Lecture 1: Basics of Geometric Algebra

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Contents: GA Course I, Session 1

- The geometric product: and how it relates to the inner and outer products.
- The mathematical framework: versors and multivectors; reversion; inversion; reflections and rotations .
- Rotations in more detail – the GA concept of a rotor.
- Reciprocal frames and how they are used.

The contents follow the notation and ordering of *Geometric Algebra for Physicists [C.J.L. Doran and A.N. Lasenby]* and the corresponding course the book was based on.

Notation

- We will see, as the course goes on, that we will be dealing with many sorts of geometric objects, not just scalars and vectors.
- Therefore, we will not, in general, use bold for vectors or any other objects (though sometimes there are exceptions).
- Use lower case roman letters for vectors, and generally lower case greek letters for scalars.

The Inner Product

To start with, let us assume a Euclidean space (all basis vectors square to +1). The inner or dot product between two vectors a and b is written as $a \cdot b$. If $a, b \neq 0$

$$a \cdot a = a^2 > 0 \quad \text{and} \quad b \cdot b = b^2 > 0$$

....the inner product can then be used to define the angle (θ) between a and b:

$$a \cdot b = |a||b| \cos\theta$$

In any space, we define an inner product via its basis vectors (for now assume they are orthogonal)

$$a \cdot b = a_i b_i$$

[repeated indices mean sum: $\sum_i$ in this case]

The Cross Product

For two vectors a and b the cross product of the two is written as $a \times b$ and only exists in 3-d space.

$$a \times b = |a||b| \sin\theta \; \hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to the plane containing a and b.

For a right handed orthonormal set of basis vectors $\{e_1, e_2, e_3\}$, we have

$$e_3 = e_1 \times e_2, \quad e_2 = e_3 \times e_1, \quad e_1 = e_2 \times e_3$$

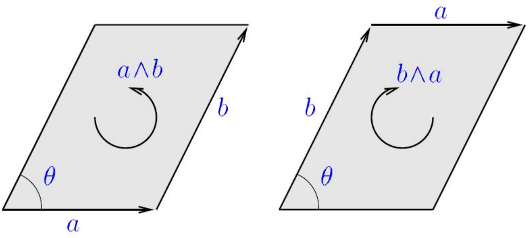
The Exterior, Outer or Wedge Product

The cross product fails in higher dimensions as there is no longer the concept of a unique vector perpendicular to the plane. It therefore seems sensible to geometrically encode the plane itself.

We write the wedge product between two vectors a and b as

$$a \wedge b$$

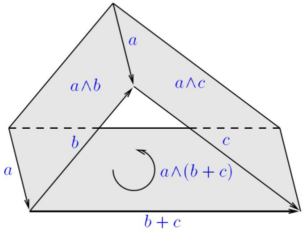
......an oriented plane – we call this a bivector



Properties of the Wedge Product

- $a \wedge b = -b \wedge a$, so the product is antisymmetric.
- $a \wedge a = 0$ as there is no plane swept out.
- The wedge product is distributive over addition

$$a \wedge (b + c) = a \wedge b + a \wedge c$$



We will see how the wedge and cross products are connected (in 3d) later.

The Geometric Product

Clifford's amazing idea was to combine the inner and outer product into a new geometric product. For two vectors a and b, we write the geometric product as ab

$$ab = a\cdot b + a\wedge b$$

...the sum of a scalar and a bivector.

The geometric product is non-commutative since

$$ba = b\cdot a + b\wedge a \equiv a\cdot b - a\wedge b$$

...recall, complex numbers also involve the addition of two fundamentally different quantities.

Inner and Outer Products from the Geometric Product

Since $ab = a \cdot b + a \wedge b$ and $ba = a \cdot b - a \wedge b$, we can write

$$a \cdot b = \frac{1}{2}(ab + ba) \quad \{\text{symmetric}\}$$

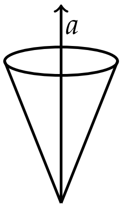
and

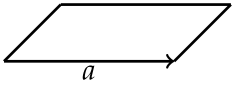
$$a \wedge b = \frac{1}{2}(ab - ba) \quad \{\text{antisymmetric}\}$$

In an axiomatic approach to GA, we can start with the geometric product and define the inner and outer products from this.

The Geometric Product is Invertible

Suppose we are given $c = a \cdot b$. If we are then given a, we cannot recover b uniquely.
Similarly, suppose we are given $c = a \wedge b$. If we are then given a, we cannot recover b uniquely.





However, suppose we are given $c = ab$. If we are then given a, we can recover b uniquely:

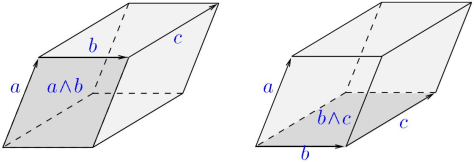
$$b = \frac{1}{a^2} ac$$

This invertibility is the key to much of the power of GA.

Higher Order Objects

In 2d, the highest order element we can have is a plane [or bivector].

In 3d, imagine sweeping a bivector $b \wedge c$ along a vector a to form a volume, $a \wedge (b \wedge c)$



..the same as sweeping bivector $a \wedge b$ along the vector c

Thus the volume or trivector formed is

$$a \wedge b \wedge c = (a \wedge b) \wedge c = a \wedge (b \wedge c)$$

An Algebra of Geometric Objects

In an n-d space, we therefore have scalars, vectors, bivectors, trivectors,....., n-vectors.

A general linear combination of these objects is called a multivector:

$$M = \langle M \rangle_0 + \langle M \rangle_1 + \langle M \rangle_2 + ...\langle M \rangle_n$$

where we use the notation $\langle M \rangle_r$ to mean the r-vector part of the multivector M.

A product of vectors is called a versor:

$$a_1 a_2 a_3 \ldots a_m$$

The highest grade object in a space, the n-vector, is unique up to scale - the 'unit' n-vector is called the Pseudoscalar and written as I_n.

Manipulating Multivectors

We can add, subtract and multiply multivectors using the geometric product.

Before we look more at this, we need to distinguish between an r-vector and an r-blade.

An r-blade, which we will call A_r, is something which can be written as the wedge product of r vectors:

$$A_r = a_1 \wedge a_2 \wedge \ldots \wedge a_r$$

An r-vector, which we will call M_r, is something which can be written as a linear combination of r blades:

$$M_r = \alpha_1 A1_r + \alpha_2 A2_r + \ldots + \alpha_m Am_r$$

Multiplying Multivectors

The geometric product is distributive over addition, so we can reduce the product of two multivectors to a sum of the products of blades, e.g.

$$P = a + (b \wedge c) \quad Q = d + (e \wedge f)$$

$$P\,Q = ad + a(e \wedge f) + (b \wedge c)d + (b \wedge c)(e \wedge f)$$

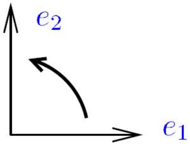
..note the order matters due to the non-commutativity.

Therefore, if we understand how to multiply blades, we can multiply multivectors.

The Geometric Algebra of 2d Euclidean Space

Consider a plane spanned by 2 orthonormal vectors $e_1,\ e_2$, such that

$$e_1{}^2 = e_2{}^2 = 1 \text{ and } e_1 \cdot e_2 = 0$$



The pseudoscalar in this 2d algebra is the bivector $e_1 \wedge e_2$ – it is a directed 'volume' element.

We call this full algebra $\mathcal{G}_2$ (sometimes written as $\mathcal{G}_{(2,0,0)}$); it has $2^2 = 4$ elements:

1	$\{e_1,\ e_2\}$	$e_1 \wedge e_2$
1 scalar	2 vectors	1 bivector

The Geometric Algebra of 2d Euclidean Space cont....

Note that, as $e_1 \cdot e_2 = 0$:

$$e_1 e_2 = e_1 \cdot e_2 + e_1 \wedge e_2 = e_1 \wedge e_2$$

and

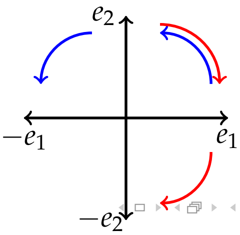
$$e_2 e_1 = e_2 \cdot e_1 + e_2 \wedge e_1 = e_2 \wedge e_1 = -e_1 e_2$$

..an example of the property that orthogonal vectors anticommute.

Now see what effect multiplying by $I_2 = e_1 \wedge e_2 = e_1 e_2$ has on vectors:

Left
$$(e_1 e_2) e_1 = -e_1 e_1 e_2 = -e_2$$
$$(e_1 e_2) e_2 = e_1 e_2 e_2 = e_1$$

Right
$$e_1 (e_1 e_2) = e_1 e_1 e_2 = e_2$$
$$e_2 (e_1 e_2) = -e_2 e_2 e_1 = -e_1$$



$\pi/2$ rotation!

The Bivector $e_1 \wedge e_2$ in 2d

Note:

$$I_2{}^2 = (e_1e_2)(e_1e_2) = -e_1e_2e_2e_1 = -1$$

We therefore have a real object, the *unit* bivector, that:

- rotates by 90° clockwise via left multiplication
- rotates by 90° anticlockwise via right multiplication
- squares to -1

Complex Numbers and the the 2d Geometric Algebra

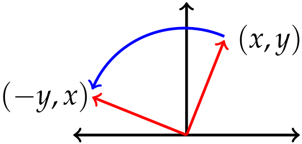
Recall that the unit imaginary, i, of complex numbers, squares to -1 and performs 90° rotations of points in the Argand plane

$$i(x+iy) = -y+ix \quad \text{so that} \quad (x,y) \longrightarrow (-y,x)$$

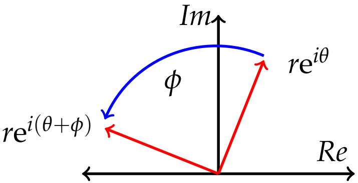
So, in $\mathcal{G}_2$, our position vector is $p = xe_1 + ye_2$ which we can write as:

$$p = e_1(x + e_1e_2y) = e_1(x + I_2y) \quad \Longrightarrow \quad e_1p = x + I_2y$$

So multiplication on the left by e_1 [which picks out the real axis] maps our position vector onto something which is analogous to the complex numbers!



Rotations in 2d



To rotate a complex number $Z = re^{i\theta}$ anticlockwise through an angle ϕ in the Argand plane we take

$$Z = re^{i\theta} \longrightarrow Z' = re^{i(\theta+\phi)} = e^{i\phi}Z$$

Now look at analogously taking $p = e_1 Z \to e_1 Z'$

$$p' = e_1 e^{I\phi} Z = e^{-I\phi} e_1 Z = e^{-I\phi} p$$

since I anticommutes with vectors [Exercise]. Giving us

$$p' = e^{-I\phi} p \equiv p e^{I\phi} \equiv e^{-I\phi/2} p e^{I\phi/2}$$

We will see that this final form is the most general – ie extends to higher dimensions.

The Geometric Algebra of 3d Euclidean Space

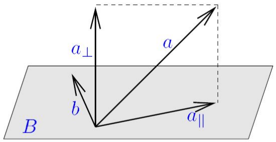
Now let our orthonormal basis vectors be e_1, e_2, e_3. Our 3d geometric algebra, $\mathcal{G}_3$, now has $2^3 = 8$ elements (with $i, j = 1, 2, 3,\ i \neq j$):

1	$\{e_i\}$	$\{e_i \wedge e_j\}$	$e_1 \wedge e_2 \wedge e_3$
1 scalar	3 vectors	3 bivectors	1 trivector

The sizes of the sets of elements are given by the binomial coefficients.

Vectors and Bivectors in 3d

Basis bivectors are $\{e_1e_2,\ e_2e_3,\ e_3e_1\}$ – all square to -1 and generate 90° rotations in their plane.



Now consider the product aB, with a a vector and B a bivector.

$$aB = (a_\perp + a_\parallel)B$$

Now write $B = a_\parallel \wedge b$, with b in the B plane and orthogonal to $a_\parallel$, so that

$$a_\parallel B = a_\parallel(a_\parallel \wedge b) = a_\parallel(a_\parallel b) = a_\parallel{}^2 b$$

$$a_\perp B = a_\perp(a_\parallel \wedge b) = a_\perp \wedge a_\parallel \wedge b$$

Vectors and Bivectors in 3d cont....

...we can write this as

$$aB = a \cdot B + a \wedge B$$

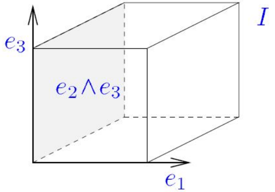
with dot and wedge now meaning the lowest and highest grade parts of the geometric product.

We therefore see that $a \cdot B$ projects onto the components of a in the plane, rotates by 90° and dilates by $|B|$.

..and that $a \wedge B$ projects onto the perpendicular component of a and forms a trivector.

The Pseudoscalar in 3d

The highest grade element is $I_3 = e_1e_2e_3$ [right handed set] – let us just use I here. It is easy to show that $I^2 = -1$.



Now take e_iI:

$$e_1I = e_1e_1e_2e_3 = e_2e_3$$

Similarly, $e_2I = e_3e_1, \; e_3I = e_1e_2$.

This is an example of a duality transformation: multiplication by I maps an r-vector onto an $(n-r)$-vector [here $r = 1, \; n = 3$].

Check that I commutes with all elements of our 3d algebra.

Cross Product vs Wedge Product

We are now able to see the connection between $a \times b$ and $a \wedge b$. Consider the product of a basis bivector and the 3d pseudoscalar, I, eg

$$I(e_1 \wedge e_2) = e_1 e_2 e_3 e_1 e_2 = -e_3$$

...ie minus the vector perpendicular to the $e_1 \wedge e_2$ plane.

We can easily generalise this to give:

$$a \times b = -I(a \wedge b)$$

We can see, therefore, that the conventional concept of axial vector or pseudovector is encoding the fact that you are actually dealing with a bivector.

Reversion

Reversion is an important operation – it reverses the order of vectors in any product.

We denote the reverse of A via a tilde, eg $\tilde{A}$.

While this operation can be performed in an algebra of any dimension, in 3d we have

$$(e_1e_2)\tilde{} = e_2e_1 = -e_1e_2$$

$$\tilde{I} = (e_1e_2e_3)\tilde{} = e_3e_2e_1 = -e_1e_2e_3 = -I$$

Since scalars and vectors remain unchanged under reversion, we have, for a general 3d multivector M

$$M = \alpha + a + B + \beta I$$

$$\tilde{M} = \alpha + a - B - \beta I$$

Rotations in 3d

We would like a 3d version of the rotation formulae in the plane (recall that Hamilton spent many years of his life looking for such a thing – he finally came up with the quaternions!).

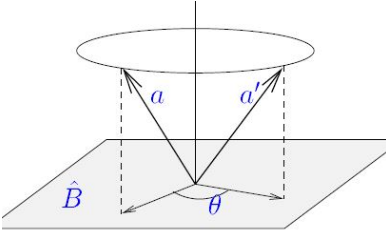
Recall that to rotate a 2d vector a through θ in the $e_1 e_2$ plane to a', we take (double-sided form shown)

$$a' = \mathrm{e}^{-e_1 e_2 \theta/2} \, a \, \mathrm{e}^{e_1 e_2 \theta/2}$$

..in 3d, this works for any a in the $e_1 e_2$ plane and additionally leaves e_3 unchanged.

Note: e_3 commutes with $\mathrm{e}^{-e_1 e_2 \theta/2}$ [Exercise]
This is why we need the two-sided formula.

Rotations in 3d cont....



Let us now rotate a general vector a through θ in the $\hat{B}$ plane (such that $\hat{B}^2 = -1$). First let $a = a_{\parallel} + a_{\perp}$:

$$\mathrm{e}^{-\hat{B}\theta/2}\,(a_{\parallel} + a_{\perp})\,\mathrm{e}^{\hat{B}\theta/2} = a'_{\parallel} + a_{\perp} = a'$$

Let $R = \mathrm{e}^{-\hat{B}\theta/2}$, then we can write our 3d rotation as

$$a' = Ra\tilde{R}$$

We call this exponentiation of a bivector, a rotor.

Note: $R\tilde{R} = \mathrm{e}^{-\hat{B}\theta/2}\,\mathrm{e}^{\hat{B}\theta/2} = 1$

Rotors in 3d

A rotor is the exponential of a bivector and rotates vectors via a double-sided formula:

$$a' = Ra\tilde{R}$$

In 3d $R = e^{-\hat{B}\theta/2}$ is a scalar + bivector:

$$e^{-\hat{B}\theta/2} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\hat{B}$$

Can we also rotate bivectors(planes) in a similar way? Let $B = a \wedge b$

$$\begin{aligned} B' &= a' \wedge b' = Ra\tilde{R} \wedge Rb\tilde{R} \\ &= \frac{1}{2}\left(Ra\tilde{R}Rb\tilde{R} - Rb\tilde{R}Ra\tilde{R}\right) \\ &= \frac{1}{2}R(ab - ba)\tilde{R} = R(a \wedge b)\tilde{R} \end{aligned}$$

In fact, we can rotate any multivector via this formula!

Rotors in Euclidean GAs

In fact, this formula of

Rotor = exponential of bivector

performs rotations in any dimension and of any object in the algebra.

We will see in future sessions, that if we do not have a euclidean space, these rotors always form transformations of a fundamental nature.

Much of the power of GA lies in its ability to nicely deal with rotations.

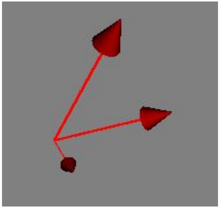
Reciprocal Frames

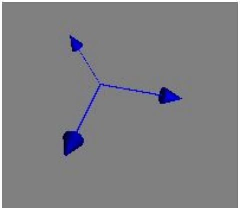
Many problems in mathematics, physics and engineering require a treatment of non-orthonormal frames.

Take a set of n linearly independent vectors $\{f_k\}$; these are not necessarily orthogonal nor of unit length.

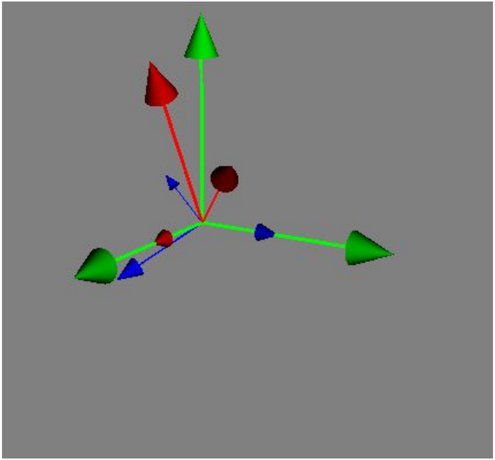
Can we find a second set of vectors (in the same space), call these $\{f^k\}$, such that

$$f^i \cdot f_j = \delta^i_j$$





Reciprocal Frames



Reciprocal Frames cont....

We call such a frame a reciprocal frame. Note that since any vector a can be written as $a = a^k f_k \equiv \sum a^k f_k$ (ie we are adopting the convention that repeated indices are summed over), we have

$$f^k \cdot a = f^k \cdot (a^j f_j) = a^j (f^k \cdot f_j) = a^j \delta_j^k = a^k$$

Similarly, since we can also write $a = a_k f^k \equiv \sum a_k f^k$

$$f_k \cdot a = f_k \cdot (a_j f^j) = a_j (f_k \cdot f^j) = a_j \delta_k^j = a_k$$

Thus we can recover the components of a given vector in a similar way to that used for orthonormal frames.

So how do we find a reciprocal frame?

Finding a Reciprocal Frame

To illustrate the process, we will find the reciprocal frame in 3d for a non-orthonormal set of basis vectors $\{f_1, f_2, f_3\}$.
Consider the quantity $f^1 = \alpha(f_2 \wedge f_3)I$:

$$f_1 \cdot f^1 = \alpha f_1 \cdot (f_2 \wedge f_3 I) = \alpha(f_1 \wedge f_2 \wedge f_3)I = \alpha E_3 I$$

(this uses a useful GA relation $a \cdot (BI) = (a \wedge B)I$) where

$$E_3 = f_1 \wedge f_2 \wedge f_3$$

Since $E_3 = \beta I$, we see that $\alpha = -1/\beta$, with $E_3{}^2 = -\beta^2$:

$$f^1 = -\frac{1}{|E_3|}(f_2 \wedge f_3)I$$

and similarly for f^2, f^3.

Example: Recovering a Rotor in 3-d

As an example of using reciprocal frames, consider the problem of recovering the rotor which rotates between two 3-d non-orthonormal frames $\{f_k\}$ and $\{f'_k\}$, ie find R such that

$$f'_k = R f_k \tilde{R}$$

It is not too hard to show that R can be written as

$$R = \beta(1 + f'_k f^k)$$

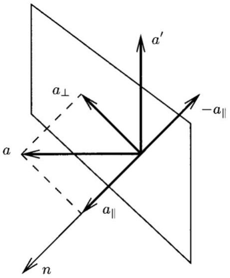
where the constant β ensures that $R\tilde{R} = 1$.

A very easy way of recovering rotations.

Reflections

Take a vector a and a unit vector n ($n^2 = 1$) – to resolve a into parts perpendicular and parallel to n we do the following:

$$\begin{aligned} a &= n^2 a = n(na) = n(n\cdot a + n\wedge a) \\ &= (n\cdot a)n + n(n\wedge a) \equiv a_{\parallel} + a_{\perp} \end{aligned}$$



If we reflect a in the plane orthogonal to n we get a' given by

$$\begin{aligned} a' &= a_{\perp} - a_{\parallel} \\ &= -n(a\wedge n) - (a\cdot n)n \\ &= -n(a\cdot n + a\wedge n) = -nan \end{aligned}$$

$a' = -nan$ is a very compact formula, afforded by the geometric product. We will see later that sandwiching like this is a very general formula for reflecting one object in another.

Generalising the Geometric Product

It can be shown (in a more general treatment) that the geometric product of an r-blade, A_r and an s-blade, B_s is given by:

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \ldots + \langle A_r B_s \rangle_{r+s}$$

We then use the dot and wedge to mean the lowest and highest grades terms in this expansion:

$$\begin{aligned} A_r \cdot B_s &= \langle A_r B_s \rangle_{|r-s|} \\ A_r \wedge B_s &= \langle A_r B_s \rangle_{r+s} \end{aligned}$$

Using the above, can make many identities very easy to prove.

Geometric Algebra on Azure Notebooks

1. Sign up for an Azure Notebook account if you don't have one (note, sometimes it does not like it if you logon with an institutional email, if your institution already has accounts with Azure – I sign on with gmail).
2. go to
   `https://notebooks.azure.com/hugohadfield/libraries/azure-clifford`
   and 'clone' the azure-clifford library (there is a 'clone' button).
3. go back to your Azure page and you should now see the azure-clifford library. Open
   clifford_example.ipynb
4. try running these examples.
5. these notebooks use the clifford package – for info on syntax, conventions etc, see

   `https://clifford.readthedocs.io/en/latest/index.html`

Exercises 1

1. Set up two vectors a, b, form $c = a \times b$, the cross product. Now form $B = a\wedge b$ and its dual IB, and show that $c = -IB$.

2. Consider the bivector $B = a\wedge b$. By writing

$$a\wedge b = ab - a\cdot b \quad \text{and} \quad a\wedge b = -b\wedge a = -(ba - b\cdot a)$$

   show that B^2 is always positive.

3. For $\{f_1, f_2, f_3\} = \{e_1, e_1 + 2e_3, e_1 + e_2 + e_3\}$ show, using the given formulae, that the reciprocal frame is given by

$$\{f^1, f^2, f^3\} = \{e_1 - \frac{1}{2}(e_2 + e_3), \frac{1}{2}(e_3 - e_2), e_2\}$$

Exercises 2

1. The quaternion algebra of Hamilton (1805-1865) has three *'unit imaginaries'*, i,j,k, which satisfy the following equations:

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 - -1 \quad \text{and} \quad \mathrm{ijk} = -1$$

Show that if we equate the quaternion imaginaries with unit bivectors as follows:

$$\mathrm{i} = e_2 e_3 \qquad \mathrm{j} = -e_3 e_1 \qquad \mathrm{k} = e_1 e_2$$

the above relations are satisfied.

2. Using the Taylor expansion, show that a rotor of the form $R = \mathrm{e}^{\hat{B}\theta/2}$ (where $\hat{B}$ is a bivector which squares to -1) can be written as

$$R = \cos\theta/2 + \hat{B}\sin\theta/2$$

Exercises 3

1. By considering the fact that any bivectors in 3d which are not the same, must have a common line of intersection, show that all bivectors in 3d are blades.

2. In a 4d Euclidean space, give an example of a bivector which cannot be written as a blade.

3. We define the exponentiation of a multivector via its Taylor series and the geometric product, ie

$$\mathrm{e}^{M} = 1 + \frac{M}{1!} + \frac{M^2}{2!} + \dots$$

   Using this, verify the 2d identity used earlier, $e_1 \mathrm{e}^{I\phi} = \mathrm{e}^{-I\phi} e_1$

Exercises 4

1. Show that $a \cdot (BI) = (a \wedge B)I$, with a a vector and B a bivector.
2. Now show that $A_r \cdot (B_s I) = (A_r \wedge B_s)I$.

[Hint: make use of the fact that $A_r \cdot (B_s I_n) = \langle A_r B_s I_n \rangle_{|r-(n-s)|}$].