Fakultät für
Mathematik und Informatik

Introduction to rough paths theory

Course 01 - Fractional Brownian motion and the Young integral

Fakultät für **Mathematik und Informatik**
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Motivation: Fractional Brownian motion

Young integral and Sewing lemma

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A stochastic process $X\colon [0,\infty) \to \mathbb{R}$ is called *Gaussian* if for every $k \in \mathbb{N}$ and every $t_1, \ldots, t_k \in [0,\infty)$, the random variable $(X_{t_1}, \ldots, X_{t_k})$ is a multivariate Gaussian random variable.

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Definition 2 (Mandelbrot, van Ness '68)

Let $H \in (0,1)$. The *fractional Brownian motion* (fBm) is a continuous zero mean Gaussian process $B^H\colon [0,\infty) \to \mathbb{R}$ with covariance function given by

$$R(s,t) := \mathbf{cov}(B_s^H, B_t^H) = \mathbb{E}(B_s^H B_t^H) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

The parameter $H \in (0,1)$ is called *Hurst parameter*.

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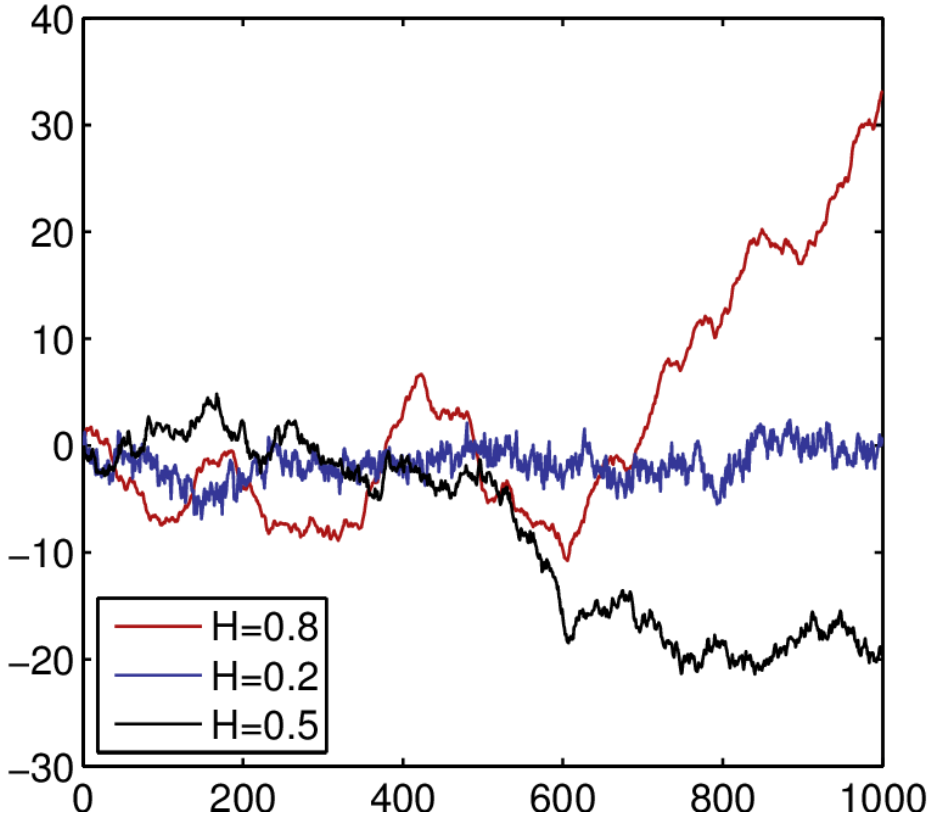

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– The *noise process* $t \mapsto \dot{B}^H$ is thus *white* only in the case $H = \frac{1}{2}$. For $H \neq \frac{1}{2}$, the noise $\dot{B}^H$ could be called *colored*.

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- Using the fBm instead of the Bm for modelling random phenomena can be more realistic in case of models with memory. For instance, it was used to model price processes in illiquid markets (electricity markets, gas markets etc.)

A generic stochastic differential equation (SDE) involving the fBm would be of the form

$$\begin{aligned} \mathrm{d}Y_t &= b(Y_t)\,\mathrm{d}t + \sum_{i=1}^{d} \sigma_i(Y_t)\,\mathrm{d}B_t^{H;i} \\ &= b(Y_t)\,\mathrm{d}t + \sigma(Y_t)\,\mathrm{d}B_t^{H}; \quad t \geq 0 \\ Y_0 &= y_0 \in \mathbb{R}^m \end{aligned} \tag{1}$$

where $B^H = (B^{H;1}, \ldots, B^{H;d})$ is a *d-dimensional fractional Brownian motion*, i.e. a vector of independent one-dimensional fBm, $b, \sigma_1, \ldots, \sigma_d \colon \mathbb{R}^m \to \mathbb{R}^m$ is a collection of vector fields and $Y \colon [0, \infty) \to \mathbb{R}^m$ is a stochastic process we aim to call a solution to (1).

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The fundamental problem is: How should we interprete (1)? What properties should the process Y satisfy to call it a *solution* to the SDE (1)?

First attempt: If the trajectories of B^H, i.e. the paths $t \mapsto B_t^H(\omega)$, $\omega \in \Omega$, were differentiable, we could interprete (1) *pathwise* as a random (non-autonomous) ordinary differential equation (ODE):

$$\frac{\mathrm{d}Y_t}{\mathrm{d}t} = b(Y_t) + \sum_{i=1}^{d} \sigma^i(Y_t) \frac{\mathrm{d}B_t^{H;i}(\omega)}{\mathrm{d}t}. \tag{2}$$

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However, the following result shows that this attempt fails:

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For every $H \in (0,1)$, the trajectories of the fBm are almost surely nowhere differentiable.

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Motivated by partial differential equations (PDE), one might have the idea to weaken the notion of differentiability in order to give a meaning to (2): we could interprete $\frac{\mathrm{d}B_t^{H;i}(\omega)}{\mathrm{d}t}$ as a *weak derivative*, i.e. as a *distribution* or *generalized function*.

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Second attempt: In stochastic analysis, the Itō-integral $\int Y \, \mathrm{d}X$ is defined in case of X being a semimartingale and Y being adapted to the filtration generated by X. We could try to interprete (1) as an integral equation

$$Y_t = Y_0 + \int_0^t b(Y_s) \, \mathrm{d}s + \sum_{i=1}^{d} \int_0^t \sigma^i(Y_s) \, \mathrm{d}B_s^{H,i}$$

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where the stochastic integral is understood in Itō-sense. Unfortunately, one can prove the following:

Proposition 8

The fBm B^H is not a semimartingale unless $H = \frac{1}{2}$.

Third attempt: In 1936, L.C. YOUNG introduced a notion of an integral that generalized Riemann-Stieltjes integration. More concretely, he defined an integral for functions $f, g\colon [0,T] \to \mathbb{R}$ that are Hölder continuous with Hölder index $\alpha \in (0,1]$ resp. $\beta \in (0,1]$ of the form $\int f \,\mathrm{d}g$ provided $\alpha + \beta > 1$.

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Theorem 9 (Kolmogorov-Chentsov)

Let $X\colon [0,T] \to \mathbb{R}$ be a continuous stochastic process, $q \geq 2$, $\beta > \frac{1}{q}$ and assume that

$$\|X_t - X_s\|_{L^q} \leq C|t-s|^{\beta}$$

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$$|X_t - X_s| \leq K_\alpha |t-s|^{\alpha}$$

for all $s, t \in [0,T]$. In particular, the trajectories of X are almost surely α-Hölder continuous.

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The trajectories of the fBm are almost surely α-Hölder continuous for every $\alpha < H$.

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For later purposes, we give a proof of the Komogorov-Chentsov theorem, too.

Proof of Theorem 9.

W.l.o.g., $T = 1$. Set $D_n = \{k2^n : k = 0, \dots, 2^n\}$ and $D = \cup_{n \geq 0} D_n$. We further define the random variables

$$K_n := \sup_{t \in D_n} |\delta X_{t,t+2^{-n}}|, \quad \delta X_{t,t+2^{-n}} := X_{t+2^{-n}} - X_t.$$

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$$\mathbb{E}(K_n^q) \leq \mathbb{E} \sum_{t \in D_n} |\delta X_{t,t+2^{-n}}|^q \leq \frac{1}{|D_n|} C^q |D_n|^{q\beta} = C^q |D_n|^{q\beta - 1}.$$

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Fix $s < t \in D$ and choose m such that $|D_{m+1}| < t - s \leq |D_m|$. Going from coarser to finer partitions successively, we can find $\tau_0, \dots, \tau_N \in \cup_{n \geq m+1} D_n$ such that

$$s = \tau_0 < \tau_1 < \dots < \tau_N = t$$

with the property that at most two intervals of the form $[\tau_i, \tau_{i+1}]$ have the same length.

Proof of Theorem 9 (cont.).

With this choice, it follows that

$$|\delta X_{s,t}| \leq \sum_{i=0}^{N-1} |\delta X_{\tau_i,\tau_{i+1}}| \leq 2 \sum_{n \geq m+1} K_n.$$

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We thus obtain

$$\frac{|\delta X_{s,t}|}{|t-s|^\alpha} \leq \sum_{n \geq m+1} \frac{2K_n}{|D_{m+1}|^\alpha} \leq \sum_{n \geq m+1} \frac{2K_n}{|D_n|^\alpha} \leq 2 \sum_{n \geq 0} \frac{K_n}{|D_n|^\alpha} =: K_\alpha.$$

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$$\|K_\alpha\|_{L^q} \leq 2 \sum_{n \geq 0} \frac{\|K_n\|_{L^q}}{|D_n|^\alpha} \leq 2C \sum_{n \geq 0} |D_n|^{\beta - \frac{1}{q} - \alpha}$$

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Remark 11

Often, the formulation of the Kolmogorov-Chentsov theorem does not assume that X is continuous. The statement then says that X has a Hölder-continuous *modification* $\tilde{X}$, i.e. $X_t = \tilde{X}_t$ almost surely for every t.

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Motivation: Fractional Brownian motion

Young integral and Sewing lemma

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1. $\mathcal{C}([0,T],W)$ will denote the space of continuous functions $f\colon [0,T] \to W$.
2. For $f \in \mathcal{C}([0,T],W)$, we define $\delta f_{s,t} := f_t - f_s$. $\mathcal{C}^\alpha([0,T],W)$ denotes the space of α-Hölder continuous functions, i.e. $f \in \mathcal{C}^\alpha([0,T],W)$ iff

$$\|f\|_\alpha := \sup_{s<t} \frac{|\delta f_{s,t}|}{|t-s|^\alpha} < \infty.$$

Definition 12 (cont.)

3. The space $\mathcal{C}_2^{\alpha,\beta}([0,T],W)$ denotes the space of functions Ξ defined on the simplex $\{(s,t) \in [0,T]^2 \ : \ s \leq t\}$ such that $\Xi_{t,t} = 0$ and

$$\|\Xi\|_{\alpha,\beta} := \|\Xi\|_\alpha + \|\delta\Xi\|_\beta < \infty$$

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and

$$\delta\Xi_{s,u,t} := \Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}, \qquad \|\delta\Xi\|_\beta := \sup_{s<u<t} \frac{|\delta\Xi_{s,u,t}|}{|t-s|^\beta}.$$

Lemma 13 (Sewing lemma)

Let $0 < \alpha \leq 1 < \beta$. *Then there exists a unique continuous linear map* $\mathcal{I}\colon \mathcal{C}_2^{\alpha,\beta}([0,T],W) \to \mathcal{C}^{\alpha}([0,T],W)$ *such that* $(\mathcal{I}\Xi)_0 = 0$ *and*

$$|\delta\mathcal{I}\Xi_{s,t} - \Xi_{s,t}| \leq C\|\delta\Xi\|_{\beta}|t-s|^{\beta} \tag{3}$$

where $C > 0$ *depends on* β.

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Proof.

Let us prove uniqueness first. Assume that I and $\tilde{I}$ both satisfy (3). Then it holds that

$$|(I-\tilde{I})_t - (I-\tilde{I})_s| \leq C|t-s|^{\beta}.$$

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$$|(I-\tilde{I})_t - (I-\tilde{I})_s| \leq C|t-s|^{\beta}.$$

Since $\beta > 1$ and $I - \tilde{I}$ is a path, $I - \tilde{I}$ is constant. Since $I_0 = \tilde{I}_0 = 0$, uniqueness follows.

Proof (cont.).

Fix an interval $[s, t]$ and a partition $\mathcal{P} = \{s = u_0 < u_1 < \ldots < u_r = t\}$ of this interval. Let $|\mathcal{P}|$ denote the maximal length between two consecutive points in $\mathcal{P}$, i.e.

$$|\mathcal{P}| := \max_i |u_{i+1} - u_i|.$$

Proof (cont.).

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$$|\mathcal{P}| := \max_i |u_{i+1} - u_i|.$$

We set

$$\int_{\mathcal{P}} \Xi := \sum_i \Xi_{u_i, u_{i+1}}.$$

The idea is now to establish a *maximal inequality* for $\int_{\mathcal{P}} \Xi$ by successively removing distinguished points from the partition $\mathcal{P}$.

Proof (cont.).

We claim that if $r \geq 2$, there exists a point $u \in \mathcal{P}$ such that for its neighbouring points $u_- < u < u_+ \in \mathcal{P}$,

$$|u_+ - u_-| \leq \frac{2}{r-1}|t-s|.$$

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which is a contradiction. With this choice, we obtain

$$\left| \int_{\mathcal{P}} \Xi - \int_{\mathcal{P} \setminus \{u\}} \Xi \right| = |\delta \Xi_{u_-, u, u_+}| \leq \|\delta \Xi\|_\beta |u_+ - u_-|^\beta \leq \|\delta \Xi\|_\beta \frac{2^\beta |t-s|^\beta}{(r-1)^\beta}.$$

Proof.

By successively removing points, we arrive at the uniform bound

$$\sup_{\mathcal{P}} \left| \int_{\mathcal{P}} \Xi - \Xi_{s,t} \right| \leq 2^{\beta} |t-s|^{\beta} \|\delta\Xi\|_{\beta} \sum_{k=1}^{\infty} \frac{1}{k^{\beta}} = 2^{\beta} |t-s|^{\beta} \|\delta\Xi\|_{\beta} \zeta(\beta) \tag{4}$$

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$$\sup_{\max\{|\mathcal{P}|, |\mathcal{P}'|\} \leq \varepsilon} \left| \int_{\mathcal{P}} \Xi - \int_{\mathcal{P}'} \Xi \right| \to 0 \quad \text{as } \varepsilon \to 0.$$

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By adding and substracting $\int_{\mathcal{P} \cup \mathcal{P}'} \Xi$, we can assume w.l.o.g. that $\mathcal{P} \subset \mathcal{P}'$.

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By adding and substracting $\int_{\mathcal{P} \cup \mathcal{P}'} \Xi$, we can assume w.l.o.g. that $\mathcal{P} \subset \mathcal{P}'$. In this case,

$$\int_{\mathcal{P}} \Xi - \int_{\mathcal{P}'} \Xi = \sum_{[u,v] \in \mathcal{P}} \left(\Xi_{u,v} - \int_{\mathcal{P}' \cap [u,v]} \Xi \right).$$

Proof (cont.).

For $\max\{|\mathcal{P}|, |\mathcal{P}'|\} = |\mathcal{P}| \leq \varepsilon$, we can use the maximal inequality (4) to see that

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For $\max\{|\mathcal{P}|, |\mathcal{P}'|\} = |\mathcal{P}| \leq \varepsilon$, we can use the maximal inequality (4) to see that

$$\left| \int_{\mathcal{P}} \Xi - \int_{\mathcal{P}'} \Xi \right| \leq 2^{\beta} \|\delta\Xi\|_{\beta} \zeta(\beta) \sum_{[u,v]\in\mathcal{P}} |v-u|^{\beta} = \mathcal{O}(|\mathcal{P}|^{\beta-1}) = \mathcal{O}(\varepsilon^{\beta-1}).$$

This finishes the proof.

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For $\max\{|\mathcal{P}|, |\mathcal{P}'|\} = |\mathcal{P}| \leq \varepsilon$, we can use the maximal inequality (4) to see that

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This finishes the proof.

□

Remark 14

In the proof of Lemma 13, we saw that

$$\delta\mathcal{I}\Xi_{s,t} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \Xi_{u,v}.$$

Theorem 15 (Young '36)

Let V and W be Banach spaces, $g \in \mathcal{C}^{\alpha}([0,T],V)$ and $f \in \mathcal{C}^{\beta}([0,T],L(V,W))$. Assume that $\alpha + \beta > 1$. Then the integral

$$\int_s^t f_u \, \mathrm{d}g_u \in W$$

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$$\int_s^t f_u \,\mathrm{d}g_u \in W$$

exists as a limit of Riemann sums for every $s<t \in [0,T]$. Moreover, we have the estimate

$$\left|\int_s^t f_u \,\mathrm{d}g_u - f_s(g_t-g_s)\right| \leq C\|f\|_{\beta}\|g\|_{\alpha}|t-s|^{\alpha+\beta} \tag{5}$$

where $C>0$ depends $\alpha+\beta$.

Proof.

Set

$$\Xi_{s,t} := f_s(g_t - g_s).$$

$\square$

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Then we have

$$\begin{aligned}\delta\Xi_{s,u,t} &= f_s(g_t - g_s) - f_s(g_u - g_s) - f_u(g_t - g_u)\\ &= -(f_u - f_s)(g_t - g_u),\end{aligned}$$

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thus $\|\Xi\|_\alpha \leq \|f\|_\infty \|g\|_\alpha < \infty$ and

$$\|\delta\Xi\|_{\alpha+\beta} \leq \|f\|_\beta \|g\|_\alpha < \infty.$$

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thus $\|\Xi\|_\alpha \leq \|f\|_\infty \|g\|_\alpha < \infty$ and

$$\|\delta\Xi\|_{\alpha+\beta} \leq \|f\|_\beta \|g\|_\alpha < \infty.$$

We can therefore apply the Sewing lemma and set

$$\int_s^t f_u \, \mathrm{d}g_u = \delta\mathcal{I}\Xi_{s,t}.$$

□

Definition 16

Let $X\colon [0,T] \to \mathbb{R}^d$, $\sigma = (\sigma_1, \dots, \sigma_d)$ a collection of vector fields $\sigma_i\colon \mathbb{R}^m \to \mathbb{R}^m$ and $y \in \mathbb{R}^m$. We call $Y\colon [0,T] \to \mathbb{R}^m$ a *solution to the Young differential equation*

$$\begin{aligned} \mathrm{d}Y_t &= \sigma(Y_t)\,\mathrm{d}X_t; \quad t \in [0,T], \\ Y_0 &= y, \end{aligned}$$

if Y satisfies the integral equation

$$Y_t = y + \sum_{i=1}^{d} \int_0^t \sigma_i(Y_s)\,\mathrm{d}X_s^i$$

where the integrals are understood as Young integrals.

Theorem 17

Let $X \in \mathcal{C}^{\alpha}([0,T],\mathbb{R}^d)$ for some $\alpha > \frac{1}{2}$ and let σ be twice continuously differentiable and bounded with bounded derivatives. Then the Young differential equation

$$\begin{aligned} \mathrm{d}Y_t &= \sigma(Y_t)\,\mathrm{d}X_t; \quad t \in [0,T], \\ Y_0 &= y, \end{aligned}$$

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The proof is classical and uses a fixed point argument. For $0 < T_0 \leq T$ and $Y \in \mathcal{C}^\alpha([0,T],\mathbb{R}^m)$ with $Y_0 = y$, we set

$$\mathcal{M}_{T_0}(Y) := \left(t \mapsto y + \int_0^t \sigma(Y_s)\,dX_s; t \in [0,T_0] \right).$$

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$$\mathcal{M}_{T_0}(Y) := \left(t \mapsto y + \int_0^t \sigma(Y_s)\,dX_s; t \in [0,T_0] \right).$$

Since σ is Lipschitz, the path $t \mapsto \sigma(Y_t)$ is α-Hölder and the integral is defined as a Young integral.

Proof (cont.).

Thus, $\mathcal{M}_{T_0}$ is in fact a map from $\mathcal{C}_y^\alpha([0, T_0], \mathbb{R}^m)$ to itself where $\mathcal{C}_y^\alpha([0, T_0], \mathbb{R}^m)$ is the space of α-Hölder paths starting in y. We aim to show that it is a contraction.

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$$\mathcal{B}_{T_0} := \{Y \in \mathcal{C}_y^\alpha([0,T_0],\mathbb{R}^m) \,:\, \|Y\|_\alpha \leq 1\}.$$

Proof (cont.).

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We first show that $\mathcal{M}_{T_0}$ leaves $\mathcal{B}_{T_0}$ invariant for $T_0 > 0$ sufficiently small, i.e. $\mathcal{M}_{T_0} : \mathcal{B}_{T_0} \to \mathcal{B}_{T_0}$.

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$$\begin{aligned}
\|\mathcal{M}_{T_0}\|_\alpha &= \|\int_0^\cdot \sigma(Y_s)\,dX_s\|_\alpha \\
&\leq C\|X\|_\alpha(T^\alpha\|\sigma(Y)\|_\alpha + \|\sigma(Y)\|_\infty) \\
&\leq C\|X\|_\alpha(T^\alpha\|\sigma\|_{\mathcal{C}^1}\|Y\|_\alpha + \|\sigma\|_{\mathcal{C}^0}) \\
&\leq C\|X\|_\alpha
\end{aligned}$$

where $\|X\|_\alpha$ denotes the α-Hölder norm on $[0,T_0]$.

Proof (cont.).

We aim to choose T_0 sufficiently small such that $C\|X\|_\alpha \leq 1$. However, it is in general *not* true that $\|X\|_\alpha$ gets small as T_0 tends to 0 (take the $1/2$-Hölder norm for the square root function, for instance).

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We aim to choose T_0 sufficiently small such that $C\|X\|_\alpha \leq 1$. However, it is in general *not* true that $\|X\|_\alpha$ gets small as T_0 tends to 0 (take the $1/2$-Hölder norm for the square root function, for instance). Therefore, we choose α' such that $\frac{1}{2} < \alpha' < \alpha$ and repeat the calculation for α'. If X is α-Hölder, it follows that $\|X\|_{\alpha'} \to 0$ as $T_0 \to 0$, thus we can choose T_0 small enough to conclude that $\|\mathcal{M}_{T_0}\|_{\alpha'} \leq 1$ and therefore $\mathcal{M}_{T_0} : \mathcal{B}_{T_0} \to \mathcal{B}_{T_0}$.

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We proceed showing that $\mathcal{M}_{T_0}$ is a contraction. For $Y, \tilde{Y} \in \mathcal{B}_{T_0}$, Theorem 15 implies that

$$\begin{aligned}
&\left|\delta\mathcal{M}_{T_0}(Y)_{s,t} - \delta\mathcal{M}_{T_0}(\tilde{Y})_{s,t}\right| \\
&= \left|\int_s^t \sigma(Y_u) - \sigma(\tilde{Y}_u)\,\mathrm{d}X_u\right| \\
&\leq C\left(\|\sigma(Y) - \sigma(\tilde{Y})\|_\infty + \|\sigma(Y) - \sigma(\tilde{Y})\|_{\alpha'}\right)\|X\|_{\alpha';[0,T_0]}|t-s|^{\alpha'}.
\end{aligned}$$

Proof (cont.).

Note that, since $Y_0 = \tilde{Y}_0$,

$$\begin{aligned}\|\sigma(Y) - \sigma(\tilde{Y})\|_\infty &\leq |\sigma(Y_0) - \sigma(\tilde{Y}_0)| + T_0^{\alpha'} \|\sigma(Y) - \sigma(\tilde{Y})\|_{\alpha'} \\ &= T_0^{\alpha'} \|\sigma(Y) - \sigma(\tilde{Y})\|_{\alpha'}.\end{aligned}$$

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With some work (cf. Lemma 7.5 in [Friz, Hairer; 2022]), it can be shown that

$$\begin{aligned}\|\sigma(Y) - \sigma(\tilde{Y})\|_{\alpha'} &\leq C_{\alpha',K} \|\sigma\|_{\mathcal{C}^2} (|Y_0 - \tilde{Y}_0| + \|Y - \tilde{Y}\|_{\alpha'}) \\ &= C_{\alpha',K} \|\sigma\|_{\mathcal{C}^2} \|Y - \tilde{Y}\|_{\alpha'}\end{aligned}$$

where $K > 0$ satisfies $\|Y\|_\alpha \vee \|\tilde{Y}\|_\alpha \leq K$.

Proof (cont.).

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where $K > 0$ satisfies $\|Y\|_\alpha \vee \|\tilde{Y}\|_\alpha \leq K$. Since $Y, \tilde{Y} \in \mathcal{B}_{T_0}$, C can be chosen independently of Y and $\tilde{Y}$. Therefore, we arrive at an estimate of the form

$$\|\mathcal{M}_{T_0}(Y) - \mathcal{M}_{T_0}(\tilde{Y})\|_{\alpha'} \leq C \|X\|_{\alpha'} \|Y - \tilde{Y}\|_{\alpha'}$$

and choosing $T_0 > 0$ smaller if necessary, we obtain $C\|X\|_{\alpha'} < 1$, i.e. $\mathcal{M}_{T_0}$ is a contraction on the space $\mathcal{B}_{T_0}$.

Proof (cont.).

It follows that the equation possesses a unique solution on the interval $[0, T_0]$. We can now repeat the argument on the interval $[T_0, 2T_0]$ with initial condition Y_{T_0} and so on and glue together the solutions. Eventually, we obtain a unique solution Y on the interval $[0, T]$.

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Proof (cont.).

It follows that the equation possesses a unique solution on the interval $[0, T_0]$. We can now repeat the argument on the interval $[T_0, 2T_0]$ with initial condition Y_{T_0} and so on and glue together the solutions. Eventually, we obtain a unique solution Y on the interval $[0, T]$. A posteriori, the estimates for the Young integral show that Y is not only α' Hölder, but even α-Hölder. This finishes the proof. $\square$

Putting things together, we can show the following:

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Theorem 18

Let B^H be a fBm with $H > \frac{1}{2}$. Assume that σ is twice continuously differentiable, bounded and has bounded derivatives. Then for every $y \in \mathbb{R}^m$, the stochastic differential equation

$$\begin{aligned} \mathrm{d}Y_t &= \sigma(Y_t)\,\mathrm{d}B_t^H(\omega); \quad t \in [0,T], \\ Y_0 &= y, \end{aligned}$$

can be interpreted as a Young differential equation and possesses a unique solution Y for almost every trajectory.

Remark 19

The solution theory just presented is *pathwise*, meaning that one can solve the SDE path-by-path. In particular, if the fBm is α-Hölder continuous outside a set $\mathcal{N} \subset \Omega$, in can be solved outside exactly that set.

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Remark 19

The solution theory just presented is *pathwise*, meaning that one can solve the SDE path-by-path. In particular, if the fBm is α-Hölder continuous outside a set $\mathcal{N} \subset \Omega$, in can be solved outside exactly that set. This is in contrast to Itō's theory of stochastic differential equations, that is not pathwise.

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The solution theory just presented is *pathwise*, meaning that one can solve the SDE path-by-path. In particular, if the fBm is α-Hölder continuous outside a set $\mathcal{N} \subset \Omega$, in can be solved outside exactly that set. This is in contrast to Itō's theory of stochastic differential equations, that is not pathwise. The solution of an Itō-SDE is defined outside a set of measure zero *that depends on the whole equation*, e.g. on y and on σ, too. Considering another initial condition $\tilde{y}$ will create a new set of measure zero outside of which the solution is defined.

Remark 19

The solution theory just presented is *pathwise*, meaning that one can solve the SDE path-by-path. In particular, if the fBm is α-Hölder continuous outside a set $\mathcal{N} \subset \Omega$, in can be solved outside exactly that set. This is in contrast to Itō's theory of stochastic differential equations, that is not pathwise. The solution of an Itō-SDE is defined outside a set of measure zero *that depends on the whole equation*, e.g. on y and on σ, too. Considering another initial condition $\tilde{y}$ will create a new set of measure zero outside of which the solution is defined. Since the set of allowed initial condtions is not countable, it is a priori not clear whether there exists a set of full measure on which an Itō-SDE can be solved for every initial condition y. For a pathwise solution theory, this is immediate.

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Thank you.