Ring of Integers of Abelian Number Fields and Algebraic Lattices

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Let $K$ be a number field of degree $n$ (over $\mathbb{Q}$). It exists $n = r_1 + 2r_2$ distinct monomorphisms $\sigma_i : K \to \mathbb{C}$, where $r_1$ is the number of real monomorphisms and $2r_2$ is the number of complex monomorphisms. The application $\sigma : K \to \mathbb{R}^{r_1} \times \mathbb{R}^{2r_2}$ given by

$$\sigma(x) = (\sigma_1(x), \ldots, \sigma_{r_1+r_2}(x)) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \cong \mathbb{R}^{r_1} \times \mathbb{R}^{2r_2}$$

is called Minkowski Homomorphism.

If $J \neq 0$ is an ideal of the ring of integers $O_K$ of $K$, $\sigma(J)$ is a lattice called algebraic lattice. The center density of $\sigma(J)$ is

$$\delta = \frac{t_J^{n/2}}{2^n \sqrt{|D(K)|} N(J)}$$

where

$$t_J = \min\{ Tr_{K:Q}(x\bar{x}) : x \in J, x \neq 0 \}$$

and $D(K)$ is the discriminant of the field $K$. 
Leopoldt-Lettl Theorem

Let $K$ be an abelian number field of conductor $n$ and $G = Gal(K : \mathbb{Q})$. The ring of integers of $K$ is

$$\mathcal{O}_K = \bigoplus_{d \in \mathcal{D}(n)} \mathbb{Z}[G] \eta_d = R_K T.$$  

where

$$G = Gal(K : \mathbb{Q}) \quad K_d = \mathbb{Q}(\zeta_d) \cap K \quad \eta_d = Tr_{\mathbb{Q}(d) : K_d}(\zeta_d)$$

$$T = \sum_{d \in \mathcal{D}(n)} \eta_d \quad R_K = \mathbb{Z}[G][\{\epsilon_d : d \in \mathcal{D}(n)\}] \subset \mathbb{Q}[G]$$

$$\mathcal{D}(n) = \{d \in \mathbb{N} : P_n \mid d, \ d \mid n \text{ e } d \not\equiv 2 \pmod{4}\}$$

in which $P_n$ is the product of the distinct primes divisors of $n$ different of 2 and $\epsilon_d$ are idempotent orthogonal elements of $\mathbb{Q}[G]$. 
Open Problem

It is known algebraic lattices of optimal density center in dimensions 2, 4, 6 and 8. For example:

- The ring of integers of $K = \mathbb{Q}(\zeta_6)$ is the ideal that minimizes the center density in dimension 2 (in this case, $\delta = \frac{1}{2\sqrt{3}}$);

- The principal ideal $(-1 - \zeta_{20} + \zeta_{20}^2 + \zeta_{20}^3 + \zeta_{20}^4)\mathcal{O}_K$ in the ring of integers of $\mathbb{Q}(\zeta_{20})$ minimizes the center density in dimension 8 (in this case, $\delta = 1/16$).

Thinking...

However, we don’t know yet an example of algebraic lattice that has optimal center density in the odd dimensions (in dimension 3, the best density center for lattices is $1/4\sqrt{2}$; in dimension 5, it is $1/8\sqrt{2}$; in the dimension 7, it is $1/16$). This is an open problem. The Leopoldt-Lettl Theorem has been used in our attempt to solve this problem.
Bibliografia


Thanks!

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