The Simplex Conjecture

Basic Formulations (WSC and SSC)

The basic formulation of the Simplex Conjecture in Communication Theory is as follows. Let $\mathcal{S} = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_{n+1}\} \subset \mathbb{R}^n$ be a signal set, used for transmission over a time-discrete Additive White Gaussian Noise (AWGN) channel. Suppose that equiprobable sets are used, i.e., $P(\boldsymbol{s}_i) = 1/(n+1)$.

Conjecture 1 (Weak Simplex Conjecture)**.** If all the vectors have *same* energy, then the average probability of correct decoding is maximized by setting $\boldsymbol{s}_i$ as the vertices of a regular simplex centered at the origin.

Conjecture 2 (Strong Simplex Conjecture)**.** If the average energy of the vectors is constant (i.e., $E[\|\boldsymbol{s}_i\|^2] = c$), then the average probability of correct decoding is maximized by setting $\boldsymbol{s}_i$ as the vertices of a regular simplex centered at the origin.

Of course the SSC implies the WSC, but the converse is not true. The SSC has been proved for $M < 5$ but is *false* for $M \geq 7$. Counter-examples for the SSC can be found in [Ste94] and [LSB12] (see also the references therein for other related results).

A geometric formulation of the (weak) conjecture was provided by Cover in [CE87] (see attachment). Let $\Omega_n \subset \mathbb{R}^n$ be the euclidean sphere

$$\Omega_n = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 = 1\right\}.$$

A spherical cap is an intersection between a translated half-space and Ω_n. A spherical simplex is the intersection of n half spaces and the surface of the unit n-sphere. Let Cap $\subset \Omega_n$ be a spherical cap. Prove that the spherical simplex that maximizes the surface "area" ($(n-1)$-dimensional volume) of the intersection with Cap is the *regular* spherical simplex centered at the center of the cap.

The Problem

Although the SSC has been disproved, the WSC remains open, and is widely believed to be true. The idea is to understand and discuss the strong conjecture. Is it possible to work out small cases ($n = 4, 5$) and see if they provide any insight on generalizations? Computer simulations may help by providing sufficient evidence that conjecture is true, or by finding candidates for counter-examples.

References

[CE87] T. Cover and B. Gopinath Eds. *Open Problems in Communication and Computation*. Springer-Verlag, New York, NY, 1987.

[LSB12] Dejan E. Lazich, Christian Senger, and Martin Bossert. Some comments on the strong simplex conjecture. *CoRR*, abs/1202.1081, 2012.

[Ste94] M. Steiner. The strong simplex conjecture is false. *IEEE Transactions on Information Theory*, 40(3):721–731, May 1994.