^k \to \infty$ or $v_i^k \to \infty$.

If $u_i^k \to \infty$, by the continuity of F, the form of $\Phi(x, u, v)$ and the fact that $\{\Phi(x^k, u^k, v^k)\}$ is bounded imposes that $v_i^k \to \infty$. Analogously, $v_i^k \to \infty$ implies that $u_i^k \to \infty$.

Now, since $\{\Phi(x^k, u^k, v^k)\}$ is bounded, both $\{\langle x^k - a, u^k\rangle^2\}$ and $\{\langle b - x^k, v^k\rangle^2\}$ are bounded. This implies, since $x_j^k - a_j \geq 0 \leq b_j - x_j^k$ for all $j = 1, \ldots, n$, that $\{(x_j^k - a_j)u_j^k\}$ and $\{(b_j - x_j^k)v_j^k\}$ are also bounded sequences. But, since $v_i^k \to \infty$ and $u_i^k \to \infty$, it follows that $x_i^k - a_i \to 0$ and $b_i - x_i^k \to 0$. So, $b_i = a_i$, which is a contradiction. Therefore, S is bounded, as we wanted to prove. □

From now on, assume that F has continuous first partial derivatives. The following result says that, under a row-sufficiency assumption on the Jacobian $F'(x)$, all the stationary points of (15) are solutions of (12)–(14).

Theorem 3. *Assume that (x^*, u^*, v^*) is a KKT point of (15) such that $F'(x^*)$ is row-sufficient. Then x^* is a solution of the MCP.*

Proof. Define

$$r^* = F(x^*) - u^* + v^*.$$

Since (x^*, u^*, v^*) is a KKT point of (15), there exist $\alpha, \alpha', \gamma, \gamma' \geq 0$ such that

$$2F'(x^*)^T r^* + 2\rho\langle x^* - a, u^*\rangle u^* - 2\rho\langle b - x^*, v^*\rangle v^* - \alpha + \alpha' = 0, \tag{16}$$

$$-r^* + 2\rho\langle x^* - a, u^*\rangle(x^* - a) - \gamma = 0, \tag{17}$$

$$r^* + 2\rho\langle b - x^*, v^*\rangle(b - x^*) - \gamma' = 0, \tag{18}$$

$$\langle \alpha, x^* - a\rangle = 0, \quad \langle \alpha', b - x^*\rangle = 0, \tag{19}$$

$$\langle \gamma, u^*\rangle = 0, \quad \langle \gamma', v^*\rangle = 0, \tag{20}$$

$$a \leq x^* \leq b, \tag{21}$$

$$\alpha \geq 0, \ \alpha' \geq 0, \ \gamma \geq 0, \ \gamma' \geq 0. \tag{22}$$

Premultiplying (16) by r_i^*, we obtain

$$r_i^*[F'(x^*)^T r^*]_i + r_i^*(2\rho\langle x^* - a, u^*\rangle u_i^* - \alpha_i) + r_i^*(-2\rho\langle b - x^*, v^*\rangle v_i^* + \alpha_i') = 0 \tag{23}$$

for all $i = 1, \ldots, n$. But, by (17)-(20) and (22), we have that

$$\begin{aligned} & r_i^*(2\rho\langle x^* - a, u^*\rangle u_i^* - \alpha_i) \\ &= (2\rho\langle x^* - a, u^*\rangle[x_i^* - a_i] - \gamma_i)(2\rho\langle x^* - a, u^*\rangle u_i^* - \alpha_i) \\ &= 4\rho^2\langle x^* - a, u^*\rangle^2[x_i^* - a_i]u_i^* + \gamma_i\alpha_i \end{aligned} \tag{24}$$

and

$$\begin{aligned} & r_i^*(-2\rho\langle b - x^*, v^*\rangle v_i^* + \alpha_i') \\ &= (-2\rho\langle b - x^*, v^*\rangle[b_i - x_i^*] + \gamma_i')(-2\rho\langle b - x^*, v^*\rangle v_i^* + \alpha_i') \\ &= 4\rho^2\langle b - x^*, v^*\rangle^2[b_i - x_i^*]v_i^* + \gamma_i'\alpha_i' \end{aligned} \tag{25}$$

for $i = 1, \ldots, n$. Now, by (23)-(25),

$$\begin{aligned} & r_i^*[F'(x^*)^T r^*]_i + 4\rho^2\langle x^* - a, u^*\rangle^2[x_i^* - a_i]u_i^* + \gamma_i\alpha_i \\ & +4\rho^2\langle b - x^*, v^*\rangle^2[b_i - x_i^*]v_i^* + \gamma_i'\alpha_i' = 0 \end{aligned} \tag{26}$$

for $i = 1, \ldots, n$. So, by (26), (21) and (22) we have that

$$r_i^*[F'(x^*)^T r^*]_i \leq 0 \tag{27}$$

for $i = 1, \ldots, n$. Since $F'(x^*)$ is row-sufficient, this implies that

$$r_i^*[F'(x^*)^T r^*]_i = 0$$

for $i = 1, \ldots, n$. So, by (26), (27), (21) and (22), we obtain:

$$\langle x^* - a, u^* \rangle = 0, \quad \langle b - x^*, v^* \rangle = 0. \tag{28}$$

Therefore, by (17) and (18), we have that

$$r^* = -\gamma \leq 0, \quad r^* = \gamma' \geq 0$$

This implies that $r^* = 0$. Therefore, by (28), we have that $\Phi(x^*, u^*, v^*) = 0$. So, x^* is a solution of the MCP. $\square$

Theorem 3 has an independent interest since it shows that bounded mixed complementarity problems can be solved using bound constrained minimization algorithms if the Jacobian matrix is row-sufficient. Since positive-semidefinite matrices and P-matrices are always row-sufficient, the scope of this result involves many interesting problems.

4 Consequences for the general (unbounded) MCP

Many times one needs to solve a general MCP but only solutions in a bounded set (say $\|x\|_\infty \leq L$) have physical meaning. In this case, we define

$$\Omega_{small} = \Omega \cap \{x \in \mathbb{R}^n \mid \|x\|_\infty \leq L\}$$

and we consider the variational inequality problem on Ω_{small} which, of course, is a bounded MCP. The reformulation (15) of this bounded MCP has, according to Theorem 2, bounded level sets. Therefore, any standard box-constraint minimization algorithm (see [6, 11] and references therein) will find a stationary point. By Theorem 3, if, at this stationary point, $F'(x)$ is row-sufficient, then we have a solution of the bounded MCP. Moreover, by Theorem 1, if Assumptions 1 and 2 are satisfied, the stationary point found by the box-constraint solver is a solution of the MCP. Summing up, the whole process consists on

1. Introducing the artificial bound $\|x\|_\infty \leq L$;

2. Solving the reformulation (15) of the resulting bounded MCP.

If a solution such that $\|x\|_\infty < L$ of the original MCP exists and F is monotone this process will be necessarily successful. The same happens if F is linear and its Jacobian is *sufficient*.

Let us comment here an interesting consequence of this state of facts. Suppose that we need to solve $F(x) = 0$, F is monotone but singular Jacobians and/or local-nonglobal minimizers of $\|F(x)\|_2^2$ are present. Take, for example, $F : I\!R \to I\!R$, $F(x) = x^3 + 1$ if $x < 0$, $F(x) = 1$ if $x > 0$. The usual globalization procedure of minimizing $\|F(x)\|_2^2$ (with or without artificial bounds) is dangerous because, besides the solution $x = -1$, all the points $x \geq 0$ are local minimizers of $\|F(x)\|_2^2$. However, according to our results, setting artificial bounds (say $-10 \leq x \leq 10$ in the example) and solving (15) using a standard box-constraint minimizer will necessarily lead to a solution of $F(x) = 0$ ($x = -1$ in the example) and the additional stationary points of $\|F(x)\|_2^2$ are not limit points of the numerical scheme at all.

Row-sufficiency and column-sufficiency (Assumption 1) play quite different roles in the reformulation process. Row-sufficiency guarantees that stationary points of the reformulation are solutions of the bounded MCP, whereas Assumption 1 legitimates the bounding process. The reformulation of the linear complementarity problem given in [13] has the property that, under row-sufficiency and feasibility, stationary points are solutions. However, this reformulation does not have bounded level sets.

Unfortunately, Theorem 1 does not hold, even in the linear case, if one only assumes that M is row-sufficient (and not column-sufficient). In fact, define

$$M = \begin{bmatrix} 0 & 0 \\ -3 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The matrix M is row-sufficient but not column-sufficient. Moreover $(0, 0)$ is a solution of the linear complementarity problem (LCP) defined by M and q. Taking $L = 2$, we have that all the points of the form $(t, 2)$ for $t \in [\frac{2}{3}, 2]$, are solutions of the MCP defined by $F(x) = Mx + q$, $a = 0, b = (L, \ldots, L)$. But, clearly, these points are not solutions of the LCP.

5 Final remarks

Up to our knowledge, the reformulation of the MCP presented in this paper is the only one that satisfies both the bounded-level-set property and the sufficiency condition given by Theorem 3.

In [4] it has been proved that the Fischer-Burmeister reformulation of the bounded MCP given in [8] (also with $3n$ variables) satisfies the bounded-level-set property if

$$\theta < \frac{1}{2} \min \{(b_i - a_i)^2, i = 1, \ldots, n\}. \tag{29}$$

Moreover, in [4] it was shown that the estimate (29) is sharp, that is, level sets of the reformulation [8] can be unbounded for larger values of θ. Curiously, if, in the original (unbounded) MCP we have $b_i - a_i = \infty$ for all

$i = 1, \ldots, n$, it is possible to choose the artificial constraint $\|x\|_\infty \leq L$ and the initial point in such a way that the corresponding level set is bounded (see [4]) but this is not possible if $b_i - a_i < \infty$ for some i.

Our reformulation is completely smooth, a property from which it is possible to take algorithmic advantage. For example, if F is polynomial, the objective function $\Phi(x, u, v)$ is polynomial too. Other reformulations create objective functions without second derivatives even if the original $F(x)$ is linear. The fact that the reformulation is a box-constrained optimization problem is interesting because we can deal with that type of problems using algorithms that do not use matrix factorizations at all, so that large-scale and non-structured problems are solvable.

Bounded-level-set results for reformulations of the nonlinear complementarity problem have been proved under hypothesis of $P-$uniformity of F in [15, 18]. In [5] a reformulation of the NCP is introduced for which the level sets are bounded when F is monotone and a strict feasibility condition holds. Therefore, our results of Sections 3 and 4 also represent advances with respect to existing NCP-reformulations.

Perhaps the bounded-level-set property is not the weakest condition which guarantees that standard minimization algorithms generate bounded sequences. In fact, to find weaker sufficient condidtions for the bounded-sequence property related to minimization methods with suitable sufficient-decrease requirements seems to represent an interesting field of research on general optimization. While an answer to the above question is not found, the discovery of reformulations with bounded level sets plays an important role, both from the theoretical and the practical point of view.

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