Introduction to rough paths theory

Course 01 - Fractional Brownian motion and the Young integral
Motivation: Fractional Brownian motion

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**Definition 1**

A stochastic process $X : [0, \infty) \to \mathbb{R}$ is called *Gaussian* if for every $k \in \mathbb{N}$ and every $t_1, \ldots, t_k \in [0, \infty)$, the random variable $(X_{t_1}, \ldots, X_{t_k})$ is a multivariate Gaussian random variable.
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Note that the law of a Gaussian process is completely determined by the means $\mathbb{E}(X_t)$ and the covariances $\text{cov}(X_s, X_t)$ of the process.
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**Definition 2 (Mandelbrot, van Ness ‘68)**
Let $H \in (0, 1)$. The *fractional Brownian motion* (fBm) is a continuous zero mean Gaussian process $B^H : [0, \infty) \to \mathbb{R}$ with covariance function given by

$$R(s, t) := \text{cov}(B^H_s, B^H_t) = \mathbb{E}(B^H_s B^H_t) = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t - s|^{2H} \right).$$

The parameter $H \in (0, 1)$ is called *Hurst parameter*.
Remark 3

For $H = \frac{1}{2}$, one obtains $R(s, t) = \min\{s, t\}$, i.e. $B^H$ is the usual Brownian motion (Bm).
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The Hurst parameter describes the behaviour of the process. One easy observation is the following:

**Lemma 4**
The increments of the fBm are
1. uncorrelated for $H = \frac{1}{2}$,
2. positively correlated for $H > \frac{1}{2}$,
3. negatively correlated for $H < \frac{1}{2}$.

**Remark 5**
– The noise process $t \mapsto \dot{B}^H$ is thus white only in the case $H = \frac{1}{2}$. For $H \neq \frac{1}{2}$, the noise $\dot{B}^H$ could be called colored.
– Using the fBm instead of the Bm for modelling random phenomena can be more realistic in case of models with memory. For instance, it was used to model price processes in illiquid markets (electricity markets, gas markets etc.)
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– Using the fBm instead of the Bm for modelling random phenomena can be more realistic in case of models with memory. For instance, it was used to model price processes in illiquid markets (electricity markets, gas markets etc.)
A generic stochastic differential equation (SDE) involving the fBm would be of the form

\[ dY_t = b(Y_t) \, dt + \sum_{i=1}^{d} \sigma_i(Y_t) \, dB_t^H; \quad t \geq 0 \]

\[ Y_0 = y_0 \in \mathbb{R}^m \]  

(1)

where \( B^H = (B^H; 1, \ldots, B^H; d) \) is a \( d \)-dimensional fractional Brownian motion, i.e. a vector of independent one-dimensional fBm, \( b, \sigma_1, \ldots, \sigma_d : \mathbb{R}^m \to \mathbb{R}^m \) is a collection of vector fields and \( Y : [0, \infty) \to \mathbb{R}^m \) is a stochastic process we aim to call a solution to (1).
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The \textbf{fundamental problem} is: How should we interprete (1)? What properties should the process \( Y \) satisfy to call it a \textit{solution} to the SDE (1)?
First attempt: If the trajectories of $B^H$, i.e. the paths $t \mapsto B^H_t(\omega)$, $\omega \in \Omega$, were differentiable, we could interpret (1) \textit{pathwise} as a random (non-autonomous) ordinary differential equation (ODE):

$$\frac{dY_t}{dt} = b(Y_t) + \sum_{i=1}^{d} \sigma^i(Y_t) \frac{dB^H_{t;i}(\omega)}{dt}.$$  \hfill (2)
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However, the following result shows that this attempt fails:

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For every $H \in (0,1)$, the trajectories of the fBm are almost surely nowhere differentiable.
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**Remark 7**

Motivated by partial differential equations (PDE), one might have the idea to weaken the notion of differentiability in order to give a meaning to (2): we could interpret $\frac{dB^H_{t;i}(\omega)}{dt}$ as a *weak derivative*, i.e. as a *distribution* or *generalized function*.
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Second attempt: In stochastic analysis, the Itô-integral $\int Y \, dX$ is defined in case of $X$ being a semimartingale and $Y$ being adapted to the filtration generated by $X$. We could try to interpret equation (1) as an integral equation

$$Y_t = Y_0 + \int_0^t b(Y_s) \, ds + \sum_{i=1}^d \int_0^t \sigma_i^i(Y_s) \, dB_{s}^{H,i},$$

where the stochastic integral is understood in Itô-sense.
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where the stochastic integral is understood in Itô-sense. Unfortunately, one can prove the following:

**Proposition 8**

*The fBm $B^H$ is not a semimartingale unless $H = \frac{1}{2}$.*
**Third attempt:** In 1936, L.C. Young introduced a notion of an integral that generalized Riemann-Stieltjes integration. More concretely, he defined an integral for functions $f, g : [0, T] \to \mathbb{R}$ that are Hölder continuous with Hölder index $\alpha \in (0, 1]$ resp. $\beta \in (0, 1]$ of the form $\int f \, dg$ provided $\alpha + \beta > 1$. 

What do we know about the regularity of the trajectories of an fBm? The following theorem is a classical result:

**Theorem 9 (Kolmogorov-Chentsov)**

Let $X : [0, T] \to \mathbb{R}$ be a continuous stochastic process, $q \geq 2$, $\beta > 1/q$ and assume that

$$\|X_t - X_s\|_{L^q} \leq C |t - s|^\beta$$

for a constant $C > 0$ and any $s, t \in [0, T]$.

Then for all $\alpha \in (0, (\beta - 1)/q)$, there is a random variable $K_\alpha \in L^q$ such that

$$|X_t - X_s| \leq K_\alpha |t - s|^\alpha$$

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In particular, the trajectories of $X$ are almost surely $\alpha$-Hölder continuous.
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The trajectories of the fBm are almost surely $\alpha$-Hölder continuous for every $\alpha < H$. 
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For later purposes, we give a proof of the Komogorov-Chentsov theorem, too.
Proof of Theorem 9.

W.l.o.g., $T = 1$. Set $D_n = \{k2^n : k = 0, \ldots, 2^n\}$ and $D = \bigcup_{n \geq 0} D_n$. We further define the random variables

$$K_n := \sup_{t \in D_n} |\delta X_{t, t+2^n}|, \quad \delta X_{t, t+2^n} := X_{t+2^n} - X_t.$$
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Set $|D_n| = 2^{-n}$. It holds that

$$\mathbb{E}(K_n^q) \leq \mathbb{E} \sum_{t \in D_n} |\delta X_{t, t+2^{-n}}|^q \leq \frac{1}{|D_n|} C^q |D_n|^{q\beta} = C^q |D_n|^{q\beta-1}.$$
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Fix $s < t \in D$ and choose $m$ such that $|D_{m+1}| < t - s \leq |D_m|$. Going from coarser to finer partitions successively, we can find $\tau_0, \ldots, \tau_N \in \bigcup_{n \geq m+1} D_n$ such that

$$s = \tau_0 < \tau_1 < \ldots < \tau_N = t$$

with the property that at most two intervals of the form $[\tau_i, \tau_{i+1}]$ have the same length.
Proof of Theorem 9 (cont.).

With this choice, it follows that

$$|\delta X_{s,t}| \leq \sum_{i=0}^{N-1} |\delta X_{\tau_i,\tau_{i+1}}| \leq 2 \sum_{n \geq m+1} K_n.$$
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We thus obtain

\[ \frac{|\delta X_{s,t}|}{|t - s|^{\alpha}} \leq \sum_{n \geq m+1} \frac{2K_n}{|D_{m+1}|^{\alpha}} \leq \sum_{n \geq m+1} \frac{2K_n}{|D_n|^{\alpha}} \leq 2 \sum_{n \geq 0} \frac{K_n}{|D_n|^{\alpha}} =: K_{\alpha}. \]
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Therefore, we have shown that

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for every $s, t \in D$. By continuity of $X$, this bound holds in fact for every $s, t \in [0, 1]$.  

Proof of Theorem 9 (cont.).
It remains to check that $K_\alpha$ is in $L^q$.

Remark 11
Often, the formulation of the Kolmogorov-Chentsov theorem does not assume that $X$ is continuous. The statement then says that $X$ has a Hölder-continuous modification $\tilde{X}$, i.e. $X_t = \tilde{X}_t$ almost surely for every $t$. Note that the proof above yields the same statement: instead of using continuity of $X$, we define a process $\tilde{X}$ to coincide with $X$ on the dyadic numbers $D$ and extend it continuously to the whole interval $[0, 1]$. One can check that $\tilde{X}$ is a modification of $X$. 
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It remains to check that $K_\alpha$ is in $L^q$. Indeed,

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\|K_\alpha\|_{L^q} \leq 2 \sum_{n \geq 0} \frac{\|K_n\|_{L^q}}{|D_n|^{\alpha}} \leq 2C \sum_{n \geq 0} |D_n|^{\beta - \frac{1}{q} - \alpha}
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Motivation: Fractional Brownian motion

Young integral and Sewing lemma
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Let $W$ be a Banach space.
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1. $C([0, T], W)$ will denote the space of continuous functions $f : [0, T] \to W$.
2. For $f \in C([0, T], W)$, we define $\delta f_{s, t} := f_t - f_s$. $C^\alpha([0, T], W)$ denotes the space of $\alpha$-Hölder continuous functions, i.e. $f \in C^\alpha([0, T], W)$ iff

$$\|f\|_\alpha := \sup_{s < t} \frac{|\delta f_{s, t}|}{|t - s|^\alpha} < \infty.$$
Definition 12 (cont.)

3. The space $C_2^{\alpha, \beta}([0, T], W)$ denotes the space of functions $\Xi$ defined on the simplex $\{(s, t) \in [0, T]^2 : s \leq t\}$ such that $\Xi_{t,t} = 0$ and

$$\|\Xi\|_{\alpha, \beta} := \|\Xi\|_\alpha + \|\delta \Xi\|_\beta < \infty$$
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3. The space $C_{2}^{\alpha,\beta}([0, T], \mathcal{W})$ denotes the space of functions $\Xi$ defined on the simplex $\{(s, t) \in [0, T]^2 : s \leq t\}$ such that $\Xi_{t,t} = 0$ and

$$
\|\Xi\|_{\alpha,\beta} := \|\Xi\|_{\alpha} + \|\delta\Xi\|_{\beta} < \infty
$$

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\|\Xi\|_{\alpha} := \sup_{s < t} \frac{|\Xi_{s,t}|}{|t - s|^\alpha}
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3. The space $C_{2}^{\alpha,\beta}([0, T], W)$ denotes the space of functions $\Xi$ defined on the simplex $\{(s, t) \in [0, T]^2 : s \leq t\}$ such that $\Xi_{t,t} = 0$ and

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where

$$\|\Xi\|_{\alpha} := \sup_{s < t} \frac{|\Xi_{s,t}|}{|t - s|^\alpha}$$

and

$$\delta \Xi_{s,u,t} := \Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}, \quad \|\delta \Xi\|_{\beta} := \sup_{s < u < t} \frac{|\delta \Xi_{s,u,t}|}{|t - s|^\beta}.$$
Lemma 13 (Sewing lemma)

Let $0 < \alpha \leq 1 < \beta$. Then there exists a unique continuous linear map
$I : C_2^{\alpha,\beta}([0, T], \mathcal{W}) \to C^\alpha([0, T], \mathcal{W})$ such that $(I \Xi)_0 = 0$ and

$$|\delta I \Xi_{s,t} - \Xi_{s,t}| \leq C \|\delta \Xi\|_\beta |t - s|^\beta$$  \hspace{1cm} (3)

where $C > 0$ depends on $\beta$. 

Lemma 13 (Sewing lemma)

Let \(0 < \alpha \leq 1 < \beta\). Then there exists a unique continuous linear map \(I : C_{2}^{\alpha,\beta}([0,T], W) \to C^{\alpha}([0,T], W)\) such that \((I\Xi)_{0} = 0\) and

\[
|\delta I\Xi_{s,t} - \Xi_{s,t}| \leq C\|\delta \Xi\|_{\beta}|t - s|^{\beta} \tag{3}
\]

where \(C > 0\) depends on \(\beta\).

Proof.

Let us prove uniqueness first. Assume that \(I\) and \(\tilde{I}\) both satisfy (3). Then it holds that

\[
|(I - \tilde{I})_{t} - (I - \tilde{I})_{s}| \leq C|t - s|^{\beta}.
\]
Lemma 13 (Sewing lemma)

Let $0 < \alpha \leq 1 < \beta$. Then there exists a unique continuous linear map $I : C^\alpha_{2,\beta}([0, T], W) \to C^\alpha([0, T], W)$ such that $(I \Xi)_0 = 0$ and

$$|\delta I \Xi_{s,t} - \Xi_{s,t}| \leq C \|\delta \Xi\|_\beta |t - s|^{\beta}$$

where $C > 0$ depends on $\beta$.

Proof.

Let us prove uniqueness first. Assume that $I$ and $\tilde{I}$ both satisfy (3). Then it holds that

$$|(I - \tilde{I})_t - (I - \tilde{I})_s| \leq C|t - s|^{\beta}.$$ 

Since $\beta > 1$ and $I - \tilde{I}$ is a path, $I - \tilde{I}$ is constant. Since $I_0 = \tilde{I}_0 = 0$, uniqueness follows.
Proof (cont.).

Fix an interval $[s, t]$ and a partition $\mathcal{P} = \{s = u_0 < u_1 < \ldots < u_r = t\}$ of this interval. Let $|\mathcal{P}|$ denote the maximal length between two consecutive points in $\mathcal{P}$, i.e.

$$|\mathcal{P}| := \max_i |u_{i+1} - u_i|.$$
Proof (cont.).

Fix an interval $[s, t]$ and a partition $\mathcal{P} = \{s = u_0 < u_1 < \ldots < u_r = t\}$ of this interval. Let $|\mathcal{P}|$ denote the maximal length between two consecutive points in $\mathcal{P}$, i.e.

$$|\mathcal{P}| := \max_i |u_{i+1} - u_i|.$$

We set

$$\int_{\mathcal{P}} \Xi := \sum_i \Xi_{u_i, u_{i+1}}.$$

The idea is now to establish a maximal inequality for $\int_{\mathcal{P}} \Xi$ by successively removing distinguished points from the partition $\mathcal{P}$. 
Proof (cont.).

We claim that if \( r \geq 2 \), there exists a point \( u \in \mathcal{P} \) such that for its neighbouring points \( u_- < u < u_+ \in \mathcal{P} \),

\[
|u_+ - u_-| \leq \frac{2}{r-1} |t - s|.
\]
Proof (cont.).

We claim that if $r \geq 2$, there exists a point $u \in \mathcal{P}$ such that for its neighbouring points $u_- < u < u_+ \in \mathcal{P}$,

$$|u_+ - u_-| \leq \frac{2}{r - 1} |t - s|.$$

Indeed, otherwise we would have

$$2|t - s| \geq \sum_{u \in \mathcal{P} \setminus \{u_0, u_r\}} |u_+ - u_-| > 2|t - s|$$

which is a contradiction.
Proof (cont.).

We claim that if $r \geq 2$, there exists a point $u \in \mathcal{P}$ such that for its neighbouring points $u_- < u < u_+ \in \mathcal{P}$,

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Indeed, otherwise we would have

$$2|t - s| \geq \sum_{u \in \mathcal{P} \setminus \{u_0, u_r\}} |u_+ - u_-| > 2|t - s|$$

which is a contradiction. With this choice, we obtain

$$\left| \int_{\mathcal{P}} \Xi - \int_{\mathcal{P} \setminus \{u\}} \Xi \right| = |\delta_{\Xi_{u-,u_+}}| \leq \|\delta_{\Xi}\|_\beta |u_+ - u_-|^\beta \leq \|\delta_{\Xi}\|_\beta \frac{2^\beta |t - s|^\beta}{(r - 1)^\beta}. $$
Proof.
By successively removing points, we arrive at the uniform bound

\[
\sup_{\mathcal{P}} \left| \int_{\mathcal{P}} \Xi - \Xi_{s,t} \right| \leq 2^\beta |t - s|^\beta \|\delta \Xi\|_\beta \sum_{k=1}^{\infty} \frac{1}{k^\beta} = 2^\beta |t - s|^\beta \|\delta \Xi\|_\beta \zeta(\beta)
\]

(4)
Proof.
By successively removing points, we arrive at the uniform bound

\[
\sup_{\mathcal{P}} \left| \int_{\mathcal{P}} \Xi - \Xi_{s,t} \right| \leq 2^\beta |t - s|^\beta \|\delta \Xi\|_\beta \sum_{k=1}^{\infty} \frac{1}{k^\beta} = 2^\beta |t - s|^\beta \|\delta \Xi\|_\beta \zeta(\beta) \quad (4)
\]

where the right hand side is finite since \( \beta > 1 \).
Proof.
By successively removing points, we arrive at the uniform bound

$$\sup_{P} \left| \int_{P} \Xi - \Xi_{s,t} \right| \leq 2^\beta |t - s|^\beta \|\delta \Xi\|_\beta \sum_{k=1}^{\infty} \frac{1}{k^\beta} = 2^\beta |t - s|^\beta \|\delta \Xi\|_\beta \zeta(\beta)$$  \hfill (4)

where the right hand side is finite since $\beta > 1$. We aim to define $\mathcal{I} \Xi$ as the limit

$$\lim_{|P| \to 0} \int_{P} \Xi$$

for which we have to prove the existence now.
Proof.
By successively removing points, we arrive at the uniform bound

$$
\sup_{\mathcal{P}} \left| \int_{\mathcal{P}} \Xi - \Xi_{s,t} \right| \leq 2^\beta |t - s|^\beta \|\delta \Xi\|_\beta \sum_{k=1}^{\infty} \frac{1}{k^\beta} = 2^\beta |t - s|^\beta \|\delta \Xi\|_\beta \zeta(\beta) \quad (4)
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where the right hand side is finite since $\beta > 1$. We aim to define $\mathcal{I} \Xi$ as the limit $\lim_{|\mathcal{P}| \to 0} \int_{\mathcal{P}} \Xi$ for which we have to prove the existence now. It suffices to show that

$$
\sup_{\max\{|\mathcal{P}|,|\mathcal{P}'|\} \leq \varepsilon} \left| \int_{\mathcal{P}} \Xi - \int_{\mathcal{P}'} \Xi \right| \to 0 \quad \text{as } \varepsilon \to 0.
$$
Proof.
By successively removing points, we arrive at the uniform bound

$$\sup_{\mathcal{P}} \left| \int_{\mathcal{P}} \Xi \right| \leq 2^\beta |t - s|^\beta \|\delta \Xi\|_\beta \sum_{k=1}^\infty \frac{1}{k^\beta} = 2^\beta |t - s|^\beta \|\delta \Xi\|_\beta \zeta(\beta)$$  \hfill (4)

where the right hand side is finite since $\beta > 1$. We aim to define $\mathcal{I}\Xi$ as the limit $\lim_{|\mathcal{P}| \to 0} \int_{\mathcal{P}} \Xi$ for which we have to prove the existence now. It suffices to show that

$$\sup_{\max\{|\mathcal{P}|,|\mathcal{P}'|\} \leq \varepsilon} \left| \int_{\mathcal{P}} \Xi - \int_{\mathcal{P}'} \Xi \right| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$  

By adding and substracting $\int_{\mathcal{P} \cup \mathcal{P}'} \Xi$, we can assume w.l.o.g. that $\mathcal{P} \subset \mathcal{P}'$. 


Proof.
By successively removing points, we arrive at the uniform bound

$$\sup_\mathcal{P} \left| \int_\mathcal{P} \Xi - \Xi_{s,t} \right| \leq 2^\beta |t - s|^\beta \|\delta \Xi\|_{\beta} \sum_{k=1}^{\infty} \frac{1}{k^\beta} = 2^\beta |t - s|^\beta \|\delta \Xi\|_{\beta} \zeta(\beta) \quad (4)$$

where the right hand side is finite since $\beta > 1$. We aim to define $\mathcal{I}\Xi$ as the limit $\lim_{|\mathcal{P}| \to 0} \int_\mathcal{P} \Xi$ for which we have to prove the existence now. It suffices to show that

$$\sup_{\max\{|\mathcal{P}|, |\mathcal{P}'| \leq \varepsilon} \left| \int_\mathcal{P} \Xi - \int_{\mathcal{P}'} \Xi \right| \to 0 \quad \text{as } \varepsilon \to 0.$$

By adding and subtracting $\int_{\mathcal{P} \cup \mathcal{P}'} \Xi$, we can assume w.l.o.g. that $\mathcal{P} \subset \mathcal{P}'$. In this case,

$$\int_\mathcal{P} \Xi - \int_{\mathcal{P}'} \Xi = \sum_{[u,v] \in \mathcal{P}} \left( \Xi_{u,v} - \int_{\mathcal{P}' \cap [u,v]} \Xi \right).$$
Proof (cont.).

For $\max\{|\mathcal{P}|, |\mathcal{P}'|\} = |\mathcal{P}| \leq \varepsilon$, we can use the maximal inequality (4) to see that

$$\left| \int_{\mathcal{P}} \Xi - \int_{\mathcal{P}'} \Xi \right| \leq 2\beta \|\delta\|_{\beta} \zeta(\beta) \sum_{[u, v] \in \mathcal{P}} |v - u|^{\beta} = O\left( |\mathcal{P}|^{\beta-1} \right) = O\left( \varepsilon^{\beta-1} \right).$$

This finishes the proof.

Remark 14

In the proof of Lemma 13, we saw that

$$\delta_{I} \Xi_{s, t} = \lim_{|\mathcal{P}| \to 0} \sum_{[u, v] \in \mathcal{P}} \Xi_{u, v}.$$
Proof (cont.).

For \( \max\{|\mathcal{P}|, |\mathcal{P}'|\} = |\mathcal{P}| \leq \varepsilon \), we can use the maximal inequality (4) to see that

\[
\left| \int_{\mathcal{P}} \Xi - \int_{\mathcal{P}'} \Xi \right| \leq 2^\beta \|\delta \Xi\|_\beta \zeta(\beta) \sum_{[u,v] \in \mathcal{P}} |v - u|^\beta = O(|\mathcal{P}|^{\beta-1}) = O(\varepsilon^{\beta-1}).
\]

This finishes the proof.

\[\square\]
Proof (cont.).
For $\max\{|\mathcal{P}|, |\mathcal{P}'|\} = |\mathcal{P}| \leq \varepsilon$, we can use the maximal inequality (4) to see that

$$\left| \int_{\mathcal{P}} \Xi - \int_{\mathcal{P}'} \Xi \right| \leq 2^\beta \|\delta \Xi\|_\beta \zeta(\beta) \sum_{[u,v] \in \mathcal{P}} |v-u|^\beta = \mathcal{O}(|\mathcal{P}|^{\beta-1}) = \mathcal{O}(\varepsilon^{\beta-1}).$$

This finishes the proof.

Remark 14
In the proof of Lemma 13, we saw that

$$\delta I_{s,t} = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \Xi_{u,v}.$$
Theorem 15 (Young ’36)

Let $V$ and $W$ be Banach spaces, $g \in C^\alpha([0, T], V)$ and $f \in C^\beta([0, T], L(V, W))$. Assume that $\alpha + \beta > 1$. Then the integral

$$\int_s^t f_u \, dg_u \in W$$

exists as a limit of Riemann sums for every $s < t \in [0, T]$. 

Moreover, we have the estimate

$$\left| \int_s^t f_u \, dg_u - f_s(g_t - g_s) \right| \leq C \|f\|_\beta \|g\|_\alpha |t - s|^{\alpha + \beta}$$

where $C > 0$ depends on $\alpha + \beta$. 


Theorem 15 (Young ‘36)

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(5)

where $C > 0$ depends $\alpha + \beta$. 
Proof.

Set

$$\Xi_{s,t} := f_s(g_t - g_s).$$
Proof.

Set

$$\Xi_{s,t} := f_s(g_t - g_s).$$

Then we have

$$\delta \Xi_{s,u,t} = f_s(g_t - g_s) - f_s(g_u - g_s) - f_u(g_t - g_u)$$

$$= -(f_u - f_s)(g_t - g_u),$$
Proof.
Set

\[ \Xi_{s,t} := f_s(g_t - g_s). \]

Then we have

\[
\delta \Xi_{s,u,t} = f_s(g_t - g_s) - f_s(g_u - g_s) - f_u(g_t - g_u) \\
= -(f_u - f_s)(g_t - g_u),
\]

thus \( \|\Xi\|_\alpha \leq \|f\|_\infty \|g\|_\alpha < \infty \) and

\[
\|\delta \Xi\|_{\alpha+\beta} \leq \|f\|_\beta \|g\|_\alpha < \infty.
\]
Proof.
Set

\[ \Xi_{s,t} := f_s(g_t - g_s). \]

Then we have

\[ \delta \Xi_{s,u,t} = f_s(g_t - g_s) - f_s(g_u - g_s) - f_u(g_t - g_u) \]
\[ = -(f_u - f_s)(g_t - g_u), \]

thus \( \|\Xi\|_\alpha \leq \|f\|_\infty \|g\|_\alpha < \infty \) and

\[ \|\delta \Xi\|_{\alpha + \beta} \leq \|f\|_\beta \|g\|_\alpha < \infty. \]

We can therefore apply the Sewing lemma and set

\[ \int_s^t f_u \, dg_u = \delta \mathcal{I}_{\Xi_{s,t}}. \]

Definition 16
Let $X : [0, T] \to \mathbb{R}^d$, $\sigma = (\sigma_1, \ldots, \sigma_d)$ a collection of vector fields $\sigma_i : \mathbb{R}^m \to \mathbb{R}^m$ and $y \in \mathbb{R}^m$. We call $Y : [0, T] \to \mathbb{R}^m$ a solution to the Young differential equation

$$dY_t = \sigma(Y_t) \, dX_t; \quad t \in [0, T],$$

$$Y_0 = y,$$

if $Y$ satisfies the integral equation

$$Y_t = y + \sum_{i=1}^d \int_0^t \sigma_i(Y_s) \, dX^i_s$$

where the integrals are understood as Young integrals.
Theorem 17

Let $X \in C^\alpha([0, T], \mathbb{R}^d)$ for some $\alpha > \frac{1}{2}$ and let $\sigma$ be twice continuously differentiable and bounded with bounded derivatives. Then the Young differential equation

$$dY_t = \sigma(Y_t) \, dX_t; \quad t \in [0, T],$$

$$Y_0 = y,$$

possesses a unique solution for every initial condition $y \in \mathbb{R}^m$. 
Theorem 17
Let \( X \in C^\alpha([0, T], \mathbb{R}^d) \) for some \( \alpha > \frac{1}{2} \) and let \( \sigma \) be twice continuously differentiable and bounded with bounded derivatives. Then the Young differential equation

\[
dY_t = \sigma(Y_t) \, dX_t; \quad t \in [0, T],
\]

\[
Y_0 = y,
\]
possesses a unique solution for every initial condition \( y \in \mathbb{R}^m \).

Proof.
The proof is classical and uses a fixed point argument.
Theorem 17
Let $X \in C^\alpha([0, T], \mathbb{R}^d)$ for some $\alpha > \frac{1}{2}$ and let $\sigma$ be twice continuously differentiable and bounded with bounded derivatives. Then the Young differential equation

$$dY_t = \sigma(Y_t) \, dX_t; \quad t \in [0, T],$$

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possesses a unique solution for every initial condition $y \in \mathbb{R}^m$.

Proof.
The proof is classical and uses a fixed point argument. For $0 < T_0 \leq T$ and $Y \in C^\alpha([0, T], \mathbb{R}^m)$ with $Y_0 = y$, we set

$$\mathcal{M}_{T_0}(Y) := \left( t \mapsto y + \int_0^t \sigma(Y_s) \, dX_s; \, t \in [0, T_0] \right).$$
Theorem 17
Let \( X \in C^\alpha ([0, T], \mathbb{R}^d) \) for some \( \alpha > \frac{1}{2} \) and let \( \sigma \) be twice continuously differentiable and bounded with bounded derivatives. Then the Young differential equation

\[
\begin{align*}
\text{d}Y_t &= \sigma(Y_t) \, \text{d}X_t; \quad t \in [0, T], \\
Y_0 &= y,
\end{align*}
\]

possesses a unique solution for every initial condition \( y \in \mathbb{R}^m \).

Proof.
The proof is classical and uses a fixed point argument. For \( 0 < T_0 \leq T \) and \( Y \in C^\alpha ([0, T], \mathbb{R}^m) \) with \( Y_0 = y \), we set

\[
\mathcal{M}_{T_0}(Y) := \left( t \mapsto y + \int_0^t \sigma(Y_s) \, \text{d}X_s; \; t \in [0, T_0] \right).
\]

Since \( \sigma \) is Lipschitz, the path \( t \mapsto \sigma(Y_t) \) is \( \alpha \)-Hölder and the integral is defined as a Young integral.
Proof (cont.).

Thus, $\mathcal{M}_{T_0}$ is in fact a map from $C^\alpha_y([0, T_0], \mathbb{R}^m)$ to itself where $C^\alpha_y([0, T_0], \mathbb{R}^m)$ is the space of $\alpha$-Hölder paths starting in $y$. We aim to show that it is a contraction.
Proof (cont.).

Thus, \( \mathcal{M}_{T_0} \) is in fact a map from \( C_y^\alpha ([0, T_0], \mathbb{R}^m) \) to itself where \( C_y^\alpha ([0, T_0], \mathbb{R}^m) \) is the space of \( \alpha \)-Hölder paths starting in \( y \). We aim to show that it is a contraction. We will not do this on the whole space, but restrict ourselves to the closed unit ball

\[
\mathcal{B}_{T_0} := \{ Y \in C_y^\alpha ([0, T_0], \mathbb{R}^m) : \| Y \|_\alpha \leq 1 \}.
\]
Proof (cont.).
Thus, $\mathcal{M}_{T_0}$ is in fact a map from $C_y^\alpha([0, T_0], \mathbb{R}^m)$ to itself where $C_y^\alpha([0, T_0], \mathbb{R}^m)$ is the space of $\alpha$-Hölder paths starting in $y$. We aim to show that it is a contraction.
We will not do this on the whole space, but restrict ourselves to the closed unit ball

$$
\mathcal{B}_{T_0} := \{ Y \in C_y^\alpha([0, T_0], \mathbb{R}^m) : \| Y \|_\alpha \leq 1 \}.
$$

We first show that $\mathcal{M}_{T_0}$ leaves $\mathcal{B}_{T_0}$ invariant for $T_0 > 0$ sufficiently small, i.e.

$\mathcal{M}_{T_0} : \mathcal{B}_{T_0} \rightarrow \mathcal{B}_{T_0}$. 

Proof (cont.).

Thus, $\mathcal{M}_{T_0}$ is in fact a map from $C_y^\alpha([0, T_0], \mathbb{R}^m)$ to itself where $C_y^\alpha([0, T_0], \mathbb{R}^m)$ is the space of $\alpha$-Hölder paths starting in $y$. We aim to show that it is a contraction. We will not do this on the whole space, but restrict ourselves to the closed unit ball

$$B_{T_0} := \{ Y \in C_y^\alpha([0, T_0], \mathbb{R}^m) : \|Y\|_\alpha \leq 1 \}.$$

We first show that $\mathcal{M}_{T_0}$ leaves $B_{T_0}$ invariant for $T_0 > 0$ sufficiently small, i.e. $\mathcal{M}_{T_0} : B_{T_0} \to B_{T_0}$. From Theorem 15,

$$\|\mathcal{M}_{T_0}\|_\alpha = \| \int_0^\cdot \sigma(Y_s) \, dX_s \|_\alpha$$

$$\leq C \|X\|_\alpha (T^\alpha \|\sigma(Y)\|_\alpha + \|\sigma(Y)\|_\infty)$$

$$\leq C \|X\|_\alpha (T^\alpha \|\sigma\|_{C^1} \|Y\|_\alpha + \|\sigma\|_{C^0})$$

$$\leq C \|X\|_\alpha$$

where $\|X\|_\alpha$ denotes the $\alpha$-Hölder norm on $[0, T_0]$. 
Proof (cont.).
We aim to choose $T_0$ sufficiently small such that $C\|X\|_\alpha \leq 1$. However, it is in general not true that $\|X\|_\alpha$ gets small as $T_0$ tends to 0 (take the $1/2$-Hölder norm for the square root function, for instance).
Proof (cont.).
We aim to choose $T_0$ sufficiently small such that $C\|X\|_\alpha \leq 1$. However, it is in general not true that $\|X\|_\alpha$ gets small as $T_0$ tends to 0 (take the $1/2$-Hölder norm for the square root function, for instance). Therefore, we choose $\alpha'$ such that $\frac{1}{2} < \alpha' < \alpha$ and repeat the calculation for $\alpha'$. If $X$ is $\alpha$-Hölder, it follows that $\|X\|_{\alpha'} \to 0$ as $T_0 \to 0$, thus we can choose $T_0$ small enough to conclude that $\|M_{T_0}\|_{\alpha'} \leq 1$ and therefore $M_{T_0} : \mathcal{B}_{T_0} \to \mathcal{B}_{T_0}$. 
Proof (cont.).

We aim to choose $T_0$ sufficiently small such that $C\|X\|_\alpha \leq 1$. However, it is in general not true that $\|X\|_\alpha$ gets small as $T_0$ tends to 0 (take the $1/2$-Hölder norm for the square root function, for instance). Therefore, we choose $\alpha'$ such that $\frac{1}{2} < \alpha' < \alpha$ and repeat the calculation for $\alpha'$. If $X$ is $\alpha$-Hölder, it follows that $\|X\|_{\alpha'} \to 0$ as $T_0 \to 0$, thus we can choose $T_0$ small enough to conclude that $\|M_{T_0}\|_{\alpha'} \leq 1$ and therefore $M_{T_0} : B_{T_0} \to B_{T_0}$.

We proceed showing that $M_{T_0}$ is a contraction. For $Y, \tilde{Y} \in B_{T_0}$, Theorem 15 implies that

$$\left| \delta M_{T_0}(Y)_{s,t} - \delta M_{T_0}(\tilde{Y})_{s,t} \right|$$

$$= \left| \int_s^t \sigma(Y_u) - \sigma(\tilde{Y}_u) \, dX_u \right|$$

$$\leq C \left( \|\sigma(Y) - \sigma(\tilde{Y})\|_\infty + \|\sigma(Y) - \sigma(\tilde{Y})\|_{\alpha'} \right) \|X\|_{\alpha';[0,T_0]} |t - s|^{\alpha'}.$$
Proof (cont.).

Note that, since $Y_0 = \tilde{Y}_0$,

$$\|\sigma(Y) - \sigma(\tilde{Y})\|_\infty \leq |\sigma(Y_0) - \sigma(\tilde{Y}_0)| + T_0^{\alpha'} \|\sigma(Y) - \sigma(\tilde{Y})\|_{\alpha'}$$

$$= T_0^{\alpha'} \|\sigma(Y) - \sigma(\tilde{Y})\|_{\alpha'}.$$
Proof (cont.).

Note that, since \( Y_0 = \tilde{Y}_0 \),

\[
\| \sigma(Y) - \sigma(\tilde{Y}) \|_\infty \leq |\sigma(Y_0) - \sigma(\tilde{Y}_0)| + T_0^{\alpha'} \| \sigma(Y) - \sigma(\tilde{Y}) \|_{\alpha'}
\]

\[= T_0^{\alpha'} \| \sigma(Y) - \sigma(\tilde{Y}) \|_{\alpha'}.\]

With some work (cf. Lemma 7.5 in [Friz, Hairer; 2022]), it can be shown that

\[
\| \sigma(Y) - \sigma(\tilde{Y}) \|_{\alpha'} \leq C_{\alpha',K} \| \sigma \|_{C^2} (|Y_0 - \tilde{Y}_0| + \| Y - \tilde{Y} \|_{\alpha'})
\]

\[= C_{\alpha',K} \| \sigma \|_{C^2} \| Y - \tilde{Y} \|_{\alpha'}
\]

where \( K > 0 \) satisfies \( \| Y \|_\alpha \vee \| \tilde{Y} \|_\alpha \leq K \).
Proof (cont.).

Note that, since $Y_0 = \tilde{Y}_0$, 

$$\|\sigma(Y) - \sigma(\tilde{Y})\|_{\infty} \leq |\sigma(Y_0) - \sigma(\tilde{Y}_0)| + T_0^{\alpha'} \|\sigma(Y) - \sigma(\tilde{Y})\|_{\alpha'} = T_0^{\alpha'} \|\sigma(Y) - \sigma(\tilde{Y})\|_{\alpha'}.$$ 

With some work (cf. Lemma 7.5 in [Friz, Hairer; 2022]), it can be shown that 

$$\|\sigma(Y) - \sigma(\tilde{Y})\|_{\alpha'} \leq C_{\alpha',K} \|\sigma\|_{C^2} (|Y_0 - \tilde{Y}_0| + \|Y - \tilde{Y}\|_{\alpha'}) = C_{\alpha',K} \|\sigma\|_{C^2} \|Y - \tilde{Y}\|_{\alpha'}$$

where $K > 0$ satisfies $\|Y\|_\alpha \vee \|\tilde{Y}\|_\alpha \leq K$. Since $Y, \tilde{Y} \in B_{T_0}$, $C$ can be chosen independently of $Y$ and $\tilde{Y}$. Therefore, we arrive at an estimate of the form 

$$\|M_{T_0}(Y) - M_{T_0}(\tilde{Y})\|_{\alpha'} \leq C \|X\|_{\alpha'} \|Y - \tilde{Y}\|_{\alpha'}$$

and choosing $T_0 > 0$ smaller if necessary, we obtain $C \|X\|_{\alpha'} < 1$, i.e. $M_{T_0}$ is a contraction on the space $B_{T_0}$. 

Proof (cont.).

It follows that the equation possesses a unique solution on the interval \([0, T_0]\). We can now repeat the argument on the interval \([T_0, 2T_0]\) with initial condition \(Y_{T_0}\) and so on and glue together the solutions. Eventually, we obtain a unique solution \(Y\) on the interval \([0, T]\).
Proof (cont.).

It follows that the equation possesses a unique solution on the interval $[0, T_0]$. We can now repeat the argument on the interval $[T_0, 2T_0]$ with initial condition $Y_{T_0}$ and so on and glue together the solutions. Eventually, we obtain a unique solution $Y$ on the interval $[0, T]$. A posteriori, the estimates for the Young integral show that $Y$ is not only $\alpha'$ Hölder, but even $\alpha$-Hölder. This finishes the proof. \qed
Putting things together, we can show the following:
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**Theorem 18**

Let $B^H$ be a fBm with $H > \frac{1}{2}$. Assume that $\sigma$ is twice continuously differentiable, bounded and has bounded derivatives. Then for every $y \in \mathbb{R}^m$, the stochastic differential equation

$$dY_t = \sigma(Y_t) dB^H_t(\omega); \quad t \in [0, T],$$

$$Y_0 = y,$$

*can be interpreted as a Young differential equation and possesses a unique solution $Y$ for almost every trajectory.*
Remark 19
The solution theory just presented is *pathwise*, meaning that one can solve the SDE path-by-path. In particular, if the fBm is \(\alpha\)-Hölder continuous outside a set \(\mathcal{N} \subset \Omega\), in can be solved outside exactly that set.
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Thank you.