Ring of Integers of Abelian Number Fields and Algebraic Lattices

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SP Coding and Information School

January 19th to 30th 2015
UNICAMP - Campinas, Brazil

Let K be a number field of degree n (over $\mathbb{Q}$). It exists $n = r_1 + 2r_2$ distinct monomorphisms $\sigma_i : K \longrightarrow \mathbb{C}$, where r_1 is the number of real monomorphisms and $2r_2$ is the number of complex monomorphisms. The application $\sigma : K \longrightarrow \mathbb{R}^{r_1} \times \mathbb{R}^{2r_2}$ given by

$$\sigma(x) = (\sigma_1(x), \ldots, \sigma_{r_1+r_2}(x)) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \simeq \mathbb{R}^{r_1} \times \mathbb{R}^{2r_2}$$

is called **Minkowski Homomorphism**.
If $J \neq 0$ is an ideal of the ring of integers $\mathcal{O}_K$ of K, $\sigma(J)$ is a lattice called **algebraic lattice**. The center density of $\sigma(J)$ is

$$\delta = \frac{t_J^{n/2}}{2^n \sqrt{|D(K)|} N(J)}$$

where

$$t_J = min\{ Tr_{K:\mathbb{Q}}(x\overline{x}) \ : \ x \in J, x \neq 0\}$$

and $D(K)$ is the discriminant of the field K.

Leopoldt-Lettl Theorem

Let K be an abelian number field of conductor n and $G = Gal(K : \mathbb{Q})$. The ring of integers of K is

$$\mathcal{O}_K = \bigoplus_{d \in \mathcal{D}(n)} \mathbb{Z}[G]\eta_d = R_K T.$$

where

$$G = Gal(K : \mathbb{Q}) \qquad K_d = \mathbb{Q}(\zeta_d) \cap K \qquad \eta_d = Tr_{\mathbb{Q}^{(d)}:K_d}(\zeta_d)$$

$$T = \sum_{d \in \mathcal{D}(n)} \eta_d \qquad R_K = \mathbb{Z}[G][\{\epsilon_d : d \in \mathcal{D}(n)\}] \subset \mathbb{Q}[G]$$

$$\mathcal{D}(n) = \{d \in \mathbb{N} : P_n \mid d,\ d \mid n \text{ e } d \not\equiv 2 \ (mod\ 4)\}$$

in which P_n is the product of the distinct primes divisors of n different of 2 and ϵ_d are idempotent orthogonal elements of $\mathbb{Q}[G]$.

Open Problem

It is known *algebraic* lattices of optimal density center in dimensions 2, 4, 6 and 8. For example:

- The ring of integers of $K = \mathbb{Q}(\zeta_6)$ is the ideal that minimizes the center density in dimension 2 (in this case, $\delta = \frac{1}{2\sqrt{3}}$);
- The principal ideal $(-1 - \zeta_{20} + \zeta_{20}^2 + \zeta_{20}^3 + \zeta_{20}^4)\mathcal{O}_{\mathbb{K}}$ in the ring of integers of $\mathbb{Q}(\zeta_{20})$ minimizes the center density in dimension 8 (in this case, $\delta = 1/16$).

Thinking...

However, we don't know yet an example of algebraic lattice that has optimal center density in the odd dimensions (in dimension 3, the best density center for lattices is $1/4\sqrt{2}$; in dimension 5, it is $1/8\sqrt{2}$; in the dimension 7, it is $1/16$). This is an open problem. The Leopoldt-Lettl Theorem has been used in our attempt to solve this problem.

Bibliografia

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Thanks!

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