Matching Channels and Metrics
Project Proposal for SPCodingSchool

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In coding theory, the Hamming metric has a prominent status, since it can be used to perform maximal likelihood (ML) decoding over a memoryless binary symmetric channel (BSC), in the sense that decoding by choosing a most probable codeword (ML decoding) or a closest codeword — nearest neighbor (NN) decoding — is actually the same decision.

Many different distances are considered in the context of coding theory (a comprehensive account may be found in [2, Chapter 16]), but not much is known about general relation between channel models and metrics and not much is known about the geometry of many important channels.

The matching problem is hence defined as follows:

Definition 1 *Let $W : \mathcal{X} \to \mathcal{X}$ be a channel with input and output alphabets $\mathcal{X}$ and let d be a metric on $\mathcal{X}$, i.e., $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is a function satisfying:*

1. *d is* symmetric*: $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;*
2. *d is* nonnegative*: $d(x, y) \geq 0$ for all $x, y \in \mathcal{X}$, with equality if and only if $x = y$; and*
3. *d satisfies the* triangle inequality*: $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in \mathcal{X}$.*

We say that W and d are matched *if maximum likelihood decoding on W coincides with nearest neighbor decoding with respect to d for every code $C \subseteq \mathcal{X}$, i.e., if for every code $C \subseteq \mathcal{X}$ and every $x \in \mathcal{X}$, we have*

$$\arg\max{}_{y \in C} \Pr(x \textit{ received} \mid y \textit{ sent}) = \arg\min_{y \in C} d(x, y). \tag{1}$$

In this project we are interested in an n-fold binary asymmetric memoryless channel (n-BAC), that is, $\mathcal{X} = \mathbb{F}_2^n$ (binary and n-fold property) and given $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$, $x, y \in \mathbb{F}_2^n$ we have that

$$\Pr(x \text{ received} \mid y \text{ sent}) = \prod_{i=1}^{n} \Pr(x_i \text{ received} \mid y_i \text{ sent})$$

(memoryless property) where

$$\begin{aligned}\Pr(0 \text{ received} \mid 0 \text{ sent}) = 1-p, \;\; \Pr(1 \text{ received} \mid 0 \text{ sent}) = p \\ \Pr(0 \text{ received} \mid 1 \text{ sent}) = q, \;\; \Pr(1 \text{ received} \mid 1 \text{ sent}) = 1-q\end{aligned}$$

with $0 \leq p \leq q \leq 1/2$ (asymmetry property).

We consider the constant $\lambda = p/q \in [0,1]$. The two extreme cases, $\lambda = 0$ and $\lambda = 1$ correspond to the Z-channel and the binary symmetric channel (BSC).

It is well known that the Hamming metric $d(x,y) := \# \{i | x_i \neq y_i\}$ is matched to the BSC. Moreover, considering an n-fold memoryless channel $\mathcal{P}$ over a finite field $\mathbb{F}_r$ other then $\mathbb{F}_2$, there exist a δ metric over $\mathbb{F}_r$ ($\delta : \mathbb{F}_r \times \mathbb{F}_r \to \mathbb{R}$) such that the sum metric on the n-fold metric

$$d(x,y) = \sum_{i=1}^{n} \delta(x_i, y_i)$$

is matched to $\mathcal{P}$ *iff* $\mathcal{P}$ is the r-ary symmetric channel (see [3]).

If we permit to consider metrics that are not defined as a sum, the situation is different. As a very simple example we consider a 2-fold BAC with $0 < \lambda < 1$ (strict inequalities) and it is possible to prove that

	00	10	01	11
00	0	1	1	5/4
10	1	0	5/4	3/4
01	1	5/4	0	3/4
11	5/4	3/4	3/4	0

(2)

is a table defining a metric matched to the channel. Indeed, if we consider the table of probabilities determined by the channel is

	00	10	01	11
00	$(1-p)^2$	$q(1-p)$	$q(1-p)$	q^2
10	$p(1-p)$	$(1-q)(1-p)$	pq	$q(1-q)$
01	$p(1-p)$	pq	$(1-q)(1-p)$	$q(1-q)$
11	p^2	$p(1-q)$	$p(1-q)$	$(1-q)^2$

(3)

and assuming $0 > p > q \leq 1/2$ we find that

$$\begin{aligned}(1-p)^2 > q(1-p) > q^2 \text{ (line 1)} \\ (1-q)(1-p) > q(1-q) > p(1-p) > pq \text{ (lines 2 \& 3)} \\ (1-q)^2 < p(1-q) < p^2 \text{ (line 4)}\end{aligned}$$

It follows that ordering the entries in each line of the table (2) from smaller to higher and (3) from higher to lower values we get the same order:

	00	10	01	11
00	1	2	2	3
10	3	1	4	2
01	3	4	1	2
11	3	2	2	1

This means that choosing a closer element (metric criterion) is equivalent to choose a more likely element (probabilistic criterion).

The main difficulty in matching a metric to the channel is the fact that a metric table (as in 2) should be symmetric and the channel table is not so. In this sense it is somehow surprising that the other extreme case of a BSC, when $\lambda = p = 0$ (the Z-channel) may be matched to a metric, as was recenetely proved by Walker and Firer ([1]).

MAIN GOAL: The main goal of this project is to determine whether there is a metric matched to any n-fold BAC.

HOW TO APPROACH THE PROBLEM;

It is possible to prove that ML decoding over a BAC depends only on λ and not on the choices of p and q. Moreover, for a given n, if we denote $\arg\max_{y\in C}^{\lambda} \Pr(x \text{ received} \mid y \text{ sent})$ the set determined by a code $C \subseteq \mathbb{F}_2^n$, an element $y \in \mathbb{F}_2^n$ and a BAC with $p/q = \lambda$ there are finitely many $\lambda_1, ..., \lambda_{l(n)}$ such that, for any $\lambda \in [0,1]$ there is a unique λ_i such that

$$\arg\max_{y\in C}^{\lambda} \Pr(x|\ y) = \arg\max_{y\in C}^{\lambda_i} \Pr(x|\ y) \tag{4}$$

for any code $C \subseteq \mathbb{F}_2^n$ and any $y \in \mathbb{F}_2^n$. If we consider the map

$$\Phi(p,q) = \left(\log\left(\frac{1}{p} - 1\right), \log\left(\frac{1}{q} - 1\right)\right)$$

we find that for each λ_i, the set of points $\Phi(p,q)$ that determines the same decoding procedure as λ_i (in the sense expressed in equation 4) is a convex cone in the Euclidean plane, contained in the positive octant. The extreme cases are singular ones, corresponding (as a limit) to the lines $y = 0$ and $y = x$, so that any n-fold BAC may be obtained as a linear combination of those extreme points. The first strategy would be to consider appropriate linear combination $((1-t)\, d_Z + t d_H)$ of metrics matched to those extreme cases and to ask whether proper choice of coefficients t leads or not to the desired matched metrics. Numerical calculations (using Mathematica, Maple, Matlab, etc.) for studying small dimensional cases (n small) is feasible and may be very useful.

PREREQUISITES AND SUGGESTED READINGS

This subject is absolutely self contained, basic definitions in coding and information theory is all that is needed. Capability to program is desirable. The problem is presented in more details in the reference ([1]) and ([3]).

References

[1] Walker, Judy and Firer, Marcelo, *Master metrics and master channels,* preprint, 2014.

[2] Michel Marie Deza and Elena Deza, *Encyclopedia of distances*, second ed., Springer, Heidelberg, 2013.

[3] Gérald Séguin, *On metrics matched to the discrete memoryless channel*, J. Franklin Inst. **309** (1980), no. 3, 179–189.