An introduction to regularity structures

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These slides can be downloaded from my home page

Plan of the course

In this course I want to discuss mainly the analytical aspects of the theory of regularity structures.

The plan for the four lectures is roughly:

- I. Reconstruction Theorem
- II. Models and modelled distributions
- III. Schauder estimates for germs
- IV. Multilevel Schauder estimates for modelled distributions
- V. Products and equations

Lecture notes and papers in collaboration with F. Caravenna and L. Broux, see my web page.

Chapter 1: The Reconstruction Theorem

A theory, a theorem

This talk is based on a paper (appeared in 2021 in the EMS Surveys in Mathematics)

- *Hairer's Reconstruction Theorem without Regularity Structures*
  by F. Caravenna and L.Z.

In this paper we have extracted a single result (the Reconstruction Theorem) from a larger theory (Regularity Structures).

We present the former in a simpler and more general version, without reference to the latter.

A later paper by Pavel Zorin-Kranich, to appear in Revista Matematica Iberoamericana, has introduced introduced further simplifications and improvements to our results.

Prelude: The Sewing Lemma

You have seen in Sebastian's lectures last week that the main tool in rough analysis is the

Theorem (Sewing Lemma)

Let $0 < \alpha \leq 1 < \beta$. *There exists a unique map* $\mathcal{I} : \mathcal{C}_2^{\alpha,\beta}([0,T] : \mathbb{R}^d) \to \mathcal{C}^\alpha([0,T] : \mathbb{R}^d)$ *s.t.*

$$(\mathcal{I}\Xi)_0 = 0, \qquad |\mathcal{I}\Xi_t - \mathcal{I}\Xi_s - \Xi_{s,t}| \lesssim |t-s|^\beta, \qquad s,t \in [0,T].$$

We recall that $\mathcal{C}_2^{\alpha,\beta}$ denotes the space of continuous $\Xi : \{(s,t) : 0 \leq s \leq t \leq T\} \to \mathbb{R}^d$ s.t.

$$\sup_{0 \leq s < t \leq T} \frac{|\Xi_{s,t}|}{|t-s|^\alpha} + \sup_{0 \leq s < u < t \leq T} \frac{|\Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}|}{|t-s|^\beta} < +\infty.$$

This theorem was proved around 2003 indipendently by Gubinelli and Feyel-de la Pradelle.

It is restricted to functions depending on a one-dimensional parameter.

It took ten years to find a version of this result in higher dimension... This is Martin's Reconstruction Theorem.

Distributions

This talk will concern the space $\mathcal{D}'(\mathbb{R}^d)$ of distributions or generalised functions.

We consider the space $\mathcal{D}(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)$ of smooth functions with compact support on $\mathbb{R}^d$.

A distribution on $\mathbb{R}^d$ is a linear functional $T : C_c^\infty(\mathbb{R}^d) \to \mathbb{R}$ such that for every compact set $K \subset \mathbb{R}^d$ there is $r = r_K \in \mathbb{N}$

$$|T(\varphi)| \lesssim \|\varphi\|_{C^r} := \max_{|k|\leq r} \|\partial^k \varphi\|_\infty, \qquad \forall\, \varphi \in C_0^\infty(K)$$

where throughout the lectures $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \leq C\, g$.

When r can be chosen uniformly over K we say that T has order r.

Distributions

Every locally integrable (in particular continuous) function $f : \mathbb{R}^d \to \mathbb{R}$ defines a distribution:

$$f(\varphi) := \int_{\mathbb{R}^d} f(x)\,\varphi(x)\,\mathrm{d}x, \qquad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

A famous example of distribution from quantum mechanics, which actually motivated the whole theory of distributions, is the Dirac measure δ_x at $x \in \mathbb{R}^d$

$$\delta_x(\varphi) = \varphi(x), \qquad \varphi \in C^\infty(\mathbb{R}^d).$$

One can also differentiate any distribution $T \in \mathcal{D}'(\mathbb{R}^d)$: for $k \in \mathbb{N}^d$

$$\partial^k T(\varphi) := (-1)^{k_1+\cdots+k_d}\, T(\partial^k \varphi).$$

Products of distributions

Distributions form a linear space. If $\varphi \in C^\infty(\mathbb{R}^d)$ and $T \in \mathcal{D}'(\mathbb{R}^d)$ then it is possible to define canonically the product $\varphi \cdot T = T \cdot \varphi$ as

$$\varphi \cdot T(\psi) = T \cdot \varphi(\psi) := T(\varphi\psi), \qquad \forall\, \psi \in C_c^\infty(\mathbb{R}^d).$$

However, if $T, T' \in \mathcal{D}'(\mathbb{R}^d)$, in general there is no canonical way of defining $T \cdot T'$.

One may use some form of regularisation of T, T' or both. Then, the result could heavily depend on the regularisation and thus be neither unique nor canonical.

For example, one can not define the square $(\delta_x)^2$ of the Dirac function.

The main question of reconstruction

For every $x \in \mathbb{R}^d$ we fix a distribution $F_x \in \mathcal{D}'(\mathbb{R}^d)$. If for all $\psi \in \mathcal{D}$ the map

$$\mathbb{R}^d \ni x \mapsto F_x(\psi)$$

is measurable, then we call $(F_x)_{x\in\mathbb{R}^d}$ a germ.

Problem:
Can we find a distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ which is locally well approximated by $(F_x)_{x\in\mathbb{R}^d}$?

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Note that for $j \in \mathbb{N}^d$, $w \in \mathbb{R}^d$, we use the notation

$$|j| := \sum_{k=1}^{d} j_k, \qquad w^j := \prod_{k=1}^{d} w_k^{j_k}, \qquad j! := \prod_{k=1}^{d} j_k!$$

with the convention $0^0 := 1$.

Taylor expansions

For example, let us fix $f \in C^\infty(\mathbb{R}^d)$, and let us define for a fixed $\gamma > 0$

$$F_x(y) := \sum_{|k|<\gamma} \partial^k f(x) \, \frac{(y-x)^k}{k!}, \qquad x, y \in \mathbb{R}^d \,.$$

Then the classical Taylor theorem says that there exists a function $R(x, y)$ such that

$$f(y) - F_x(y) = R(x, y), \qquad |R(x, y)| \lesssim |x - y|^\gamma$$

uniformly for every x, y on compact sets of $\mathbb{R}^d$.

We say that the distribution f is locally well approximated by the germ $(F_x)_{x \in \mathbb{R}^d}$.

Scaling

Let us introduce now the following fundamental tool:

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\lambda > 0$ and $y \in \mathbb{R}^d$

$$\varphi_y^\lambda(w) := \frac{1}{\lambda^d}\,\varphi\left(\frac{w-y}{\lambda}\right), \qquad w \in \mathbb{R}^d .$$

Then the local approximation property

$$f(y) - F_x(y) = R(x,y), \qquad |R(x,y)| \lesssim |x-y|^\gamma$$

implies for any $\varphi \in \mathcal{D}$, uniformly for y in compact sets of $\mathbb{R}^d$, $\lambda \in]0,1]$.

$$\begin{aligned}\left|(f - F_y)(\varphi_y^\lambda)\right| &= \left|\int_{\mathbb{R}^d} R(y,w)\,\varphi_y^\lambda(w)\,\mathrm{d}w\right| \\ &\lesssim \frac{1}{\lambda^d}\int_{B_y(\lambda)} |w-y|^\gamma\,\mathrm{d}w \lesssim \lambda^\gamma\end{aligned}$$

Taylor expansions

Another simple formula in this context is

$$\left|(F_z - F_y)(\varphi_y^\lambda)\right| \lesssim (|y-z|+\lambda)^\gamma,$$

for any $\varphi \in \mathcal{D}$, uniformly for y, z in compact sets of $\mathbb{R}^d$, $\lambda \in]0,1]$.

We call this property coherence, see below.

This comes from a simple estimate of $F_z(w) - F_y(w)$.

Coherence of Taylor expansions

Let us note that we can Taylor expand also the derivatives of f: for $|k| < \gamma$

$$\partial^k f(y) = \sum_{|\ell|<\gamma-|k|} \partial^{k+\ell} f(z) \, \frac{(y-z)^\ell}{\ell!} + R^k(y,z), \qquad |R^k(y,z)| \lesssim |y-z|^{\gamma-|k|}.$$

Then we can write

$$\begin{aligned} F_y(w) &= \sum_{|k|<\gamma} \partial^k f(y) \, \frac{(w-y)^k}{k!} \\ &= \sum_{|k|<\gamma} \left(\sum_{|\ell|<\gamma-|k|} \partial^{k+\ell} f(z) \, \frac{(y-z)^\ell}{\ell!} + R^k(y,z) \right) \frac{(w-y)^k}{k!} \\ &= F_z(w) + \sum_{|k|<\gamma} R^k(y,z) \, \frac{(w-y)^k}{k!}. \end{aligned}$$

Coherence of Taylor expansions

Therefore

$$F_z(w) - F_y(w) = -\sum_{|k|<\gamma} R^k(y,z)\,\frac{(w-y)^k}{k!}.$$

In particular

$$\begin{aligned}
|F_z(w) - F_y(w)| &\le \sum_{|k|<\gamma} |R^k(y,z)|\,\frac{|w-y|^k}{k!} \\
&\lesssim \sum_{|k|<\gamma} |y-z|^{\gamma-|k|}\,|w-y|^k \\
&\lesssim (|y-z| + |w-y|)^\gamma
\end{aligned}$$

since $a^t b^s \le (a+b)^t (a+b)^s$ for $a, b, t, s \ge 0$.

Coherence of Taylor expansions

Now recall that

$$\varphi_y^\lambda(w) := \frac{1}{\lambda^d}\,\varphi\left(\frac{w-y}{\lambda}\right), \qquad w \in \mathbb{R}^d\,.$$

Then

$$\begin{aligned}\left|\int_{\mathbb{R}^d} (F_z(w) - F_y(w))\ \varphi_y^\lambda(w)\,\mathrm{d}w\right| &\lesssim \frac{1}{\lambda^d}\ \int_{B_y(\lambda)} (|y-z| + |w-y|)^\gamma\,\mathrm{d}w \\ &\lesssim (|y-z|+\lambda)^\gamma.\end{aligned}$$

We have obtained for the germ $(F_y)_{y\in\mathbb{R}^d}$ and for any $\varphi \in \mathcal{D}$, $y, z \in \mathbb{R}^d$

$$\left|(F_z - F_y)(\varphi_y^\lambda)\right| \lesssim (|y-z|+\lambda)^\gamma.$$

Coherence

Let us set from now on

$$\varepsilon_n := 2^{-n}, \qquad n \in \mathbb{N}.$$

In particular for the germ related to a Taylor expansion we have for $\lambda \in \{\varepsilon_n : n \in \mathbb{N}\}$

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim (|y-z| + \varepsilon_n)^\gamma, \qquad \left|(f - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\gamma,$$

for any $\varphi \in \mathcal{D}$, uniformly for y, z in compact sets of $\mathbb{R}^d$ and $n \in \mathbb{N}$.

We say that a germ $(F_z)_{z\in\mathbb{R}^d} \subset \mathcal{D}'$ is (α, γ)-coherent for $\alpha, \gamma \in \mathbb{R}$ with $\alpha \le \gamma$, if there exists $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\int \varphi \neq 0$ and

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\alpha (|y-z| + \varepsilon_n)^{\gamma-\alpha}$$

uniformly for z, y in compact sets of $\mathbb{R}^d$ and $n \in \mathbb{N}$.

Hairer's Reconstruction Theorem (without regularity structures)

Theorem (Hairer 14, Caravenna-Z. 20)

Consider a (α,γ)-coherent germ with $\gamma > 0$, namely we suppose that there exists $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\int \varphi \neq 0$ and

$$|(F_y - F_x)(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\alpha}(|x-y| + \varepsilon_n)^{\gamma-\alpha},$$

uniformly for x, y in compact sets of $\mathbb{R}^d$ and $n \in \mathbb{N}$ (coherence condition). Then there exists a unique $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ such that

$$|(\mathcal{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\gamma}$$

uniformly for x in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$, $\{\psi \in \mathcal{D}(B(0,1)) : \|\psi\|_{C^r} \leq 1\}$ with a fixed $r > -\alpha$.

Comments

- This result was stated and proved by Martin in [Hai14] for a subclass of germs related to regularity structures. He used wavelets.
- Later Otto-Weber proposed an approach based on a semigroup. This corresponds to a special choice of the test functions φ, ψ.
- Our statement is more general and requires no knowledge of regularity structures.
- This result can be seen as a generalisation of the Sewing Lemma in rough paths (Gubinelli, Feyel-de La Pradelle).
- The construction is completely local: constants and even the exponent α can depend on the compact set.
- We also cover the case $\gamma \leq 0$ (see below).
- Pavel Zorin-Kranich recently showed how to simplify, shorten and (slightly) improve our proof.

Proof for $\gamma > 0$: Uniqueness

Suppose we have two distributions $f, g \in \mathcal{D}'$ which satisfy, uniformly for $x \in K$ for any compact $K \subset \mathbb{R}^d$,

$$\lim_{n\to+\infty} |(f - F_x)(\varphi_x^{\varepsilon_n})| = \lim_{n\to+\infty} |(g - F_x)(\varphi_x^{\varepsilon_n})| = 0\,. \tag{1}$$

We may assume that $c := \int \varphi = 1$ (otherwise just replace φ by $c^{-1}\,\varphi$).

We set $T := f - g$, we fix a test function $\psi \in \mathcal{D}$. We recall the definition of the convolution

$$\psi * \varphi(w) = \int_{\mathbb{R}^d} \psi(y)\,\varphi(w-y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \psi(w-y)\,\varphi(y)\,\mathrm{d}y\,,$$

for $w \in \mathbb{R}^d$. This implies

$$T(\psi * \varphi) = \int_{\mathbb{R}^d} \psi(y)\,T(\varphi(\cdot - y))\,\mathrm{d}y = \int_{\mathbb{R}^d} T(\psi(\cdot - y))\,\varphi(y)\,\mathrm{d}y\,. \tag{2}$$

Proof for $\gamma > 0$: Uniqueness

It follows that

$$T(\psi) = \lim_{n\to+\infty} T(\psi * \varphi_0^{\varepsilon_n}).$$

Moreover

$$T(\psi * \varphi_0^{\varepsilon_n}) = \int_{\mathbb{R}^d} T(\varphi_0^{\varepsilon_n}(\cdot - y))\,\psi(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} T(\varphi_y^{\varepsilon_n})\,\psi(y)\,\mathrm{d}y\,,$$

$$|T(\psi * \varphi_0^{\varepsilon_n})| = \left| \int_{\mathbb{R}^d} T(\varphi_y^{\varepsilon_n})\,\psi(y)\,\mathrm{d}y \right| \le \|\psi\|_{L^1} \sup_{y\in \mathrm{supp}(\psi)} \left|T(\varphi_y^{\varepsilon_n})\right| .$$

It remains to show that $\lim_{n\to+\infty} \sup_{y\in\mathrm{supp}(\psi)} \left|T(\varphi_y^{\varepsilon_n})\right| = 0$. Now

$$|T(\varphi_y^{\varepsilon_n})| = |f(\varphi_y^{\varepsilon_n}) - g(\varphi_y^{\varepsilon_n})| \le |(f - F_y)(\varphi_y^{\varepsilon_n})| + |(g - F_y)(\varphi_y^{\varepsilon_n})|$$

which vanishes as $n \to +\infty$ uniformly for $y \in \mathrm{supp}(\psi)$, by the reconstruction bound (1).

Proof for $\gamma > 0$: Existence

We fix a test function $\varphi \in \mathcal{D}$ with $\int \varphi \neq 0$ which makes the germ F coherent.

We can find in an elementary way a related $\hat{\varphi} \in \mathcal{D}(B(0,1))$ such that

$$\int_{\mathbb{R}^d} \hat{\varphi}(y)\, \mathrm{d}y = 1\,, \qquad \int_{\mathbb{R}^d} y^k \hat{\varphi}(y)\, \mathrm{d}y = 0\,, \qquad \forall\, k \in \mathbb{N}_0^d:\ 1 \leq |k| \leq r-1\,,$$

for a given $r > -\alpha$. Then we define

$$\rho := \hat{\varphi}^2 * \hat{\varphi} \qquad \text{and} \qquad \check{\varphi} := \hat{\varphi}^{\frac{1}{2}} - \hat{\varphi}^2,$$

where by $\hat{\varphi}^{\frac{1}{2}}, \hat{\varphi}^2$ we mean $\hat{\varphi}^{\lambda}(z) = \lambda^{-d}\hat{\varphi}(\lambda^{-1}z)$ for $\lambda = \frac{1}{2}, 2$, respectively.

This peculiar choice of ρ ensures that the difference $\rho^{\frac{1}{2}} - \rho$ is a convolution:

$$\rho^{\frac{1}{2}} - \rho = \hat{\varphi} * \check{\varphi}\,.$$

It follows that

$$\rho^{\varepsilon_{n+1}} - \rho^{\varepsilon_n} = (\rho^{\frac{1}{2}} - \rho)^{\varepsilon_n} = \hat{\varphi}^{\varepsilon_n} * \check{\varphi}^{\varepsilon_n}\,.$$

Proof for $\gamma > 0$: Existence

Finally we define

$$f_n(z) := F_z(\rho_z^{\varepsilon_n}), \qquad f_n(\psi) := \int_{\mathbb{R}^d} F_z(\rho_z^{\varepsilon_n})\,\psi(z)\,\mathrm{d}z\,, \qquad z \in \mathbb{R}^d,\ \psi \in \mathcal{D}\,.$$

Then we want to prove that $f_n(\psi) \to f(\psi)$ and $|(f - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\gamma$ for all $\psi \in \mathcal{D}$, namely that

$$\mathcal{R}F = \lim_{n\to+\infty} f_n \qquad \text{in } \mathcal{D}'.$$

We study the function

$$f_{x,n}(z) := f_n(z) - F_x(\rho_z^{\varepsilon_n}) = (F_z - F_x)(\rho_z^{\varepsilon_n}), \qquad x, z \in \mathbb{R}^d. \tag{3}$$

Proof for $\gamma > 0$: Existence

We write $f_{x,n}$ as a telescoping sum:

$$
\begin{aligned}
f_{x,k+1}(z) - f_{x,k}(z) &= (F_z - F_x)(\rho_z^{\varepsilon_{k+1}} - \rho_z^{\varepsilon_k}) \\
&= (F_z - F_x)(\hat{\varphi}^{\varepsilon_n} * \check{\varphi}_z^{\varepsilon_n}) = \int_{\mathbb{R}^d} (F_z - F_x)(\hat{\varphi}_y^{\varepsilon_k})\, \check{\varphi}^{\varepsilon_k}(y - z)\, \mathrm{d}y \\
&= \underbrace{\int_{\mathbb{R}^d} (F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})\, \check{\varphi}^{\varepsilon_k}(y - z)\, \mathrm{d}y}_{g'_{x,k}(z)} + \underbrace{\int_{\mathbb{R}^d} (F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k})\, \check{\varphi}^{\varepsilon_k}(y - z)\, \mathrm{d}y}_{g''_k(z)}, \qquad (4)
\end{aligned}
$$

where again we use (2). By coherence we have

$$
|g''_k(z)| \le \|\check{\varphi}^{\varepsilon_k}\|_{L^1} \sup_{|y-z| \le \varepsilon_k} |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k})| \lesssim \varepsilon_k^{\alpha}\, \varepsilon_k^{\gamma-\alpha} = \varepsilon_k^{\gamma},
$$

$$
\left| \int_{\mathbb{R}^d} g'_{x,k}(z)\, \psi(z)\, \mathrm{d}z \right| \le \sup_{y \in \bar{K}_1} |(F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})|\, \|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^1} \lesssim \varepsilon_k^{\alpha} \|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^1}.
$$

Proof for $\gamma > 0$: Existence

By the properties of $\check\varphi$ we can write

$$(\check\varphi^\varepsilon * \psi)(y) = \int_{\mathbb{R}^d} \check\varphi^\varepsilon(y-z)\left\{\psi(z) - p_y(z)\right\} \mathrm{d}z\,,$$

where $p_y(z) := \sum_{|k|\leq r-1} \frac{\partial^k \psi(y)}{k!}(z-y)^k$ is the Taylor polynomial of ψ of order $r-1$ based at y; since $|\psi(z) - p_y(z)| \lesssim \|\psi\|_{C^r} |z-y|^r$, we obtain

$$\|\check\varphi^{\varepsilon_k} * \psi\|_{L^1} \lesssim \int_{\mathbb{R}^d} |\check\varphi^{\varepsilon_k}(y-z)|\, |z-y|^r \,\mathrm{d}z \lesssim \varepsilon_k^r\,.$$

We obtain

$$\left|\int_{\mathbb{R}^d} g'_{x,k}(z)\,\psi(z)\,\mathrm{d}z\right| \lesssim \varepsilon_k^{\alpha+r}\,, \qquad \left|\int_{\mathbb{R}^d} g''_k(z)\,\psi(z)\,\mathrm{d}z\right| \lesssim \varepsilon_k^{\gamma}\,.$$

Now we have by assumptions $\gamma > 0$ and $\alpha + r > 0$.

Proof for $\gamma > 0$: Existence

In particular, as $n \to +\infty$,

$$f_{x,n}(\psi) = f_{x,0}(\psi) + \sum_{k=0}^{n-1} \left[g'_{x,k}(\psi) + g''_k(\psi) \right]$$

converges to a distribution of order r. Now that $F_x(\rho_\cdot^{\varepsilon_n})$ converges to F_x in $\mathcal{D}'$. We obtain $f_n = f_{x,n} + F_x(\rho_\cdot^{\varepsilon_n})$ converges to a distribution $\mathcal{R}F$ in $\mathcal{D}'$. We also obtain for all ℓ

$$\mathcal{R}F(\psi) = F_x(\psi) + f_{x,\ell}(\psi) + \sum_{k=\ell}^{\infty} \left[g'_{x,k}(\psi) + g''_k(\psi) \right] ,$$

and the latter formula yields similarly the reconstruction bound $|(f - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\gamma$.

The Reconstruction Theorem for $\gamma \leq 0$.

Theorem (Hairer 14, Caravenna-Z. 20)

Let $F : \mathbb{R}^d \to \mathcal{D}'(\mathbb{R}^d)$ *be a* (α, γ)*-coherent germ, with* $\alpha \leq \gamma \leq 0$*, namely there exists a* $\varphi \in \mathcal{D}(\mathbb{R}^d)$ *with* $\int \varphi \neq 0$ *s.t.*

$$|(F_y - F_x)(\varphi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\alpha}(|x-y| + \varepsilon_n)^{\gamma-\alpha}, \qquad n \in \mathbb{N},\ x, y \in \mathbb{R}^d,$$

(coherence condition). Then there exists a non-unique $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ *such that*

$$|(\mathcal{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \begin{cases} \varepsilon_n^{\gamma} & \text{if } \gamma < 0 \\ (1 + |\log \varepsilon_n|) & \text{if } \gamma = 0 \end{cases}.$$

uniformly for x *in compact sets of* $\mathbb{R}^d$, $n \in \mathbb{N}$, $\{\psi \in \mathcal{D}(B(0,1)) : \|\psi\|_{C^r} \leq 1\}$ *with a fixed* $r > -\alpha$.

Proof for $\gamma \le 0$

In the proof with $\gamma > 0$, we wrote, see (4) and (3),

$$f_{x,n} := f_n - F_x(\rho_\cdot^{\varepsilon_n}) = f_{x,0} + \sum_{k=0}^{n-1} \left[g'_{x,k} + g''_k \right], \qquad |g'_{x,n}| \lesssim \varepsilon_n^{\alpha+r}, \qquad |g''_n| \le \varepsilon_n^\gamma.$$

Now we can choose r such that $\alpha + r > 0$, but $\gamma \le 0$ is fixed.

The solution is to define a different approximation sequence, eliminating the term g''_n whose convergence depends on $\gamma > 0$, and the proof follows with the same estimates. Namely

$$\bar{f}_n := f_n - \sum_{k=0}^{n-1} g''_k, \qquad \bar{f}_{x,n}(\psi) := \bar{f}_n(\psi) - F_x(\rho^{\varepsilon_n} * \psi) = f_{x,0}(\psi) + \sum_{k=0}^{n-1} g'_{x,k}(\psi).$$

Then with the same arguments $\bar{f}_n(\psi) \to \bar{f}(\psi)$ and $|(\bar{f} - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\gamma$.

Homogeneity

The coherence assumption only concerns $F_z - F_y$, never F_y alone.

Under coherence alone, the reconstruction $\mathcal{R}F$ exists in $\mathcal{D}'$ but we have little more information.

Another crucial notion for germs is homogeneity (with exponent $\bar{\alpha}$)

$$|F_x(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\bar{\alpha}}$$

uniformly for x in compact sets, $n \in \mathbb{N}$ and $\psi \in \mathcal{D}(B(0,1))$ with $\|\psi\|_{C^r} \leq 1$, for some fixed $r > -\bar{\alpha}$.

Negative Hölder (Besov) spaces

Given $\bar\alpha \in]-\infty, 0[$, we define $\mathcal{C}^{\bar\alpha} = \mathcal{C}^{\bar\alpha}(\mathbb{R}^d)$ as the space of distributions $T \in \mathcal{D}'$ such that for all $\psi \in \mathcal{D} \setminus \{0\}$

$$\frac{|T(\psi_x^\varepsilon)|}{\|\psi\|_{C^{r_{\bar\alpha}}}} \lesssim \varepsilon^{\bar\alpha}$$

$$\text{uniformly for } x \text{ in compact sets and } \varepsilon \in (0,1]\,,$$

where we define $r_{\bar\alpha}$ as the smallest integer $r \in \mathbb{N}$ such that $r > -\bar\alpha$.

Theorem

The reconstruction $\mathcal{R}F$ of a (α, γ)-coherent germ F with homogeneity exponent $\bar\alpha$ is in $\mathcal{C}^{\bar\alpha}$ (and the map $F \mapsto \mathcal{R}F \in \mathcal{C}^{\bar\alpha}$ is linear continuous).

Sewing versus reconstruction

In dimension $d = 1$, the Sewing Lemma and the Reconstruction are almost equivalent.

For a continuous $\Xi : \{(s,t) : 0 \leq s \leq t \leq T\} \to \mathbb{R}$ which vanishes on the diagonal we can define the germ $F_t(\cdot) := \partial_s \Xi_{\cdot,t}$.

Let $z > y > x$ and $\varphi := \mathbb{1}_{(-1,0)}$, so that $\varphi_y^{y-x} = \frac{1}{y-x}\mathbb{1}_{(x,y)}$. Then

$$\begin{aligned}(F_z - F_y)(\varphi_y^{y-x}) &= \frac{1}{y-x}\int_x^y \left(\partial_s \Xi_{s,z} - \partial_s \Xi_{s,y}\right) \mathrm{d}s \\ &= -\frac{1}{y-x}\left(\Xi_{x,z} - \Xi_{x,y} - \Xi_{y,z}\right).\end{aligned}$$

Then

$$\left|(F_z - F_y)(\varphi_y^{y-x})\right| \lesssim |y-x|^{-1}(|z-y| + |y-x|)^{\beta-1+1} \iff |\Xi_{x,z} - \Xi_{x,y} - \Xi_{y,z}| \lesssim |z-x|^{\beta}$$

namely $(-1, \beta - 1)$-coherence of F is equivalent to $\delta\Xi \in \mathcal{C}_3^{\beta}$.

Sewing versus reconstruction

In particular, we can interpret the conditions

$$\underbrace{\sup_{0\le s<t\le T} \frac{|\Xi_{s,t}|}{|t-s|^\alpha} < +\infty}_{\text{homogeneity}} \qquad \underbrace{\sup_{0\le s<u<t\le T} \frac{|\Xi_{s,t}-\Xi_{s,u}-\Xi_{u,t}|}{|t-s|^\beta} < +\infty}_{\text{coherence}} .$$

As for reconstruction, also Sewing is possible under mere coherence

- coherence implies existence of $\mathcal{I}\Xi$
- homogeneity implies that $\mathcal{I}\Xi \in \mathcal{C}^\alpha$.

Moreover for $\beta \le 1$ we still have a version of the Sewing Lemma, as for Reconstruction with $\gamma = \beta - 1 \le 0$ (see Broux/Z.).

Singular product

Let $f \in \mathcal{C}^\alpha$ with $\alpha > 0$ and $F_y(w) = \sum_{|k|<\alpha} \partial^k f(y) \frac{(w-y)^k}{k!}$.

Let also $g \in \mathcal{C}^\beta$ with $\beta \leq 0$. We define the germ $P = (P_x := g \cdot F_x)_{x \in \mathbb{R}^d}$, that is

$$P_x(\varphi) = (g \cdot F_x)(\varphi) := g(\varphi F_x), \qquad \varphi \in \mathcal{D}.$$

Theorem

If $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$, with $\alpha > 0$ and $\beta \leq 0$, then the germ $P = (P_x)_{x \in \mathbb{R}^d}$ is $(\beta, \alpha + \beta)$-coherent, namely

$$\left|(P_z - P_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^\beta (|y - z| + \varepsilon_n)^\alpha.$$

If $\alpha + \beta > 0$, the unique distribution $\mathcal{R}P$ can be used to construct a canonical product of f and g. Moreover $\mathcal{R}P \in \mathcal{C}^\beta$.

If $\alpha + \beta \leq 0$, the (non-unique) distribution $\mathcal{R}P$ can be used to construct a non-canonical product of f and g. Moreover $\mathcal{R}P \in \mathcal{C}^\beta$.

Recent developments

- *Reconstruction Theorem for Germs of Distributions on Smooth Manifolds*
  by Paolo Rinaldi and Federico Sclavi
- *On a Microlocal Version of Young's Product Theorem*
  by Claudio Dappiaggi, Paolo Rinaldi and Federico Sclavi
- *Besov Reconstruction*
  by Lucas Broux and David Lee
- *Reconstruction theorem in quasinormed spaces*
  by Pavel Zorin-Kranich
- *A stochastic reconstruction theorem*
  by Hannes Kern
- *The Sewing lemma for* $0 < \gamma \leq 1$
  by Lucas Broux and L.Z.

What we did yesterday

We defined the notion of coherent germs: $(F_x)_{x\in\mathbb{R}^d} \subset \mathcal{D}'(\mathbb{R}^d)$ such that

$$\left|(F_z - F_y)(\varphi_y^{\varepsilon_n})\right| \lesssim \varepsilon_n^{\alpha}(|y-z| + \varepsilon_n)^{\gamma-\alpha},$$

where for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$, $\lambda > 0$ and $y \in \mathbb{R}^d$

$$\varphi_y^{\varepsilon_n}(w) := \frac{1}{\varepsilon_n^d}\,\varphi\left(\frac{w-y}{\lambda}\right), \qquad w \in \mathbb{R}^d\,.$$

Here $\gamma, \alpha \in \mathbb{R}$ and $\alpha \leq \gamma$.

We stated the Reconstruction Theorem: there exists $\mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$ such that

$$|(\mathcal{R}F - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^{\gamma}$$

(with a log-correction for $\gamma = 0$) and $\mathcal{R}F$ is unique if $\gamma > 0$.

An important special case of reconstruction

Let F be a (α, γ)-coherent germ with $\gamma > 0$.

We know that the (unique) reconstruction $\mathcal{R}F$ satisfies

$$\mathcal{R}F(\psi) = \lim_{n\to+\infty} \int_{\mathbb{R}^d} F_z(\rho_z^{\varepsilon_n})\, \psi(z)\, \mathrm{d}z, \qquad \forall\, \psi \in \mathcal{D}.$$

Let us suppose now that $(x, y) \mapsto F_x(y)$ is continuous.

Then by dominated convergence we obtain

$$\mathcal{R}F(\psi) = \int_{\mathbb{R}^d} F_z(z)\, \psi(z)\, \mathrm{d}z, \qquad \forall\, \psi \in \mathcal{D},$$

namely the reconstruction $\mathcal{R}F$ is equal to the function $z \mapsto F_z(z)$.

This includes the Taylor polynomial example where $F_x(x) = f(x)$.

Non-uniqueness for $\gamma \leq 0$

Let F be a (α, γ)-coherent germ with $\alpha \leq \gamma < 0$.

Suppose that $T \in \mathcal{D}'$ is a reconstruction of F, namely

$$|(T - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\gamma$$

uniformly for x in compact sets etc.

Then for any $D \in \mathcal{C}^\gamma$, the distribution $T + D$ is also a reconstruction of F.

Viceversa, if T' is a reconstruction of F, then

$$|(T - T')(\psi_x^{\varepsilon_n})| \leq |(T - F_x)(\psi_x^{\varepsilon_n})| + |(T' - F_x)(\psi_x^{\varepsilon_n})| \lesssim \varepsilon_n^\gamma$$

so that $T - T' \in \mathcal{C}^\gamma$.

Therefore, for $\gamma < 0$, the reconstruction of F is unique up to an element of $\mathcal{C}^\gamma$.

Again on singular products

Let us go back to the singular product between $f \in \mathcal{C}^\alpha$ with $\alpha > 0$ and $g \in \mathcal{C}^\beta$ with $\beta \leq 0$.

We defined a germ P which is $(\alpha, \alpha + \beta)$-coherent.

If $\alpha + \beta > 0$ then the product $f\, g = \mathcal{R}P$ is canonical (we can call it the Young product).

If $\alpha + \beta < 0$ then the reconstruction $\mathcal{R}P$ is unique up to an element of $\mathcal{C}^{\alpha+\beta}$.

Chapter 2: Models and modelled distributions

More on germs

The reconstruction theorem can be applied to coherent germs, which form a large (vector) space.

However this space is too large. When we want to solve SPDEs, we are going to use a much smaller space to set up a fixed point.

We are going to study germs which can be written as suitable linear combinations of a fixed finite family of germs.

An example in one-dimension

You saw in Theorem 55 of Riedel3.pdf that given

- $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$
- $\mathbf{X} = (X, \mathbb{X})$ a α-rough path
- $Y \in \mathcal{D}_X^\alpha([0,T])$ a controlled path

then setting

$$\Xi_{u,v} := Y_u \,\delta X_{u,v} + Y'_u \,\mathbb{X}_{u,v}$$

one obtains $\delta\Xi \in C_3^{3\alpha}$ and one can apply the Sewing Lemma to define the rough integral

$$I_t = \int_0^t Y_u \,\mathrm{d}\mathbf{X}_u,$$

which is the unique continuous function $I : [0,T] \to \mathbb{R}$ s.t.

$$I_0 = 0, \quad |I_t - I_s - \Xi_{s,t}| \lesssim |t-s|^{3\alpha}.$$

For the reconstruction theorem, we want analogs of $\mathbf{X}$ and Y to build coherent germs.

Pre-models

Definition

A *pre-model* is a pair (Π, Γ) s.t.

1. $\Pi = (\Pi^i)_{i\in I}$ is a family of germs $\Pi^i = (\Pi^i_x)_{x\in\mathbb{R}^d}$ labelled by a finite index set I,
2. $\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \mapsto (\Gamma^{ij}_{xy})_{i,j\in I}$ is a matrix-valued function such that

$$\Pi^j_y = \sum_{i\in I} \Pi^i_x \, \Gamma^{ij}_{xy} \,, \qquad j \in I, \; x, y \in \mathbb{R}^d,$$

3. there exist $(\alpha_i)_{i\in I} \subset \mathbb{R}$ and a $\varphi \in \mathcal{D}(\mathbb{R}^d)$ with $\int \varphi \neq 0$ such that

$$|\Pi^i_x(\varphi^{\epsilon_n}_x)| \lesssim \epsilon_n^{\alpha_i},$$

uniformly over x in compact sets of $\mathbb{R}^d$, $n \in \mathbb{N}$.

We denote $\bar\alpha := \min(\alpha_i, i \in I)$.

An example

For a fixed $\gamma > 0$, the family of classical monomials

$$\Pi_y^j(w) = \frac{(w-y)^j}{j!}, \qquad j \in \mathbb{N}^d, \quad y, w \in \mathbb{R}^d, \quad j \in I := \{i \in \mathbb{N}^d : |i| \le \gamma\},$$

with $\alpha_i = |i|$, any $\varphi \in \mathcal{D}$ and

$$\Gamma_{xy}^{ij} = \mathbb{1}_{(i \le j)} \frac{(x-y)^{j-i}}{(j-i)!}, \qquad i, j \in I,$$

forms a pre-model.

Modelled distributions

Definition

Let (Π, Γ) be a pre-model, and let $\gamma > \max(\alpha_i, i \in I)$.
If $f : \mathbb{R}^d \to \mathbb{R}^I$ is measurable and satisfies for all $i \in I$

$$\left|f_x^i\right| \lesssim 1, \qquad \left| f_x^i - \sum_{j\in I} \Gamma_{xy}^{ij} f_y^j \right| \lesssim |x-y|^{\gamma-\alpha_i},$$

uniformly for x, y in compact subsets of $\mathbb{R}^d$, then we call f a *distribution modelled* by (Π, Γ), or simply a *modelled distribution*, and we write $f \in \mathcal{D}^\gamma_{(\Pi,\Gamma)}$.

Given a pre-model (Π, Γ) and a modelled distribution $f \in \mathcal{D}^\gamma_{(\Pi,\Gamma)}$, we define the germ

$$\langle \Pi, f\rangle_x := \sum_{i\in I} \Pi_x^i f_x^i, \qquad x \in \mathbb{R}^d.$$

Coherence of $\langle \Pi, f \rangle$

We want to show that $\langle \Pi, f \rangle$ is $(\bar\alpha, \gamma)$-coherent, where $\bar\alpha := \min(\alpha_i, i \in I)$. Using the reexpansion property $\Pi^j_z = \sum_{i\in I} \Pi^i_y \, \Gamma^{ij}_{yz}$ we have

$$\langle \Pi, f \rangle_z - \langle \Pi, f \rangle_y = \sum_{j\in I} \Pi^j_z f^j_z - \sum_{i\in I} \Pi^i_y f^i_y = -\sum_{i\in I} \Pi^i_y \left(f^i_y - \sum_{j\in I} \Gamma^{ij}_{yz} f^j_z \right) .$$

Therefore

$$(\langle \Pi, f \rangle_z - \langle \Pi, f \rangle_y)(\varphi^\varepsilon_y) = -\sum_{i\in I} \Pi^i_y(\varphi^\varepsilon_y) \left(f^i_y - \sum_{j\in I} \Gamma^{ij}_{yz} f^j_z \right) ,$$

namely

$$\left| (\langle \Pi, f \rangle_z - \langle \Pi, f \rangle_y)(\varphi^\varepsilon_y) \right| \lesssim \sum_{i\in I} \varepsilon^{\alpha_i} |z-y|^{\gamma-\alpha_i} \lesssim \varepsilon^{\bar\alpha} (\varepsilon + |z-y|)^{\gamma-\bar\alpha},$$

uniformly for y, z in compact sets.

Moreover

$$\left|\langle \Pi, f\rangle_y(\varphi_y^\varepsilon)\right| \leq \sum_{i\in I} f_y^i\,|\Pi_y^i(\varphi_y^\varepsilon)| \lesssim \sum_{i\in I} \epsilon^{\alpha_i} \lesssim \varepsilon^{\bar{\alpha}},$$

uniformly over y in compact subsets of $\mathbb{R}^d$. In other words we have proved that

Theorem

If (Π, Γ) is a pre-model and $f \in \mathcal{D}^\gamma_{(\Pi,\Gamma)}$, then $\langle \Pi, f\rangle$ is a $(\bar{\alpha}, \gamma)$-coherent germs with uniform homogeneity bound with exponent $\bar{\alpha}$.

Note that here $\alpha = \bar{\alpha}$.

Hölder functions as modelled distributions

We have see that the classical polynomial family

$$\Pi_y^i(w) = \frac{(w-y)^i}{i!}, \quad \Gamma_{xy}^{ij} = \mathbb{1}_{(i\le j)} \frac{(x-y)^{j-i}}{(j-i)!}, \quad i,j \in \mathbb{N}^d,$$

forms a pre-model. It is an interesting exercise to check that modelled distributions with respect to this pre-model are actually classical Hölder functions.

Let us consider for simplicity the case $\gamma \notin \mathbb{N}$. Now, a modelled distribution $f \in \mathcal{D}^\gamma_{(\Pi,\Gamma)}$ satisfies by definition

$$\left| f_x^i - \sum_{j\ge i,\, |j|<\gamma} \frac{(x-y)^{j-i}}{(j-i)!} f_y^j \right| \lesssim |x-y|^{\gamma-|i|}, \qquad \forall\, |i| < \gamma.$$

This is in fact a Taylor expansion of f^i at order $\lfloor \gamma - |i| \rfloor$ with a remainder of order $\gamma - |i|$, and this implies that f^i is of class $C^{\gamma-|i|}$ and

$$f^j = \partial_{j-i} f^i, \qquad \forall\, j \ge i.$$

Hölder functions as modelled distributions

In particular, for $i = 0$ we see that f^0 is of class C^γ and satisfies

$$f^0(y) - F_x(y) = R(x,y), \qquad |R(x,y)| \lesssim |x-y|^\gamma$$

Then f^0 is a reconstruction of $\langle \Pi, f \rangle$, and since $\gamma > 0$ it is the unique reconstruction. In other words we have seen that

$$f^0 = \mathcal{R}\langle \Pi, f \rangle \in C^\gamma, \qquad f^i = \partial_i f^0, \quad \forall |i| < \gamma.$$

The fact that f^0 is the reconstruction of $\langle \Pi, f \rangle$ is also a consequence of $\mathcal{R}\langle \Pi, f \rangle = \{x \mapsto \langle \Pi, f \rangle_x(x)\} = \{x \mapsto f^0_x\}$.

Semi-norms

Back to the general case, for a fixed pre-model (Π, Γ) we can interpret, by analogy with the case of Hölder functions of the previous section, the space $\mathcal{D}^\gamma_{(\Pi,\Gamma)}$ of all distributions modelled by (Π, Γ) as the collection of *generalised derivatives* of $u := \mathcal{R}\langle \Pi, f\rangle$ with respect to the pre-model (Π, Γ).

We can define a system of seminorms for $f \in \mathcal{D}^\gamma_{(\Pi,\Gamma)}$

$$[f]_{\mathcal{D}^\gamma_{(\Pi,\Gamma)},K} = \sup_{i\in I} \sup_{x,y\in K,\, x\neq y} \frac{\left| f^i_x - \sum_{j\in I} \Gamma^{ij}_{xy} f^j_y \right|}{|x-y|^{\gamma-\alpha_i}},$$

where K is a compact subset of $\mathbb{R}^d$.

This is rather original with respect to the standard situation in ODEs or PDEs, where one sets an equation in a fixed Banach space. Here the Banach (Fréchet) space depends on an external parameter, the pre-model (Π, Γ). For SDEs and SPDEs, the pre-model (or rough path) (Π, Γ) is actually *random*.

Models

Definition

A *model* is a pre-model (Π, Γ), such that moreover

1. $\Gamma^{ii}_{xy} = 1$ for all $i \in I$,
2. $\Gamma^{ij}_{xy} = 0$ if $\alpha_i \geq \alpha_j$ and $i \neq j$,
3. $|\Gamma^{ij}_{xy}| \lesssim |x-y|^{\alpha_j - \alpha_i}$ if $\alpha_i < \alpha_j$.

For a fixed $\gamma > 0$, the family of classical monomials

$$\Pi^j_y(w) = \frac{(w-y)^j}{j!}, \qquad j \in \mathbb{N}^d, \quad y, w \in \mathbb{R}^d, \quad j \in I := \{i \in \mathbb{N}^d : |i| \leq \gamma\},$$

$$\Gamma^{ij}_{xy} = \mathbb{1}_{(i \leq j)} \frac{(x-y)^{j-i}}{(j-i)!}, \qquad i, j \in I,$$

with $\alpha_i = |i|$, forms a model.

Lemma

Let (Π, Γ) *be a model. Fix an exponent* $\gamma > \max(\alpha_i : i \in I)$ *and set* $\bar{\alpha} := \min(\alpha_i : i \in I)$. *Then*

1. *The space* $\mathcal{D}^{\gamma}_{(\Pi,\Gamma)}$ *is not reduced to the null vector.*
2. *For any* $\gamma' > \bar{\alpha}$, *the restricted family* $(\Pi', \Gamma') := (\Pi^i, \Gamma^{ij})_{i,j \in I'}$ *labelled by* $I' := \{i \in I : \alpha_i < \gamma'\}$ *is a model. If* $\gamma > \gamma'$, *the projection*

$$f = (f^i)_{i \in I} \mapsto f' = (f^i)_{i \in I'}$$

maps $\mathcal{D}^{\gamma}_{(\Pi,\Gamma)}$ *to* $\mathcal{D}^{\gamma'}_{(\Pi',\Gamma')}$.

Proof.

For the first assertion, we consider an element Π^i_x of minimal homogeneity $\bar{\alpha} = \min_I \alpha$. In this case we see that $\Gamma^{ij}_{xy} = \delta_{ij}$ for all $j \in I$, where δ is the Kronecker symbol, and the function $f^j_x = \delta_{ij}$ is a modelled distribution. □

Chapter 3: The Schauder estimates for germs

A theory, a theorem

This lecture and the next are based on work with L. Broux and F. Caravenna (see the Lecture Notes and a forthcoming paper). We discuss one of the most important operations on coherent germs: the convolution with a regularising integration kernel.

The temptative title for this paper is

- *Hairer's multilevel Schauder estimates without Regularity Structures*

In this paper we have extracted a single result (the multilevel Schauder estimates) from a larger theory (Regularity Structures).

We present the former in a simpler and more general version, without reference to the latter.

Definition (Regularising kernel)

Fix a dimension $d \in \mathbb{N}$ and an exponent $\beta \in (0, d)$. A measurable function $\mathsf{K} : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ is called a *β-regularizing kernel up to degree $m \in \mathbb{N}$* if the following conditions hold:

- the function $x \mapsto \mathsf{K}(x)$ is of class C^m on $\mathbb{R}^d \setminus \{0\}$;
- the following upper bound holds:

$$\forall\, k \in \mathbb{N}^d \;\text{ with }\; |k| \leq m : \qquad |\partial^k \mathsf{K}(x)| \lesssim \frac{1}{|x|^{d-\beta+|k|}} \mathbb{1}_{\{|x|\leq 1\}} \qquad (5)$$

uniformly for x in compact sets .

By the way, let us introduce the notations

$$\mathcal{G}^{\alpha,\gamma} := \{(H_x)_{x\in\mathbb{R}^d} : \; H \text{ is } \; (\alpha,\gamma)\text{-coherent}\}$$
$$\mathcal{G}^{\bar{\alpha};\alpha,\gamma} := \{(H_x)_{x\in\mathbb{R}^d} \in \mathcal{G}^{\alpha,\gamma} : \; H \text{ has homogeneity bound with exponent } \bar{\alpha}\}$$

Classical Schauder Estimates

Theorem

Let $\gamma \in \mathbb{R}$ *and* $\beta > 0$.
Let K *be a* β*-regularising kernel up to degree* $m > \gamma + \beta$.
Suppose that $\{\gamma, \gamma + \beta\} \cap \mathbb{Z} = \emptyset$.
Then, the convolution by K *defines a continuous linear map from* $\mathcal{C}^\gamma$ *to* $\mathcal{C}^{\gamma+\beta}$.

We want to lift this result to coherent germs, in a way which is compatible with the reconstruction.

Partition of unity

With a partition of unity, it is possible to decompose

$$\mathsf{K}(x) = \sum_{n=0}^{\infty} \mathsf{K}_n(x) \quad \forall\, x \in \mathbb{R}^d \setminus \{0\},$$

where $\mathsf{K}_n : \mathbb{R}^d \to \mathbb{R}$ is of class C^m and is supported in the annulus $\{\frac{1}{2}\epsilon_n \leq |x| \leq 2\epsilon_n\}$. Moreover

$$\forall k \in \mathbb{N}^d \text{ with } |k| \leq m :$$

$$|\partial^k \mathsf{K}_n(x)| \lesssim \frac{1}{|x|^{d-\beta-|k|}} \mathbb{1}_{\{\frac{1}{2}\epsilon_n \leq |x| \leq 2\epsilon_n\}}$$

$$\lesssim \epsilon_n^{\beta-d-|k|} \mathbb{1}_{\{\frac{1}{2}\epsilon_n \leq |x| \leq 2\epsilon_n\}}$$

$$\text{uniformly for } n \in \mathbb{N}.$$

Singular convolution

We want to consider the convolution $\mathsf{K} * f \in \mathcal{D}'$ between K and $f \in \mathcal{D}'$. This is *formally* defined by

$$(\mathsf{K} * f)(x) := f(\mathsf{K}(x - \cdot)) = \int_{\mathbb{R}^d} \mathsf{K}(x-y) f(\mathrm{d}y) ,$$

but we stress that in general $\mathsf{K} * f$ is ill-defined. Under suitable conditions, $\mathsf{K} * f$ can be defined as a distribution by duality: for a test function $\psi \in \mathcal{D}$ we set

$$(\mathsf{K} * f)(\psi) := f(\mathsf{K}^* \psi) \quad \text{where} \quad (\mathsf{K}^* \psi)(y) := \int_{\mathbb{R}^d} \psi(x)\, \mathsf{K}(x-y)\, \mathrm{d}x .$$

The delicate point is that $\mathsf{K}^* \psi$ *needs not be smooth*, hence we cannot hope to define $f(\mathsf{K}^* \psi)$ for arbitrary $(f, \psi) \in \mathcal{D}' \times \mathcal{D}$.

This delicate point is hidden under the carpet in these slides, but its solution is explained in the lecture notes.

Convolution with coherent germs

Fix two real numbers α, γ such that

$$\alpha \leq \gamma, \quad \gamma \neq 0.$$

We define r_α as the smallest integer larger than $-\alpha$, namely

$$r_\alpha := \min\{k \in \mathbb{N} : \ k > -\alpha\}.$$

Let $F = (F_x)_{x\in\mathbb{R}^d}$ be a (α, γ)-coherent germ. We now want to *lift the convolution with* K *on the space of coherent germs*, i.e. to find a coherent germ $H = (H_x)_{x\in\mathbb{R}^d}$ with the property

$$\mathcal{R}H = \mathsf{K} * \mathcal{R}F.$$

A simple solution is the constant germ $H_x \equiv \mathsf{K} * \mathcal{R}F$, which is trivially coherent, but this does not allow to construct a fixed-point theory for PDEs.

Convolution with coherent germs

The naive guess $H_x = \mathsf{K} * F_x$ needs not give a coherent germ, therefore we need to enrich it. To this purpose, we look for H_x of the following special form:

$$\forall x \in \mathbb{R}^d : \qquad H_x = \mathsf{K} * F_x + R_x \qquad \text{where } R_x(\cdot) \text{ is a } \textit{polynomial} .$$

Remarkably, this is possible with the following explicit solution:

$$H_x := \mathsf{K} * F_x + \underbrace{\sum_{|\ell|<\gamma+\beta} (\mathcal{R}F - F_x)\left(\partial^\ell \mathsf{K}(x-\cdot)\right) \mathbb{X}_x^\ell}_{R_x(\cdot)} ,$$

where we denote for $x \in \mathbb{R}^d$, $\ell \in \mathbb{N}^d$ the classical monomials

$$\mathbb{X}_x^\ell : \mathbb{R}^d \to \mathbb{R}, \qquad \mathbb{X}_x^\ell(w) := \frac{(w-x)^\ell}{\ell!}$$

and where we agree that

$$R_x(\cdot) \equiv 0 \qquad \text{if} \qquad \gamma + \beta \leq 0.$$

Schauder estimates on coherent germs

Theorem

Fix $\alpha, \gamma, \beta \in \mathbb{R}$ *such that*

$$\alpha \le \gamma, \qquad \gamma \neq 0, \qquad \beta > 0\,,$$

where we further assume for simplicity that $\{\alpha+\beta,\ \gamma+\beta\} \cap \mathbb{N} = \emptyset$. *Consider*

- $F = (F_x)_{x\in\mathbb{R}^d} \in \mathcal{G}^{\alpha,\gamma}$ *is a* (α,γ)*-coherent germ;*
- K *is a* β*-regularizing kernel up to degree* $m > \gamma + \beta + r_\alpha$.

Then

1. *the germ* $H = (H_x)_{x\in\mathbb{R}^d}$ *is well-defined.*
2. H *is* $((\alpha+\beta)\wedge 0, \gamma+\beta)$*-coherent, namely* $H \in \mathcal{G}^{(\alpha+\beta)\wedge 0,\gamma+\beta}$.
3. H *satisfies* $\mathcal{R}H = \mathsf{K} * \mathcal{R}F$.

Schauder estimates on coherent germs

In other words, setting $\mathcal{K}F := H$, with

$$H_x := \mathsf{K} * F_x + \sum_{|\ell|<\gamma+\beta} (\mathcal{R}F - F_x)\left(\partial^\ell \mathsf{K}(x-\cdot)\right) \mathbb{X}_x^\ell ,$$

we have a well-defined linear operator satisfying

$$\mathcal{K} : \mathcal{G}^{\alpha,\gamma} \to \mathcal{G}^{(\alpha+\beta)\wedge 0,\gamma+\beta}, \qquad \mathcal{R}\circ\mathcal{K} = \mathsf{K} * \mathcal{R}.$$

Let us define the new germ

$$J_x := F_x - \mathcal{R}F ,$$

which allows to rewrite H as

$$\begin{aligned} H_x &= \mathsf{K} * F_x - \sum_{|\ell|<\gamma+\beta} J_x\left(\partial^\ell \mathsf{K}(x-\cdot)\right) \mathbb{X}_x^\ell \\ &= \mathsf{K} * \mathcal{R}F + L_x, \qquad \text{where} \qquad L_x := \mathsf{K} * J_x - \sum_{|\ell|<\gamma+\beta} J_x\left(\partial^\ell \mathsf{K}(x-\cdot)\right) \mathbb{X}_x^\ell . \end{aligned}$$

Sketch of the proof

The proof is based on two steps:

- L is $((\alpha+\beta)\wedge 0, \gamma+\beta)$-coherent,
- L has homogeneity bound with exponent $\gamma+\beta$.

In other words we show that $L \in \mathcal{G}^{\gamma+\beta;(\alpha+\beta)\wedge 0,\gamma+\beta}$.

(Recall that we did not assume homogeneity of F. Indeed, $H_x = \mathsf{K} * \mathcal{R}F + L_x$ is not homogeneous either, in general.)

Then 0 is a $(\gamma+\beta)$-reconstruction of L, i.e. $\mathsf{K} * \mathcal{R}F$ is a $(\gamma+\beta)$-reconstruction of H, namely

$$\mathcal{R} \circ \mathcal{K} = \mathsf{K} * \mathcal{R}.$$

Chapter 4: The Schauder estimates for modelled distributions