Fakultät für
Mathematik und Informatik

Introduction to rough paths theory

Course 03 - Geometric rough paths and controlled paths

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In the last lecture, we defined the set $\mathscr{C}^\alpha([0,T],\mathbb{R}^d)$ of α-Hölder rough paths. Note that there is no meaningful notion of a sum of two rough paths, i.e. $\mathscr{C}^\alpha$ is not a linear space. We will see now that it is still a metric space.

Definition 40

Let $\mathbf{X},\mathbf{Y}\in\mathscr{C}^\alpha$. Then we define

$$\varrho_\alpha(\mathbf{X},\mathbf{Y}) := \sum_{n=1}^{\lfloor 1/\alpha\rfloor} \sup_{0\le s<t\le T} \frac{|\mathbb{X}^{(n)}_{s,t}-\mathbb{Y}^{(n)}_{s,t}|}{|t-s|^{n\alpha}}.$$

It is not hard to see that ϱ_α is a metric on $\mathscr{C}^\alpha$. Moreover, one can prove the following:

Proposition 41

For every $\alpha\in(0,1]$, the space $(\mathscr{C}^\alpha,\varrho_\alpha)$ is a complete metric space.

Sometimes, it is desirable to work with separable rough paths spaces. However, since Hölder spaces are not separable, we can not expect that the spaces $\mathscr{C}^\alpha$ are separable. To solve this issue for Hölder spaces, one often considers *little Hölder spaces* that are defined as the closure of the space of smooth functions in the α-Hölder metric. A similar definition works for rough paths spaces, too.

Definition 42

Let $X\colon [0,T] \to \mathbb{R}^d$ be smooth (e.g. piecewise $\mathcal{C}^1$) and $\alpha \in (0,1]$. Then we call $\mathbb{X} \in \mathscr{C}^\alpha$ with

$$\mathbb{X}^{(n)}_{s,t} = \int_{s<u_1<\ldots<u_n<t} \mathrm{d}X_{u_1} \otimes \cdots \otimes \mathrm{d}X_{u_n}$$

the *canonical lift* of X to an α-Hölder rough path. Rough paths $\mathbf{X} \in \mathscr{C}^\alpha$ of this form are also called *smooth rough paths*. The space $\mathscr{C}^\alpha_g$ is defined as the closure of smooth rough paths in the metric ϱ_α. The elements in $\mathscr{C}^\alpha_g$ are called *geometric rough paths*.

Proposition 43

For every $\alpha \in (0,1]$, the space $(\mathscr{C}_g^\alpha, \varrho_\alpha)$ is a complete separable metric, i.e. Polish space.

There is also an important algebraic property satisfied by geometric rough paths that is inherited from smooth rough paths. In fact, multiplying two iterated integrals of smooth paths yields a linear combination of iterated integrals. For example,

$$\int_0^T \mathrm{d}X_s \cdot \int_0^T \mathrm{d}Y_s = \int_{0<s_1<s_2<T} \mathrm{d}X_{s_1}\, \mathrm{d}Y_{s_2} + \int_{0<s_1<s_2<T} \mathrm{d}Y_{s_1}\, \mathrm{d}X_{s_2}.$$

Note that this is a property that does not hold for every rough path. For example, the Itō integral satisfies the identity

$$\int_0^T B_t \, \mathrm{d}B_t = \frac{B_t^2}{2} - \frac{t}{2}.$$

We aim to give a more detailed description of the product of iterated integrals of smooth paths. To do this, we have to introduce some more notation.

Definition 44

The direct sum

$$T(\mathbb{R}^d) := \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d) \oplus \cdots = \bigoplus_{n=0}^{\infty} (\mathbb{R}^d)^{\otimes n}$$

is called *tensor algebra*.

One can show that the extended tensor algebra is the (algebraic) dual of the tensor algebra.

We will identify the basis elements $e_{i_1} \otimes \cdots \otimes e_{i_n}$ in the tensor algebra $T(\mathbb{R}^d)$ with the words $i_1 \cdots i_n$ composed by the letters $1, \ldots, d$. The empty word will be denoted by ϵ.

For two words, we can define their shuffle product:

Definition 45

Let u, v be words and a, b be letters. The *shuffle product* is defined recursively by

$$u \sqcup\!\sqcup \epsilon = \epsilon \sqcup\!\sqcup u = u,$$
$$ua \sqcup\!\sqcup vb = (a \sqcup\!\sqcup vb)a + (ua \sqcup\!\sqcup v)b.$$

The shuffle product is extended bilinearly to a product

$$\sqcup\!\sqcup \colon T(\mathbb{R}^d) \times T(\mathbb{R}^d) \to T(\mathbb{R}^d).$$

Example 46

$$12 \sqcup\!\sqcup 3 = 123 + 132 + 312,$$
$$12 \sqcup\!\sqcup 24 = 2 \cdot 1224 + 1242 + 2124 + 2142 + 2412.$$

Let $X\colon [0,T] \to \mathbb{R}^d$ be a smooth path (e.g. $\mathcal{C}^1$) and

$$\mathbf{X}_{0,T} := \left(1, \int_0^T \mathrm{d}X_s, \dots, \int_{0<s_1<\dots<s_n<T} \mathrm{d}X_{s_1} \otimes \dots \otimes \mathrm{d}X_{s_n}, \dots\right) \in T((\mathbb{R}^d)).$$

With the notation we introduced above, we have, for example,

$$\langle 121, \mathbf{X}_{0,T}\rangle = \int_{0<s_1<s_2<s_3<T} \mathrm{d}X^1_{s_1}\, \mathrm{d}X^2_{s_2}\, \mathrm{d}X^1_{s_3},$$

$$\langle \sqrt{3}\cdot 12 - 2\cdot 21, \mathbf{X}_{0,T}\rangle = \sqrt{3}\int_{0<s_1<s_2<T} \mathrm{d}X^1_{s_1}\, \mathrm{d}X^2_{s_2} - 2\int_{0<s_1<s_2<T} \mathrm{d}X^2_{s_1}\, \mathrm{d}X^1_{s_2} \quad \text{etc.}$$

The main observation is the following:

Theorem 47

For every $l_1, l_2 \in T(\mathbb{R}^d)$,

$$\langle l_1, \mathbf{X}_{0,T}\rangle\langle l_2, \mathbf{X}_{0,T}\rangle = \langle l_1 \sqcup\!\sqcup l_2, \mathbf{X}_{0,T}\rangle.$$

Corollary 48

Let $\mathbf{X} \in \mathscr{C}_g^\alpha$ be a geometric rough path. We identify $\mathbf{X}$ with its Lyons-lift to a path with values in $T((\mathbb{R}^d))$. Then for every $l_1, l_2 \in T(\mathbb{R}^d)$,

$$\langle l_1, \mathbf{X}_{0,T} \rangle \langle l_2, \mathbf{X}_{0,T} \rangle = \langle l_1 \sqcup\!\sqcup l_2, \mathbf{X}_{0,T} \rangle.$$

Remark 49

Let $\mathbf{X}$ be a rough path. As usual, we identify $\mathbf{X}$ with its Lyons-lift to a path with values in $T((\mathbb{R}^d))$. Then the element $\mathbf{X}_{0,T} \in T((\mathbb{R}^d))$ is called the *signature* of the rough path $\mathbf{X}$. The signature is important since it contains all (necessary) information about the rough path. In a series of papers, it was shown that the signature determines a geometric rough path completely up to "tree-like" excursions. The (truncated) signature plays an important role in machine learning.

Proposition 50

The process $\mathbf{B}^{\mathrm{Strat}} = (1, B, \mathbb{B}^{\mathrm{Strat}})$ *takes values in the space* $\mathscr{C}_g^\alpha$ *for every* $\frac{1}{3} < \alpha < \frac{1}{2}$ *almost surely.*

Proof.

For simplicity, $T = 1$. Choose α' such that $\alpha < \alpha' < \frac{1}{2}$. We know that

$$\left|\!\left|\!\left| \mathbf{B}^{\mathrm{Strat}} \right|\!\right|\!\right|_{\alpha'} < \infty.$$

For $n \in \mathbb{N}$, we define $B(n)$ to be the piecewise-linear approximation of B at the dyadic points $0 < 2^{-n} < \cdots < (2^n - 1)2^{-n} < 1$, i.e.

$$B_t(n) = B_{k2^{-n}} + 2^n(t - k2^{-n})(B_{(k+1)2^{-n}} - B_{k2^{-n}}), \qquad t \in [k2^{-n}, (k+1)2^{-n}].$$

Let $\mathbf{B}(n)$ be the canonical lift of $B(n)$ to an α-Hölder rough path.

Proof (cont.).

With some basic calculations, one can show that

$$\|\delta B(n)_{s,t}\|_{L^2} \leq C|t-s|^{\frac{1}{2}} \quad \text{and} \quad \|\mathbb{B}(n)_{s,t}\|_{L^2} \leq C|t-s|$$

holds for every $s < t$ for a constant C that is independent of n. Since B is Gaussian, the same estimates also hold for the L^q-norm for every $q \geq 2$. The Kolmogorov-Chentsov theorem for multiplicative functionals implies that

$$sup_{n\in\mathbb{N}} |||\mathbf{B}(n)|||_{\alpha'} < \infty.$$

To prove that $\varrho_\alpha(\mathbf{B}^{\text{Strat}}, \mathbf{B}(n)) \to 0$, by the Arzelà-Ascoli theorem, it is sufficient to show that $B(n) \to B$ and $\mathbb{B}(n) \to \mathbb{B}^{\text{Strat}}$ pointwise as $n \to \infty$. The first statement is clear. For the second, we first note that

$$\int_s^t (B_u^i(n) - B_s^i(n))\, \mathrm{d}B_u^i(n) = \frac{(B_t(n) - B_s(n))^2}{2} \to \frac{(B_t - B_s)^2}{2} = \int_s^t (B_u^i - B_s^i) \circ \mathrm{d}B_u^i.$$

Proof (cont.).

Define

$$\mathcal{F}_n := \sigma(B_k \,:\, k \in \{0, 2^{-n}, \dots, (2^n - 1)2^{-n}, 1\}).$$

Then $(\mathcal{F}_n)_{n\geq 1}$ is a filtration. Fix $t \in [0, T]$. By Gaussian conditioning, one can check that $B_t(n) = \mathbb{E}[B_t \mid \mathcal{F}_n]$.| From the martingale convergence theorem, it follows that

$$B_t(n) = \mathbb{E}[B_t \mid \mathcal{F}_n] \to B_t$$

almost surely and in L^p for any $p \geq 1$ as $n \to \infty$ (which yields an alternative proof of what we already know). For $i \neq j$,

$$\begin{aligned}\mathbb{E}\left(\int_0^t B_s^i \,\mathrm{d}B_s^j \mid \mathcal{F}_n\right) &= \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \mathbb{E}\left(B_u^i \delta B_{u,v}^j \mid \mathcal{F}_n\right) = \sum_{[u,v]\in\mathcal{P}} B_u^i(n)\delta B_{u,v}^j(n) \\ &= \int_0^t B_s^i(n)\,\mathrm{d}B_s^j(n).\end{aligned}$$

Proof (cont.).

Therefore, the martingale convergence theorem yields that

$$\int_0^t B_s^i(n)\,\mathrm{d}B_s^j(n) = \mathbb{E}\left(\int_0^t B_s^i\,\mathrm{d}B_s^j \mid \mathcal{F}_n\right) \to \int_0^t B_s^i\,\mathrm{d}B_s^j$$

almost surely and in L^p for any $p \geq 1$ as $n \to \infty$ which finishes the proof. $\square$

Exercise: work out the details.

The space of rough paths as a metric space and geometric rough paths

Controlled paths

Controlled paths as a field of Banach spaces

Our goal is to solve non-linear rough differential equations of the form

$$\begin{aligned} \mathrm{d}Y_t &= \sigma(Y_t)\,\mathrm{d}\mathbf{X}_t; \quad t \in [0,T] \\ Y_0 &= y \in \mathbb{R}^m. \end{aligned}$$

As for the Young case, we want to interprete the equation as an integral equation:

$$Y_t = y + \int_0^t \sigma(Y_s)\,\mathrm{d}\mathbf{X}_s.$$

But is there a good notion of an integral we can use?

It is natural to have a notion of an integral that coincides with the usual integral in case the integrand is smooth. That is, for a smooth function f and a Brownian motion B, we would like to have that

$$\int_0^T f(s)\,\mathrm{d}\mathbf{B}_s^{\text{Itō}} = \int_0^T f(s)\,\mathrm{d}B_s.$$

If we want to perform a fixed point argument to solve the equation, it is desirable to look for a Banach space E containing smooth functions such that the map

$$f \mapsto \left(t \mapsto \int_0^t f(s)\,\mathrm{d}\mathbf{B}_s^{\text{Itō}}\right)$$

is a continuous map from E to itself. A minimal requirement for E would be that it contains the trajectories of the Brownian motion, otherwise we would not be able to integrate constant functions. However, one can show that such a space E does not exist:

Theorem 51

There is no space of functions E carrying the Wiener measure on which we can define a continuous map $I\colon E \to E$ that coincides with the pathwise defined integral

$$I(f) = (t \mapsto \int_0^t f(s)\,\mathrm{d}B_s)$$

for smooth functions f on a set of full measure.

Proof.

Same idea as in Theorem 24 in Lecture 2. □

The solution to this issue proposed by rough paths theory is that we allow the space E to depend on the trajectory of the Brownian motion, i.e. we will define spaces $(E_\omega)_{\omega\in\Omega}$ for which $B(\omega) \in E_\omega$ and such that

$$E_\omega \ni f \mapsto (t \mapsto \int_0^t f(s)\,\mathrm{d}B_s(\omega)) \in E_\omega$$

extends the integral map on smooth paths and is continuous.

Our goal is to define a "rough integral" of the form

$$\int_0^T Y_t \, \mathrm{d}\mathbf{X}_t$$

for a given rough path $\mathbf{X} \in \mathscr{C}^\alpha$. To keep it simple, we will assume $\alpha \in (1/3, 1/2]$ from now on. In Lecture 2, we deduced the regularity of a 3-times iterated Young integral by introducing a "compensator":

$$\int_0^T \mathbb{X}^{(2)}_{0,t} \, \mathrm{d}X_t = \sum_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \mathbb{X}^{(2)}_{0,u} \otimes \delta X_{u,v} + \delta X_{0,u} \otimes \mathbb{X}^{(2)}_{u,v}.$$

This motivates the following ansatz: for given $\mathbf{X} = (1, X, \mathbb{X}) \in \mathscr{C}^\alpha$ and $Y \colon [0,T] \to L(\mathbb{R}^d, \mathbb{R}^m)$, we assume that there exists a path $Y' \colon [0,T] \to L(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^m)$ for which we can define the limit

$$\int_0^T Y_t \, \mathrm{d}\mathbf{X}_t = \sum_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} Y_u \delta X_{u,v} + Y'_u \mathbb{X}_{u,v}.$$

We use the Sewing lemma to prove the existence of the limit. Set

$$\Xi_{u,v} := Y_u \delta X_{u,v} + Y'_u \mathbb{X}_{u,v}.$$

Clearly, $\|\Xi\|_\alpha < \infty$. We have to make sure that $\|\delta\Xi\|_\beta < \infty$ for some $\beta > 1$. After some lines of calculations, we see that

$$\delta\Xi_{s,u,t} = -(\delta Y_{s,u} - Y'_s \delta X_{s,u}) X_{u,t} - \delta Y'_{s,u} \mathbb{X}_{u,t}.$$

Therefore, we arrive at the conditions

$$|\delta Y'_{s,t}| = \mathcal{O}(|t-s|^\gamma), \quad \gamma + 2\alpha > 1,$$
$$|\delta Y_{s,t} - Y'_s \delta X_{s,t}| = \mathcal{O}(|t-s|^{\tilde{\gamma}}), \quad \tilde{\gamma} + \alpha > 1.$$

These conditions are in particular satisfied for $\gamma = \alpha$ and $\tilde{\gamma} = 2\alpha$. This motivates the following definition:

Definition 52

Let $\mathbf{X} \in \mathscr{C}^\alpha([0,T],\mathbb{R}^d)$ for $\alpha \in (1/3, 1/2]$. A path $Y \in \mathcal{C}^\alpha([0,T],W)$ is said to be *controlled by* $\mathbf{X}$ if there exists a path $Y' \in \mathcal{C}^\alpha([0,T],L(\mathbb{R}^d,W))$ such that the remainder term R^Y given by

$$R^Y_{s,t} := \delta Y_{s,t} - Y'_s \delta X_{s,t}$$

satisfies $\|R^Y\|_{2\alpha} < \infty$. The path Y' is called a *Gubinelli-derivative* of Y. The set of all controlled paths (Y, Y') is denoted by $\mathscr{D}^\alpha_X([0,T],W)$. If $(Y,Y') \in \mathscr{D}^\alpha_X([0,T],W)$, we set

$$\|Y,Y'\|_{X,\alpha} := \|Y'\|_\alpha + \|R^Y\|_{2\alpha}.$$

It is easily seen that the spaces of controlled paths are linear spaces. Moreover, one can prove the following:

Proposition 53

The spaces $\mathscr{D}_X^\alpha([0,T],W)$ are Banach spaces with a norm given by

$$(Y,Y') \mapsto |Y_0| + |Y_0'| + \|Y,Y'\|_{X,\alpha} =: \left|\left|\left| Y,Y' \right|\right|\right|_{X,\alpha}.$$

Remark 54

Gubinelli derivatives are not unique, in general. Indeed, if Y is smooth, we can choose $Y' = 0$. But if X is smooth, too, we can in fact choose any $\mathcal{C}^\alpha$-path as a Gubinelli-derivative.

The most important fact about controlled paths is that they are good integrands.

Theorem 55

Let $\mathbf{X} \in \mathscr{C}^\alpha([0,T],\mathbb{R}^d)$ *and* $(Y,Y') \in \mathscr{D}_X^\alpha([0,T],L(\mathbb{R}^d,\mathbb{R}^m))$.

1. *The integral*

$$\int_s^t Y_u \,\mathrm{d}\mathbf{X}_u := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} Y_u \delta X_{u,v} + Y'_u \mathbb{X}_{u,v}$$

exists and satisfies the bound

$$\left| \int_s^t Y_u \,\mathrm{d}\mathbf{X}_u - Y_s \delta X_{s,t} + Y'_s \mathbb{X}_{s,t} \right| \leq C(\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha}\|Y\|_\alpha)|t-s|^{3\alpha}. \tag{1}$$

Theorem 55 (cont.)

2. *The path* $t \mapsto \int_0^t Y_u \, \mathrm{d}\mathbf{X}_u$ *is a controlled path with Gubinelli-derivative* Y*. The map*

$$(Y, Y') \mapsto \left(\int_0^{\cdot} Y_u \, \mathrm{d}\mathbf{X}_u, Y \right) =: (Z, Z')$$

is a continuous linear map from $\mathscr{D}_X^\alpha([0,T], L(\mathbb{R}^d, \mathbb{R}^m))$ *to* $\mathscr{D}_X^\alpha([0,T], \mathbb{R}^m)$*. Moreover, we have the bound*

$$\|Z, Z'\|_{X,\alpha} \leq \|Y\|_\alpha + \|Y'\|_\infty \|\mathbb{X}\|_{2\alpha} + CT^\alpha(\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha} \|Y'\|_\alpha).$$

Proof.

As already indicated above, we use the Sewing lemma with

$$\Xi_{u,v} := Y_u \delta X_{u,v} + Y'_u \mathbb{X}_{u,v}.$$

Proof (cont.).

From

$$\delta\Xi_{s,u,t} = -R^Y_{s,u}X_{u,t} - \delta Y'_{s,u}\mathbb{X}_{u,t},$$

we see that

$$\|\delta\Xi\|_{3\alpha} \leq \|R^Y\|_{2\alpha}\|X\|_\alpha + \|Y'\|_\alpha\|\mathbb{X}\|_{2\alpha}$$

and the first assertion follows. For the second assertion, we have to prove that

$$R^Z_{s,t} := \delta Z_{s,t} - Z'_s\delta X_{s,t} = \int_s^t Y_u \,\mathrm{d}\mathbf{X}_u - Y_s\delta X_{s,t}$$

is 2α-Hölder which follows from (1) and the triangle inequality. Calculating the bound for $\|Z, Z'\| = \|Z'\|_\alpha + \|R^Z\|_{2\alpha}$ is straightforward.

$\square$

The space of rough paths as a metric space and geometric rough paths

Controlled paths

Controlled paths as a field of Banach spaces

Recall that we defined a Banach space of controlled paths for every Hölder path X. The question we would like to answer now is whether the indexed spaces $\{\mathscr{D}_X^\alpha([0,T],W)\}_{X\in\mathcal{C}^\alpha}$ have more structure than being just a collection of isolated spaces. This will also have practical relevance. Recall that rough integration induces bounded linear maps

$$\Phi(\mathbf{X},\cdot)\colon \mathscr{D}_X^\alpha([0,T],W)\to\mathscr{D}_X^\alpha([0,T],\bar{W}).$$

If $\mathbf{X}$ is a stochastic process (i.e. a random rough path), the operator norm

$$\|\Phi(\omega)\| := \sup_{\substack{(Z,Z')\in\mathscr{D}_{X(\omega)}^\alpha([0,T],W)\\ (Z,Z')\neq 0}} \frac{\|\Phi(\mathbf{X}(\omega),(Z,Z'))\|}{\|Z,Z'\|}$$

is a natural quantity to consider. One seemingly basic question to answer first is the measurability of this random number. We will see that having some additional structure on the space of controlled paths will help us to answer this question.

We make the following definition:

Definition 56

Let $\mathcal{X}$ be a topological space and $\{E_x\}_{x\in\mathcal{X}}$ a collection of Banach spaces. $\{E_x\}_{x\in\mathcal{X}}$ is called a *separable continuous field of Banach spaces* if there exists a countable set of sections $\Delta \in \prod_{x\in\mathcal{X}}$, i.e. every $g \in \Delta$ is a map $g\colon \mathcal{X} \to \bigcup_{x\in\mathcal{X}} E_x$ with $g(x) \in E_x$ for every $x \in \mathcal{X}$, that has the following properties:

1. For every $g \in \Delta$, $x \mapsto \|g(x)\|_{E_x} \in \mathbb{R}$ is continuous.
2. For every $x \in \mathcal{X}$, the set $\{g(x) \,:\, g \in \Delta\}$ is dense in E_x.

Remark 57

The usual definition in the literature differs slightly from the one above. In [Dixmier '77], the definition of a continuous field of Banach spaces assumes the existence of a *linear subspace* of sections Δ' satisfying (1.) and (2.). Separability means that there is a countable subset $\Delta \subset \Delta'$ satisfying (2.). It is clear that our definition yields also a linear subspace of sections by considering the linear span of Δ. Also, [Dixmier '77] assumes a third property for Δ' that is as follows:

3'. Let $\tilde{g} \in \prod_{x \in \mathcal{X}} E_x$. If for every $y \in \mathcal{X}$ and $\varepsilon > 0$, there exists $g_y \in \Delta'$ such that $\|\tilde{g}(x) - g_y(x)\|_{E_x} \leq \varepsilon$ in some neighbourhood of y in $\mathcal{X}$, then $\tilde{g} \in \Delta'$.

However, one can show that having a Δ' satisfying only (1.) and (2.), one can take some "completion" of Δ' that satisfies (3'.), too. Therefore, the definition we gave here could also be called a separable continuous *pre-field* of Banach spaces.

Do the spaces of controlled paths form a separable continuous field of Banach spaces? We will see that a slightly weaker result holds. Inspired by the little Hölder and geometric rough paths spaces, we make the following definition:

Definition 58

Let $X \in \mathcal{C}^\beta$ and $\alpha \leq \beta$. We define $\mathscr{D}_X^{\alpha,\beta}([0,T],W)$ to be the closure of the space $\mathscr{D}_X^{\beta}([0,T],W)$ in the $|||\cdot|||_\alpha$-Norm.

The key result is the following lemma.

Lemma 59

Let $X \in \mathcal{C}^\gamma$ and $\alpha < \beta \leq \gamma \leq \frac{1}{2}$. Then the set

$$\mathcal{Z} := \Big\{(Z,Z') \,:\, Z_t = \int_0^t \phi_u \,\mathrm{d}X_u + \psi_t,\ Z_t' = \phi_t \ : \phi \in \mathcal{C}^\infty([0,T],L(\mathbb{R}^d,W)),$$
$$\psi \in \mathcal{C}^\infty([0,T],W)\Big\}$$

is dense in $\mathscr{D}_X^{\alpha,\beta}([0,T],W)$. The integral here is defined as a Young-integral.

Proof.

It suffices to proof that $\mathcal{Z}$ is dense in $\mathscr{D}_X^\beta([0,T],W)$ equipped with the norm $|||\cdot|||_\alpha$. Let $(\xi,\xi') \in \mathscr{D}_X^\beta([0,T],W)$ with remainder R^ξ, i.e. $\|\xi'\|_\beta < \infty$ and $\|R^\xi\|_{2\beta} < \infty$. Let

$$\mathcal{P} = \{0 = t_0 < t_1 < \ldots < t_n = T\}$$

be a partition with $\mathcal{P} = |t_{i+1} - t_i| = \theta > 0$ for all $i = 0, \ldots, n-1$. Define $\bar{\xi}' \colon [0,T] \to W$ to be the piecewise-linear approximation of ξ' w.r.t. to $\mathcal{P}$, i.e.

$$\bar{\xi}'_t := \xi'_{t_i} + \frac{t - t_i}{\theta}(\xi'_{t_{i+1}} - \xi'_{t_i}), \quad t \in [t_i, t_{i+1}].$$

Our goal is to find a function ψ with $\psi_0 = \xi_0$ such that for

$$\bar{\xi}_t := \int_0^t \bar{\xi}'_u \, \mathrm{d}X_u + \psi_t,$$

we have $\left|\left|\left|(\xi,\xi') - (\bar{\xi},\bar{\xi}')\right|\right|\right|_\alpha = \|(\xi,\xi') - (\bar{\xi},\bar{\xi}')\|_{X,\alpha} \leq \varepsilon$ for any given $\varepsilon > 0$ as $\theta \to 0$.

Proof (cont.).

Set $\eta := \xi' - \bar{\xi}'$. We leave it as an exercise to prove that $\|\eta\|_\alpha \to 0$ as $\theta \to 0$. It remains to show that

$$\|R^\xi - R^{\bar{\xi}}\|_{2\alpha} \to 0$$

as $\theta \to 0$ where

$$\begin{aligned} R^{\bar{\xi}}_{s,t} &= \delta\bar{\xi}_{s,t} - \bar{\xi}'_s \delta X_{s,t} \\ &= \int_s^t \bar{\xi}'_u \, \mathrm{d}X_u - \bar{\xi}'_s \delta X_{s,t} + \delta\psi_{s,t} \end{aligned}$$

for a ψ still to be chosen. For $s, t \in [0, T]$, we define

$$\rho_{s,t} := \int_s^t \bar{\xi}'_u \, \mathrm{d}X_u - \bar{\xi}'_s \delta X_{s,t} = \int_s^t \delta\bar{\xi}'_{s,u} \, \mathrm{d}X_u.$$

Proof (cont.).

If $s, t \in [t_i, t_{i+1}]$, we have

$$\rho_{s,t} = \frac{\delta \xi'_{t_i,t_{i+1}}}{\theta} \int_s^t (u-s)\,\mathrm{d}X_u.$$

Using the estimate for the Young integral in Theorem 15, we see that

$$\|\rho\|_{2\alpha;[t_i,t_{i+1}]} \leq C\|\xi'\|_\beta \|X\|_\gamma \theta^{\gamma+\beta-2\alpha}.$$

Now we take $t_j, t_k \in \mathcal{P}$, $k < j$. Then,

$$\begin{aligned}
\rho_{t_k,t_j} &= \sum_{k\leq i<j} \left[\int_{t_i}^{t_{i+1}} \bar{\xi}'_{t_i,u}\,\mathrm{d}X_u + \delta\bar{\xi}'_{t_k,t_i}\delta X_{t_i,t_{i+1}} \right] \\
&= \sum_{k\leq i<j} \left[\rho_{t_i,t_{i+1}} - R^\xi_{t_i,t_{i+1}} + \delta\xi_{t_i,t_{i+1}} - \xi_{t_k}\delta X_{t_i,t_{i+1}} \right] \\
&= \sum_{k\leq i<j} \left[\rho_{t_i,t_{i+1}} - R^\xi_{t_i,t_{i+1}} \right] + R^\xi_{t_k,t_j}.
\end{aligned}$$

Proof (cont.).

Setting $\tilde{\rho}_{s,t} := R^{\xi}_{s,t} - \rho_{s,t}$, the calculation above implies that

$$\tilde{\rho}_{t_k,t_j} = \sum_{k \le i < j} \left[\rho_{t_i,t_{i+1}} - R^{\xi}_{t_i,t_{i+1}} \right]. \tag{2}$$

We define $\tilde{\psi}$ to be the continuous, piecewise-linear function satisfying $\tilde{\psi}_0 = \xi_0$ and

$$\delta\tilde{\psi}_{s,t} = \frac{t-s}{t_{i+1}-t_i}(R^{\xi}_{t_i,t_{i+1}} - \rho_{t_i,t_{i+1}}), \quad s,t \in [t_i, t_{i+1}].$$

With this choice,

$$\begin{aligned} R^{\bar{\xi}}_{s,t} &= \int_s^t \bar{\xi}'_u \, \mathrm{d}X_u - \bar{\xi}'_s \delta X_{s,t} + \delta\tilde{\psi}_{s,t} \\ &= \rho_{s,t} + \delta\tilde{\psi}_{s,t}. \end{aligned}$$

Proof (cont.).

Now let $s, t \in \mathcal{P}$ with $t_k \leq s \leq t_{k+1} \leq \cdots \leq t_j \leq t \leq t_{j+1}$. By (2),

$$\delta\tilde{\psi}_{s,t} = \delta\tilde{\psi}_{s,t_{k+1}} + \delta\tilde{\psi}_{t_{k+1},t_{k+2}} + \ldots + \delta\tilde{\psi}_{t_j,t} = \delta\tilde{\psi}_{s,t_{k+1}} + \delta\tilde{\psi}_{t_j,t} + \tilde{\rho}_{t_{k+1},t_j}.$$

Furthermore,

$$\rho_{s,t} = \rho_{s,t_{k+1}} + \rho_{t_{k+1},t_j} + \rho_{t_j,t} + \delta\bar{\xi}'_{s,t_{k+1}}\delta X_{t_{k+1},t_j} + \delta\bar{\xi}'_{t_{k+1},t_j}\delta X_{t_j,t}$$

and

$$\tilde{\rho}_{s,t} = \tilde{\rho}_{s,t_{k+1}} + \tilde{\rho}_{t_{k+1},t_j} + \tilde{\rho}_{t_j,t} + \delta\eta_{s,t_{k+1}}\delta X_{t_{k+1},t_j} + \delta\eta_{t_{k+1},t_j}\delta X_{t_j,t}.$$

Thus, we obtain that

$$\begin{aligned} R^{\bar{\xi}}_{s,t} - R^{\xi}_{s,t} &= \delta\tilde{\psi}_{s,t} - \tilde{\rho}_{s,t} \\ &= \delta\tilde{\psi}_{s,t_{k+1}} + \delta\tilde{\psi}_{t_j,t} - \tilde{\rho}_{s,t_{k+1}} - \tilde{\rho}_{t_j,t} - \delta\eta_{s,t_{k+1}}\delta X_{t_{k+1},t_j} - \delta\eta_{t_{k+1},t_j}\delta X_{t_j,t}. \end{aligned}$$

Proof (cont.).

Each term can now be estimated separately and we can conclude that indeed

$$\|R^{\bar{\xi}} - R^{\xi}\|_{2\alpha} \to 0$$

as $\theta \to 0$. It remains to argue that we can replace the piecewise smooth functions $\bar{\xi}'$ and $\tilde{\psi}$ by genuine smooth functions. This, however, does not cause any problems since we can approximate any continuous function arbitrarily close my smooth functions in the Hölder metric. We omit the details here.

$\square$

Proposition 60

Let $\alpha < \beta \leq \gamma \leq \frac{1}{2}$. Then the family $\{\mathscr{D}_X^{\alpha,\beta}\}_{X \in \mathcal{C}^\gamma}$ is a separable continuous field of Banach spaces.

Proof.

Let $\mathcal{S}$ and $\mathcal{S}'$ be a countable dense subsets of $\mathcal{C}^\infty([0,T], L(\mathbb{R}^d, W))$ resp. $\mathcal{C}^\infty([0,T], W)$. Then we can define Δ as the set of maps $g\colon \mathcal{C}^\gamma \to \bigcup_{X \in \mathcal{C}^\gamma} \mathscr{D}_X^{\alpha,\beta}$ given by $g(X) = (Z, Z')$ where

$$Z_t = \int_0^t \phi_u \, \mathrm{d}X_u + \psi_t, \; Z'_t = \phi_t$$

with $\phi \in \mathcal{S}$ and $\psi \in \mathcal{S}'$. The claimed properties now follow from continuity of the Young integral, cf. Theorem 15, and Lemma 59.

$\square$

- T. Lyons, M. Caruana, T. Lévy. Differential equations driven by rough paths. St. Flour Lecture notes. Springer (2007).
- C. Reutenauer. Free Lie Algebras. Oxford University Press (1993).
- H. Boedihardjoa, X. Geng, T. Lyons, D. Yang. The signature of a rough path: Uniqueness. Advances in Mathematics (2016).
- I. Chevyrev, A. Kormilitzin. A primer on the signature method in machine learning. arXiv:1603.03788 (2016).
- T. Lyons, A. D. McLeod. Signature methods in machine learning. arXiv:2206.14674 (2022).
- P. K. Friz, N. Victoir. Multidimensional stochastic processes as rough paths: theory and applications. Cambridge University Press (2010).

M. Gubinelli. Controlling rough paths. JFA (2004).

P. K. Friz, M. Hairer. A course on rough paths. Springer (2020).

J. Dixmier. C^*-algebras. Elsevier (North Holland) (1977).

M. Ghani Varzaneh. Dynamical properties of rough delay equations. PhD thesis, TU Berlin (2021).

Thank you.