Fakultät für
Mathematik und Informatik

Introduction to rough paths theory

Course 04 - Stability of the rough integral and nonlinear rough differential equations

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Controlled paths as a field of Banach spaces

Nonlinear rough differential equations

Rough differential equations driven by stochastic processes

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Last time, we made the following definition:

Definition 56

Let $\mathcal{X}$ be a topological space and $\{E_x\}_{x\in\mathcal{X}}$ a collection of Banach spaces. $\{E_x\}_{x\in\mathcal{X}}$ is called a *separable continuous field of Banach spaces* if there exists a countable set of sections $\Delta \in \prod_{x\in\mathcal{X}}$, i.e. every $g \in \Delta$ is a map $g\colon \mathcal{X} \to \bigcup_{x\in\mathcal{X}} E_x$ with $g(x) \in E_x$ for every $x \in \mathcal{X}$, that has the following properties:

1. For every $g \in \Delta$, $x \mapsto \|g(x)\|_{E_x} \in \mathbb{R}$ is continuous.
2. For every $x \in \mathcal{X}$, the set $\{g(x) \,:\, g \in \Delta\}$ is dense in E_x.

Lemma 59

Let $X \in \mathcal{C}^\gamma$ and $\alpha < \beta \leq \gamma \leq \frac{1}{2}$. Then the set

$$\mathcal{Z} := \Big\{ (Z, Z') \,:\, Z_t = \int_0^t \phi_u \, \mathrm{d}X_u + \psi_t,\ Z'_t = \phi_t \ : \phi \in \mathcal{C}^\infty([0,T], L(\mathbb{R}^d, W)), \\ \psi \in \mathcal{C}^\infty([0,T], W) \Big\}$$

is dense in $\mathscr{D}_X^{\alpha,\beta}([0,T], W)$. The integral here is defined as a Young-integral.

Proof.

It suffices to proof that $\mathcal{Z}$ is dense in $\mathscr{D}_X^\beta([0,T],W)$ equipped with the norm $|||\cdot|||_\alpha$. Let $(\xi,\xi') \in \mathscr{D}_X^\beta([0,T],W)$ with remainder R^ξ, i.e. $\|\xi'\|_\beta < \infty$ and $\|R^\xi\|_{2\beta} < \infty$. Let

$$\mathcal{P} = \{0 = t_0 < t_1 < \ldots < t_n = T\}$$

be a partition with $\mathcal{P} = |t_{i+1} - t_i| = \theta > 0$ for all $i = 0, \ldots, n-1$. Define $\bar{\xi}' \colon [0,T] \to W$ to be the piecewise-linear approximation of ξ' w.r.t. to $\mathcal{P}$, i.e.

$$\bar{\xi}'_t := \xi'_{t_i} + \frac{t - t_i}{\theta}(\xi'_{t_{i+1}} - \xi'_{t_i}), \quad t \in [t_i, t_{i+1}].$$

Our goal is to find a function ψ with $\psi_0 = \xi_0$ such that for

$$\bar{\xi}_t := \int_0^t \bar{\xi}'_u \, \mathrm{d}X_u + \psi_t,$$

we have $\left|\left|\left|(\xi,\xi') - (\bar{\xi},\bar{\xi}')\right|\right|\right|_\alpha = \|(\xi,\xi') - (\bar{\xi},\bar{\xi}')\|_{X,\alpha} \leq \varepsilon$ for any given $\varepsilon > 0$ as $\theta \to 0$.

Proof (cont.).

Set $\eta := \xi' - \bar{\xi}'$. We leave it as an exercise to prove that $\|\eta\|_\alpha \to 0$ as $\theta \to 0$. It remains to show that

$$\|R^\xi - R^{\bar{\xi}}\|_{2\alpha} \to 0$$

as $\theta \to 0$ where

$$\begin{aligned} R^{\bar{\xi}}_{s,t} &= \delta\bar{\xi}_{s,t} - \bar{\xi}'_s \delta X_{s,t} \\ &= \int_s^t \bar{\xi}'_u \, \mathrm{d}X_u - \bar{\xi}'_s \delta X_{s,t} + \delta\psi_{s,t} \end{aligned}$$

for a ψ still to be chosen. For $s, t \in [0, T]$, we define

$$\rho_{s,t} := \int_s^t \bar{\xi}'_u \, \mathrm{d}X_u - \bar{\xi}'_s \delta X_{s,t} = \int_s^t \delta\bar{\xi}'_{s,u} \, \mathrm{d}X_u.$$

Proof (cont.).

If $s, t \in [t_i, t_{i+1}]$, we have

$$\rho_{s,t} = \frac{\delta \xi'_{t_i,t_{i+1}}}{\theta} \int_s^t (u-s)\,\mathrm{d}X_u.$$

Using the estimate for the Young integral in Theorem 15, we see that

$$\|\rho\|_{2\alpha;[t_i,t_{i+1}]} \leq C\|\xi'\|_\beta \|X\|_\gamma \theta^{\gamma+\beta-2\alpha}.$$

Now we take $t_j, t_k \in \mathcal{P}$, $k < j$. Then,

$$\begin{aligned}
\rho_{t_k,t_j} &= \sum_{k \leq i < j} \left[\int_{t_i}^{t_{i+1}} \bar{\xi}'_{t_i,u}\,\mathrm{d}X_u + \delta \bar{\xi}'_{t_k,t_i} \delta X_{t_i,t_{i+1}} \right] \\
&= \sum_{k \leq i < j} \left[\rho_{t_i,t_{i+1}} - R^\xi_{t_i,t_{i+1}} + \delta \xi_{t_i,t_{i+1}} - \xi_{t_k} \delta X_{t_i,t_{i+1}} \right] \\
&= \sum_{k \leq i < j} \left[\rho_{t_i,t_{i+1}} - R^\xi_{t_i,t_{i+1}} \right] + R^\xi_{t_k,t_j}.
\end{aligned}$$

Proof (cont.).

Setting $\tilde{\rho}_{s,t} := R^{\xi}_{s,t} - \rho_{s,t}$, the calculation above implies that

$$\tilde{\rho}_{t_k,t_j} = \sum_{k \le i < j} \left[\rho_{t_i,t_{i+1}} - R^{\xi}_{t_i,t_{i+1}} \right]. \tag{1}$$

We define $\tilde{\psi}$ to be the continuous, piecewise-linear function satisfying $\tilde{\psi}_0 = \xi_0$ and

$$\delta\tilde{\psi}_{s,t} = \frac{t-s}{t_{i+1}-t_i}(R^{\xi}_{t_i,t_{i+1}} - \rho_{t_i,t_{i+1}}), \quad s,t \in [t_i, t_{i+1}].$$

With this choice,

$$\begin{aligned} R^{\bar{\xi}}_{s,t} &= \int_s^t \bar{\xi}'_u \, \mathrm{d}X_u - \bar{\xi}'_s \delta X_{s,t} + \delta\tilde{\psi}_{s,t} \\ &= \rho_{s,t} + \delta\tilde{\psi}_{s,t}. \end{aligned}$$

Proof (cont.).

Now let $s, t \in \mathcal{P}$ with $t_k \leq s \leq t_{k+1} \leq \cdots \leq t_j \leq t \leq t_{j+1}$. By (1),

$$\delta\tilde{\psi}_{s,t} = \delta\tilde{\psi}_{s,t_{k+1}} + \delta\tilde{\psi}_{t_{k+1},t_{k+2}} + \ldots + \delta\tilde{\psi}_{t_j,t} = \delta\tilde{\psi}_{s,t_{k+1}} + \delta\tilde{\psi}_{t_j,t} + \tilde{\rho}_{t_{k+1},t_j}.$$

Furthermore,

$$\rho_{s,t} = \rho_{s,t_{k+1}} + \rho_{t_{k+1},t_j} + \rho_{t_j,t} + \delta\bar{\xi}'_{s,t_{k+1}}\delta X_{t_{k+1},t_j} + \delta\bar{\xi}'_{t_{k+1},t_j}\delta X_{t_j,t}$$

and

$$\tilde{\rho}_{s,t} = \tilde{\rho}_{s,t_{k+1}} + \tilde{\rho}_{t_{k+1},t_j} + \tilde{\rho}_{t_j,t} + \delta\eta_{s,t_{k+1}}\delta X_{t_{k+1},t_j} + \delta\eta_{t_{k+1},t_j}\delta X_{t_j,t}.$$

Thus, we obtain that

$$\begin{aligned} R^{\bar{\xi}}_{s,t} - R^{\xi}_{s,t} &= \delta\tilde{\psi}_{s,t} - \tilde{\rho}_{s,t} \\ &= \delta\tilde{\psi}_{s,t_{k+1}} + \delta\tilde{\psi}_{t_j,t} - \tilde{\rho}_{s,t_{k+1}} - \tilde{\rho}_{t_j,t} - \delta\eta_{s,t_{k+1}}\delta X_{t_{k+1},t_j} - \delta\eta_{t_{k+1},t_j}\delta X_{t_j,t}. \end{aligned}$$

Proof (cont.).

Each term can now be estimated separately and we can conclude that indeed

$$\|R^{\bar{\xi}} - R^{\xi}\|_{2\alpha} \to 0$$

as $\theta \to 0$. It remains to argue that we can replace the piecewise smooth functions $\bar{\xi}'$ and $\tilde{\psi}$ by genuine smooth functions. This, however, does not cause any problems since we can approximate any continuous function arbitrarily close my smooth functions in the Hölder metric. We omit the details here.

$\square$

Proposition 60

Let $\alpha < \beta \leq \gamma \leq \frac{1}{2}$. Then the family $\{\mathscr{D}_X^{\alpha,\beta}\}_{X \in \mathcal{C}^\gamma}$ is a separable continuous field of Banach spaces.

Proof.

Let $\mathcal{S}$ and $\mathcal{S}'$ be a countable dense subsets of $\mathcal{C}^\infty([0,T], L(\mathbb{R}^d, W))$ resp. $\mathcal{C}^\infty([0,T], W)$. Then we can define Δ as the set of maps $g\colon \mathcal{C}^\gamma \to \bigcup_{X \in \mathcal{C}^\gamma} \mathscr{D}_X^{\alpha,\beta}$ given by $g(X) = (Z, Z')$ where

$$Z_t = \int_0^t \phi_u \, \mathrm{d}X_u + \psi_t, \; Z'_t = \phi_t$$

with $\phi \in \mathcal{S}$ and $\psi \in \mathcal{S}'$. The claimed properties now follow from continuity of the Young integral, cf. Theorem 15, and Lemma 59.

$\square$

In the last lecture, we considered the measurability question of the operator norm of a family of linear mappings

$$\Phi(\mathbf{X}(\omega), \cdot) \colon \mathscr{D}^{\alpha}_{X(\omega)}([0,T], W) \to \bar{W}$$

(like rough integration, for instance). We will formulate a corresponding result now.

Proposition 61

Let $\alpha < \beta \leq \gamma \leq \frac{1}{2}$ and let Δ be the set of sections given in the definition of a continuous measurable field of Banach spaces. Assume that for every rough path $\mathbf{X} \in \mathscr{C}^\gamma$, we have a bounded linear map

$$\Phi(\mathbf{X}, \cdot) \colon \mathscr{D}_X^{\alpha,\beta}([0,T], W) \to \bar{W}$$

that satisfies the property that $\mathbf{X} \mapsto \|\Phi(\mathbf{X}, g(X))\|_{\bar{W}}$ is continuous for every $g \in \Delta$. Let $\mathbf{X}(\omega)$ be a random rough path with the property that $\omega \mapsto \mathbf{X}(\omega)$ is measurable. Then the operator norm

$$\|\Phi(\omega)\| = \sup_{\substack{(Z,Z') \in \mathscr{D}_{X(\omega)}^{\alpha,\beta}([0,T],W) \\ (Z,Z') \neq 0}} \frac{\|\Phi(\mathbf{X}(\omega), (Z,Z'))\|}{\|Z, Z'\|}$$

is measurable.

Proof.

We have for every $\omega \in \Omega$

$$\|\Phi(\omega)\| = \sup_{\substack{(Z,Z') \in \mathscr{D}^{\alpha,\beta}_{X(\omega)}([0,T],W) \\ (Z,Z') \neq 0}} \frac{\|\Phi(\mathbf{X}(\omega),(Z,Z'))\|}{\|Z,Z'\|} = \sup_{g \in \Delta} \frac{\|\Phi(\mathbf{X}(\omega),g(X(\omega)))\|}{\|g(X(\omega))\|}.$$

By our assumptions, $\omega \mapsto \frac{\|\Phi(\mathbf{X}(\omega),g(X(\omega)))\|}{\|g(X(\omega))\|}$ is measurable for every fixed $g \in \Delta$. Since Δ is countable, the result follows.

$\square$

To apply Proposition 61 to the rough integration map, we still have to prove that

$$\mathbf{X} \to \left\| \int g(X)\,\mathrm{d}\mathbf{X} \right\|_{\mathscr{D}^{\alpha,\beta}_X}$$

is continuous. This will follow by a more general result on rough integration.

In the literature, it is known that a continuous field of Banach spaces $\{E_x\}_{x\in\mathcal{X}}$ induces a natural topology on the total space $E := \sqcup_{x\in\mathcal{X}} E_x$. To describe it, we introduce the projection $p\colon E \to \mathcal{X}$, i.e. if $Z \in E_x$, $p(Z) = x$. We define for $g \in \Delta$, an open set $U \subset \mathcal{X}$ and $\varepsilon > 0$ the *tube*

$$W(g, U, \varepsilon) = \{Z \in E \,:\, p(Z) \in U,\ \|Z - g(p(Z))\| < \varepsilon\}.$$

The topology defined on E is the smallest one containing these sets as open sets. It is also called *tube topology*. Fortunately, in the case of controlled paths, the tube topology is completely metrizable with an explicit metric. We state this result now.

Proposition 62

Let $\alpha < \beta \leq \frac{1}{2}$ and $\mathscr{D} := \sqcup_{\mathbf{X}\in\mathscr{C}^\beta} \mathscr{D}_X^{\alpha,\beta}([0,T], W)$. Then the tube topology on $\mathscr{D}$ is completely metrizable with metric given by

$$d^\flat_{\alpha,\beta}((Y,Y'),(\tilde{Y},\tilde{Y}')) := \varrho_\beta(p(Y,Y'),p(\tilde{Y},\tilde{Y}')) + \|Y' - \tilde{Y}'\|_\alpha + \|R^Y - R^{\tilde{Y}}\|_{2\alpha} + |Y_0 - \tilde{Y}_0| + |Y_0' - \tilde{Y}_0'|.$$

If we replace $\mathscr{C}^\beta$ by $\mathscr{C}_g^\beta$, $\mathscr{D}$ is also separable, i.e. Polish.

We can now prove an important stability result for rough integration.

Theorem 63

Let $\mathbf{X}, \tilde{\mathbf{X}} \in \mathscr{C}^\beta$, $(Y, Y') \in \mathscr{D}_X^{\alpha,\beta}$ *and* $(\tilde{Y}, \tilde{Y}') \in \mathscr{D}_{\tilde{X}}^{\alpha,\beta}$. *Set*

$$Z := \int_0^\cdot Y_u \, \mathrm{d}\mathbf{X}_u, \quad Z' := Y$$

and define $(\tilde{Z}, \tilde{Z}')$ *similarly. Then, locally,*

$$d^\flat_{\alpha,\beta}((Z, Z'), (\tilde{Z}, \tilde{Z}')) \leq C \, d^\flat_{\alpha,\beta}((Y, Y'), (\tilde{Y}, \tilde{Y}')).$$

In other words: the integration map

$$(Y, Y') \mapsto \left(\int Y \, \mathrm{d}p(Y, Y'), Y \right)$$

is locally Lipschitz continuous.

Proof.

It suffices to establish a bound for $\|R^Z - R^{\tilde{Z}}\|_{2\alpha}$. Recall that

$$R^Z_{s,t} = \int_s^t Y_u \, \mathrm{d}\mathbf{X}_u - Y_s \delta X_{s,t} = (\mathcal{I}\Xi)_{s,t} - \Xi_{s,t} + Y'_s \mathbb{X}_{s,t}$$

where $\Xi_{u,v} = Y_u \delta X_{u,v} + Y'_u \mathbb{X}_{u,v}$ and $\mathcal{I}$ is the integration map provided by the Sewing lemma. A similar decomposition holds for $R^{\tilde{Z}}_{s,t}$ with Ξ replaced by $\tilde{\Xi}_{u,v} = \tilde{Y}_u \delta \tilde{X}_{u,v} + \tilde{Y}'_u \tilde{\mathbb{X}}_{u,v}$. Setting $\Psi := \Xi - \tilde{\Xi}$, linearity of $\mathcal{I}$ yields

$$|R^Z_{s,t} - R^{\tilde{Z}}_{s,t}| \leq |(\mathcal{I}\Psi)_{s,t} - \Psi_{s,t}| + |Y'_s \mathbb{X}_{s,t} - \tilde{Y}'_s \tilde{\mathbb{X}}_{s,t}|.$$

The Sewing lemma gives us the bound

$$|(\mathcal{I}\Psi)_{s,t} - \Psi_{s,t}| \leq C \|\delta\Psi\|_{3\alpha} |t-s|^{3\alpha}.$$

Proof (cont.).

We have

$$\delta\Psi_{s,u,t} = \delta\Xi_{s,u,t} - \delta\tilde{\Xi}_{s,u,t} = R^{\tilde{Y}}_{s,u}\tilde{X}_{u,t} + \delta\tilde{Y}'_{s,u}\tilde{\mathbb{X}}_{u,t} - R^{Y}_{s,u}X_{u,t} - \delta Y'_{s,u}\mathbb{X}_{u,t}.$$

Therefore, using the triangle inequality,

$$\|\delta\Psi\|_{3\alpha} \leq C d^{\flat}_{\alpha,\beta}((Y,Y'),(\tilde{Y},\tilde{Y}')).$$

The triangle inequality also yields

$$|Y'_s\mathbb{X}_{s,t} - \tilde{Y}'_s\tilde{\mathbb{X}}_{s,t}| \leq C|t-s|^{2\alpha} d^{\flat}_{\alpha,\beta}((Y,Y'),(\tilde{Y},\tilde{Y}'))$$

which concludes the proof.

□

Corollary 64

For every $g \in \Delta$, the map

$$\mathbf{X} \to \left\| \int g(X)\,\mathrm{d}\mathbf{X} \right\|_{\mathscr{D}_X^{\alpha,\beta}}$$

is continuous.

Proof.

For $\mathbf{X}, \tilde{\mathbf{X}} \in \mathscr{C}^\beta$, the reverse triangle inequality for Hölder norms gives

$$\begin{aligned}
&\left| \left\| \int g(X)\,\mathrm{d}\mathbf{X} \right\|_{\mathscr{D}_X^{\alpha,\beta}} - \left\| \int g(\tilde{X})\,\mathrm{d}\tilde{\mathbf{X}} \right\|_{\mathscr{D}_{\tilde{X}}^{\alpha,\beta}} \right| \\
&\leq d^\flat_{\alpha,\beta}\left(\int g(X)\,\mathrm{d}\mathbf{X}, \int g(\tilde{X})\,\mathrm{d}\tilde{\mathbf{X}} \right) \\
&\leq C\, d^\flat_{\alpha,\beta}(g(X), g(\tilde{X}))
\end{aligned}$$

locally.

Proof (cont.).

Recall that

$$g(X) = \left(\int \phi \, \mathrm{d}X + \psi, \phi \right)$$

for some smooth functions ϕ and ψ. Therefore, we can use continuity of the Young integral to see that

$$d^{\flat}_{\alpha,\beta}(g(X), g(\tilde{X})) \leq C \varrho_{\beta}(\mathbf{X}, \tilde{\mathbf{X}})$$

locally and continuity follows.

$\square$

Controlled paths as a field of Banach spaces

Nonlinear rough differential equations

Rough differential equations driven by stochastic processes

We want to solve rough differential equations now.

Definition 65

Let $\mathbf{X} \in \mathscr{C}^\alpha$, $\frac{1}{3} < \alpha \leq \frac{1}{2}$, $\sigma = (\sigma_1, \dots, \sigma_d)$ a collection of vector fields $\sigma_i \colon \mathbb{R}^m \to \mathbb{R}^m$ and $y \in \mathbb{R}^m$. We call $Y \colon [0, T] \to \mathbb{R}^m$ a *solution to the rough differential equation*

$$\begin{aligned} \mathrm{d}Y_t &= \sigma(Y_t)\,\mathrm{d}\mathbf{X}_t; \quad t \in [0, T], \\ Y_0 &= y, \end{aligned}$$

if Y satisfies the integral equation

$$Y_t = y + \int_0^t \sigma(Y_s)\,\mathrm{d}\mathbf{X}_s$$

where the integral is understood as a rough integral.

For the integral $\int \sigma(Y)\,\mathrm{d}\mathbf{X}$ to exist, one needs that $\sigma(Y)$ is controlled by X. Since rough integrals are also controlled paths, the solution Y, if it exists, will be controlled, too. A natural candidate for the Gubinelli derivative of Y is $\sigma(Y)$. We would therefore like to consider the map

$$\mathcal{M}(Y, Y') := \left(y + \int_0^{\cdot} \sigma(Y_s)\,\mathrm{d}\mathbf{X}_s, \sigma(Y) \right)$$

as a map from the space of controlled paths to itself and try show that that it is a contraction on a small time interval. To properly define this map, one has to show that the composition of a controlled path with a sufficiently smooth function σ is again controlled.

Lemma 66

Let $X \in \mathcal{C}^\alpha$, $(Y, Y') \in \mathscr{D}_X^\alpha([0,T], W)$ and let $\varphi\colon W \to \bar{W}$ be twice continuously differentiable. Then the path $t \mapsto \varphi(Y_t)$ is again controlled by X with a Gubinelli derivative given by $\varphi(Y)'_t = D\varphi(Y_t)Y'_t$. Moreover, if σ is bounded with bounded derivatives, the estimate

$$\|\varphi(Y), \varphi(Y)'\|_{X,\alpha} \leq C(\|Y\|_\alpha + \|Y\|_\alpha^2 + \|Y, Y'\|_{X,\alpha})$$

holds where C depends on $\|\sigma\|_{\mathcal{C}^2}$.

Proof.

It suffices to consider the case of σ being bounded with bounded derivatives, the general case follows by localization. We have

$$\|\varphi(Y)\|_\alpha \leq \|D\sigma\|_\infty \|Y\|_\alpha$$

and

$$\begin{aligned}\|\varphi(Y)'\|_\alpha = \|D\varphi(Y)Y'\|_\alpha &\leq \|D\varphi(Y)\|_\alpha \|Y'\|_\infty + \|D\varphi(Y)\|_\infty \|Y'\|_\alpha \\ &\leq \|D^2\sigma\|_\infty \|Y\|_\alpha + \|D\sigma\|_\infty \|Y'\|_\alpha.\end{aligned}$$

Proof (cont.).

This shows that $\varphi(Y), \varphi(Y)' \in \mathcal{C}^\alpha$. We have to prove that

$$\begin{aligned} R^{\varphi}_{s,t} := R^{\varphi(Y)}_{s,t} &:= \delta\varphi(Y)_{s,t} - \sigma(Y)'_s \delta X_{s,t} \\ &= \delta\varphi(Y)_{s,t} - D\varphi(Y_s) Y'_s \delta X_{s,t} \end{aligned}$$

is 2α-Hölder. Since

$$R^{\varphi}_{s,t} = \varphi(Y_t) - \varphi(Y_s) - D\varphi(Y_s)\delta Y_{s,t} + D\varphi(Y_s) R^{Y}_{s,t},$$

Taylor's theorem yields the bound

$$\|R^{\varphi}\|_{2\alpha} \leq \frac{1}{2}\|D^2\varphi\|_\infty \|Y\|^2_\alpha + \|D\varphi\|_\infty \|R^Y\|_{2\alpha}$$

which shows that indeed $(\varphi(Y), \varphi(Y)')$ is controlled by X and the desired bound.

$\square$

Next, we formulate the main theorem about non-linear rough differential equations.

Theorem 67

Let $\mathbf{X} \in \mathscr{C}^\alpha([0,T],\mathbb{R}^d)$ *for* $\frac{1}{3} < \alpha \leq \frac{1}{2}$, $y \in \mathbb{R}^m$ *and* $\sigma \in \mathcal{C}_b^3(\mathbb{R}^m, L(\mathbb{R}^d,\mathbb{R}^m))$. *Then there exists a unique controlled path* $(Y,Y') \in \mathscr{D}_X^\alpha([0,T],\mathbb{R}^m)$ *with* $Y' = \sigma(Y)$ *that satisfies*

$$Y_t = y + \int_0^t \sigma(Y_s)\,\mathrm{d}\mathbf{X}_s; \quad t \in [0,T].$$

Proof.

The idea is the same as for the Young equation, i.e. we will show that an appropriate mapping has a fixed point. For $(Y,Y') \in \mathscr{D}_X^\alpha$ and $0 < T_0 \leq T$, we define

$$(Z_t, Z_t') := (\sigma(Y_t), D\sigma(Y_t)Y_t') \in \mathscr{D}_X^\alpha; \quad t \in [0,T_0].$$

Proof (cont.).

We define the map

$$\mathcal{M}(Y, Y') := \left(y + \int_0^t Z_s \, \mathrm{d}\mathbf{X}_s, Z_t; \; t \in [0, T_0] \right).$$

The expected solution will be a fixed point of this map. We will not define this map on the whole space of controlled paths but rather on the closed unit ball

$$\mathcal{B}_{T_0} := \left\{ (Y, Y') \in \mathscr{D}_X^\alpha \; : \; Y_0 = y, Y_0' = \sigma(y), \|Y, Y'\|_{X,\alpha} \leq 1 \right\}$$

of controlled paths starting in $(y, \sigma(y))$. We will have to prove two things:

1. $\mathcal{M}$ leaves $\mathcal{B}_{T_0}$ invariant, i.e. $\mathcal{M} \colon \mathcal{B}_{T_0} \to \mathcal{B}_{T_0}$ is a well defined map,
2. $\mathcal{M}$ is a contraction.

Proof (cont.).

We start with the first point. Clearly, $\mathcal{M}(Y, Y')_0 = (y, \sigma(y))$. To prove that $\|Y, Y'\|_{X,\alpha} \leq 1$, we use the estimate for the rough integral given in Theorem 55:

$$\begin{aligned}
\|\mathcal{M}\|_{X,\alpha} &= \| \int_0^{\cdot} Z_s \, \mathrm{d}\mathbf{X}_s, Z\|_{X,\alpha} \\
&\leq \|Z\|_\alpha + \|Z'\|_\alpha \|\mathbb{X}\|_{2\alpha} + CT_0^\alpha (\|X\|_\alpha \|R^Z\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha} \|Z'\|_\alpha) \\
&\leq \|Z\|_\alpha + \|Z, Z'\|_{X,\alpha} \|\mathbb{X}\|_{2\alpha} + CT^\alpha \|\!|\mathbf{X}|\!\|_\alpha \|Z, Z'\|_{X,\alpha}.
\end{aligned}$$

We have $\|Z\|_\alpha \leq C\|Y\|_\alpha$ and

$$\begin{aligned}
\|Y\|_\alpha &\leq \|Y'\|_\infty \|X\|_\alpha + T_0^\alpha \|R^Y\|_{2\alpha} \\
&\leq |Y_0'| \|X\|_\alpha + T_0^\alpha \|Y'\|_\alpha \|X\|_\alpha + T^\alpha \|R^Y\|_{2\alpha} \\
&\leq C\|X\|_\alpha + T_0^\alpha (1 + \|X\|_\alpha) \|Y, Y'\|_{X,\alpha} \\
&\leq C\|X\|_\alpha + T_0^\alpha (1 + \|X\|_\alpha).
\end{aligned}$$

Proof (cont.).

To estimate $\|Z, Z'\|_{X,\alpha}$, we use Lemma 66:

$$\begin{aligned}\|Z, Z'\|_{X,\alpha} &\leq C(\|Y\|_\alpha + \|Y\|_\alpha^2 + \|Y, Y'\|_{X,\alpha}) \\ &\leq C(1 + \|Y\|_\alpha + \|Y\|_\alpha^2).\end{aligned}$$

Note that we already estimated $\|Y\|_\alpha$ above. To summarize, we see that $\|\mathcal{M}\|_{X,\alpha}$ gets small if T_0 and $|||\mathbf{X}|||_\alpha$ are getting small. As we already did for Young equations, we will therefore assume first that $\mathbf{X}$ is smoother than only being α-Hölder continuous to assure that $|||\mathbf{X}|||_\alpha$ gets small as $T_0 \to 0$. In total, we can thus guarantee that $\mathcal{M}$ leaves $\mathcal{B}_{T_0}$ invariant for a sufficiently small $T_0 > 0$.

It remains to prove that $\mathcal{M}$ is a contraction on $\mathcal{B}_{T_0}$. To do this, we have to estimate the difference of two rough integrals in the $|||\cdot|||_{X,\alpha}$-norm. Note that we do not have to use Theorem 63 since the driving rough path $\mathbf{X}$ is fixed. The complete proof for the contraction property is a bit long, but does not provide many new insights, that is why we will not present it here. It can be found in [Friz, Hairer; 2020], Theorem 8.3.

□

Let $(Y, Y') \in \mathscr{D}_\beta^\alpha$ be the solution of a rough differential equation (RDE)

$$\begin{aligned} \mathrm{d}Y_t &= \sigma(Y_t)\,\mathrm{d}\mathbf{X}_t; \quad t \in [0, T], \\ Y_0 &= y, \end{aligned}$$

with $Y'_t = \sigma(Y_t)$. Using estimates for the rough integral, we can also find a bound for this solution.

Proposition 68 (A priori estimate on RDE solutions)

It holds that

$$\|Y\|_\alpha \leq C\left((\|\sigma\|_{\mathcal{C}^2}|||\mathbf{X}|||_\alpha) \vee (\|\sigma\|_{\mathcal{C}^2}|||\mathbf{X}|||_\alpha)^{\frac{1}{\alpha}}\right).$$

Proof.

Proposition 8.2 in [Friz Hairer; 2020]. □

There is also a stability result for RDEs. To formulate it, we define the metric

$$\begin{aligned} d_\alpha^\flat((Y,Y'),(\tilde{Y},\tilde{Y}')) &:= d_{\alpha,\alpha}^\flat((Y,Y'),(\tilde{Y},\tilde{Y}')) \\ &:= \varrho_\alpha(\mathbf{X},\tilde{\mathbf{X}}) + \|Y'-\tilde{Y}'\|_\alpha + \|R^Y - R^{\tilde{Y}}\|_{2\alpha} \\ &\quad + |Y_0-\tilde{Y}_0| + |Y_0'-\tilde{Y}_0'| \end{aligned}$$

for $(Y,Y') \in \mathscr{D}_X^\alpha$ and $(\tilde{Y},\tilde{Y}') \in \mathscr{D}_{\tilde{X}}^\alpha$ which is a metric on the total space $\mathscr{D} = \sqcup_{\mathbf{X}\in\mathscr{C}^\alpha} \mathscr{D}_X^\alpha$.

Theorem 69 (Stability of RDE solutions)

Let (Y,Y') and $(\tilde{Y},\tilde{Y}')$ be solutions to

$$\mathrm{d}Y_t = \sigma(Y_t)\,\mathrm{d}\mathbf{X}_t;\ Y_0 = y \quad \textit{resp.} \quad \mathrm{d}\tilde{Y}_t = \sigma(\tilde{Y}_t)\,\mathrm{d}\tilde{\mathbf{X}}_t;\ \tilde{Y}_0 = \tilde{y}$$

with $Y' = \sigma(Y)$ and $\tilde{Y}' = \sigma(\tilde{Y})$. Then

$$d_\alpha^\flat((Y,Y'),(\tilde{Y},\tilde{Y}')) \le C(|y-\tilde{y}| + \varrho_\alpha(\mathbf{X},\tilde{\mathbf{X}}))$$

locally.

Controlled paths as a field of Banach spaces

Nonlinear rough differential equations

Rough differential equations driven by stochastic processes

Proposition 70

Let $(B = (B^1, \dots, B^d)$ be a d-dimensional Brownian motion and $\mathbf{B}^{Itō}$ resp. $\mathbf{B}^{Strat}$ its Itō resp. Stratonovich lift to a rough paths valued process. For $\frac{1}{3} < \alpha < \frac{1}{2}$, assume that $(Y(\omega), Y'(\omega)) \in \mathscr{D}^{\alpha}_{X(\omega)}$ almost surely and that (Y, Y') is adapted to the filtration generated by B. Then

$$\int_0^T Y \, \mathrm{d}B = \int_0^T Y \, \mathrm{d}\mathbf{B}^{Itō} \quad \text{and} \quad \int_0^T Y \circ \mathrm{d}B = \int_0^T Y \, \mathrm{d}\mathbf{B}^{Strat}$$

almost surely.

Proof.

We will only prove the Itō-case, the identity for the Stratonovich integral can be found in [Friz-Hairer; 2020] Corollary 5.2. It is known that

$$\int_0^T Y \, \mathrm{d}B = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} Y_u \delta B_{u,v}$$

in probability.

Proof (cont.).

Passing to a subsequence, we may assume that there is a sequence of partitions such that the convergence holds almost surely. It suffices to prove that

$$\lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} Y'_u \mathbb{B}^{\text{Itō}}_{u,v} = 0$$

in $L^2(\Omega)$. We will assume that $\|Y'(\omega)\| \leq M$ almost surely, the general case follows by a stopping argument. Fix a partition $\mathcal{P} = \{0 = \tau_0 < \ldots \tau_N = T\}$. One can check that (S_k) with $S_0 = 0$ and $S_{k+1} - S_k = Y'_{\tau_k} \mathbb{B}^{\text{Itō}}_{\tau_{k+1},\tau_k}$ is a discrete martingale. Since its increments are uncorrelated,

$$\left\| \sum_{[u,v]\in\mathcal{P}} Y'_u \mathbb{B}^{\text{Itō}}_{u,v} \right\|_{L^2}^2 = \sum_{[u,v]\in\mathcal{P}} \left\| Y'_u \mathbb{B}^{\text{Itō}}_{u,v} \right\|_{L^2}^2 \leq M \sum_{[u,v]\in\mathcal{P}} \left\| \mathbb{B}^{\text{Itō}}_{u,v} \right\|_{L^2}^2 = \mathcal{O}(|\mathcal{P}|)$$

and the claim follows.

□

Corollary 71

For $\sigma \in \mathcal{C}_b^3$, the solutions to

$$\mathrm{d}Y = \sigma(Y)\,\mathrm{d}B \quad \textit{and} \quad \mathrm{d}Y = \sigma(Y)\,\mathrm{d}\mathbf{B}^{\textit{Itō}}$$

resp.

$$\mathrm{d}Y = \sigma(Y) \circ \mathrm{d}B \quad \textit{and} \quad \mathrm{d}Y = \sigma(Y)\,\mathrm{d}\mathbf{B}^{\textit{Strat}}$$

coincide almost surely.

We can now prove an important theorem in stochastic analysis:

Theorem 72 (Wong-Zakai)

Let $\sigma \in \mathcal{C}_b^3$, $y \in \mathbb{R}^m$, $B = (B^1, \dots, B^d)$ be a Brownian motion defined on $[0,1]$ and $B(n)$ and be its piecewise-linear approximation at the the dyadic points $0 < 2^{-n} < \dots (2^n - 1)2^{-n} < 1$. Then the solutions $Y(n)$ to the random ODEs

$$\mathrm{d}Y(n) = \sigma(Y(n))\,\mathrm{d}B(n); \quad Y_0(n) = y$$

converge in the α-Hölder metric for any $\frac{1}{3} < \alpha < \frac{1}{2}$ to the solution of the Stratonovich SDE

$$\mathrm{d}Y = \sigma(Y) \circ \mathrm{d}B; \quad Y_0 = y$$

almost surely as $n \to \infty$.

Proof.

Let $\mathbf{B}(n)$ be the canonical lift of $B(n)$ to an α-Hölder rough path. Then the solutions $Y(n)$ to the random ODEs coincides with the solutions to the RDEs

$$\mathrm{d}Y(n) = \sigma(Y(n))\,\mathrm{d}\mathbf{B}(n); \quad Y_0(n) = y.$$

From Corollary 71, the solution Y of the Stratonovich SDE coincides almost surely with the solution to the RDE

$$\mathrm{d}Y = \sigma(Y)\,\mathrm{d}\mathbf{B}^{\mathrm{Strat}}; \quad Y_0 = y.$$

In Proposition 50, we have seen that $\varrho_\alpha(\mathbf{B}(n), \mathbf{B}^{\mathrm{Strat}}) \to 0$ as $n \to \infty$. From the stability result on RDE solutions (Theorem 69), it follows that

$$d_\alpha^\flat((Y, Y'), (Y(n), Y'(n))) \to 0$$

almost surely as $n \to \infty$. In particular, $Y(n) \to Y$ almost surely as $n \to \infty$ in the α-Hölder metric.

$\square$

Remark 73

What about the fractional Brownian motion? Let B^H be a d-dimensional fBm with Hurst parameter $\frac{1}{4} < H < 1$. Let $B^H(n)$ be a piecewise-linear approximation of B^H. Let $\mathbf{B}^H(n)$ be the canonical lift to an α-Hölder rough path. With a much more involved argument than in Proposition 50 [Coutin-Qian 2002, Friz-Victoir 2007], one can show that $\mathbf{B}^H(n)$ converges in the space $\mathscr{C}^\alpha$ almost surely provided $\alpha < H$. The limit is denoted by $\mathbf{B}^H$ and is called the *natural lift* of the fBm B^H to a rough paths valued process. The above cited results allow to solve and study stochastic differential equations driven by a fBm with rough paths theory in the regime $\frac{1}{4} < H \leq \frac{1}{2}$. For $H \leq \frac{1}{4}$, this approach does not work anymore, one can show that the canonical lift $\mathbf{B}^H(n)$ diverges in this case.

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Thank you.